Int. J. Math. And Appl., 6(2–A)(2018), 83–92
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Fixed Points of Generalized $\alpha - \beta - \psi-$Contractive Mappings in Partially Ordered Complete Metric Spaces

Venkata Ravindranadh Babu Gutti[1], Sudheer Kumar Pathina[1,*] and Satyanarayana Gedala[1]

1 Department of Mathematics, Andhra University, Visakhapatnam, Andhra Pradesh, India.

Abstract: In this paper, we introduce the notion of generalized $\alpha - \beta - \psi-$contractive mappings and provide sufficient conditions for the existence and uniqueness of a fixed points for such mappings in partially ordered metric spaces. We deduce some corollaries and provide examples in support of our main results. Our results generalize the results of Asgari and Badehian [4].

MSC: 47H10, 54H25.

Keywords: Fixed point, generalized $\alpha - \beta - \psi-$ contractive map, $\alpha - \beta-$admissible map, partially ordered complete metric space.

 Accepted on: 21.03.2018

1. Introduction

Fixed point theory plays a vital role in nonlinear analysis. Most of the generalizations of fixed point theorems in metric spaces originated from Banach contraction principle [5]. In 2012, Samet, Vetro, Vetro [11] introduced the concept of $\alpha - \psi-$ contractive mappings in metric spaces as a generalization of contraction maps. Recently, in 2015, Asgari and Badehian [4] introduced the concept of $\alpha - \beta - \psi-$contractive mappings in partially ordered complete metric spaces. In recent times, fixed point theory has been developed in partially ordered metric spaces rapidly and for more literature, we see the references [1–3, 6, 8–10]. In this paper, we denote

$$\Psi = \{\psi : [0, \infty) \to [0, \infty)/\psi \text{ is nondecreasing and } \sum_{n=1}^{\infty} \psi^n < +\infty \text{ for } t > 0 \text{ where } \psi^n \text{ is the } n\text{-th iterate of } \psi\}.$$

If $\psi \in \Psi$ then we have the following.

(1). $\psi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$;

(2). $\psi(t) < t$ for all $t > 0$ and

(3). $\psi(0) = 0$.

Definition 1.1. *Let $(X, \preceq)$ be a partially ordered set. We say that $f : X \to X$ is monotone nondecreasing if $x \preceq y \Rightarrow f(x) \preceq f(y)$ for all $x, y \in X$.*

* *E-mail: sudheer232.maths@hotmail.com*

In this paper, $(X, \preceq, d)$ denotes a partially ordered metric space, where d is a metric on X together with $\preceq$ is a partial order on X. If X is complete with respect to the metric d then we call $(X, \preceq, d)$ a partially ordered complete metric space.

Definition 1.2 ([4]). *Let $(X, \preceq, d)$ be a partially ordered metric space. We say that $f : X \to X$ is an $\alpha-\beta-\psi-$contractive mapping if there exist three functions $\alpha, \beta : X \times X \to [0, \infty), \psi \in \Psi$ such that for all $x, y \in X$ with $x \preceq y$,*

$$\alpha(x, y)d(f(x), f(y)) \le \beta(x, y)\psi(d(x, y)). \tag{1}$$

Definition 1.3 ([4]). *Let $f : X \to X$, $\alpha, \beta : X \times X \to [0, \infty)$ and $C_\alpha > 0,\ C_\beta \ge 0$. We say that f is an $\alpha - \beta-$ admissible mapping, if for all $x, y \in X$ with $x \preceq y$.*

(1). $\alpha(x, y) \ge C_\alpha \implies \alpha(fx, fy) \ge C_\alpha$,

(2). $\beta(x, y) \le C_\beta \implies \beta(fx, fy) \le C_\beta$,

(3). $0 \le \frac{C_\beta}{C_\alpha} \le 1$.

Example 1.4. *Let $X = [1, 3]$ with partial order $\preceq$ defined by $x \preceq y$ if and only if $x \ge y$ in the usual sense. We define $f : X \to X$ and $\alpha, \beta : X \times X \to [0, \infty)$ by $f(x) = \frac{x}{1+x}$ for all $x \in X$ and*

$$\alpha(x, y) = \begin{cases} 1 + \frac{y}{x} & \text{if } x \preceq y \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \beta(x, y) = \begin{cases} \frac{x}{y} & \text{if } x \preceq y \\ 0 & \text{otherwise.} \end{cases}$$

Then f is $\alpha - \beta-$admissible map with $C_\alpha = 2$ and $C_\beta = 1$.

Theorem 1.5 ([4]). *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $f : X \to X$ be a nondecreasing, $\alpha-\beta-\psi-$ contractive mapping satisfying the following conditions:*

(1). f is continuous,

(2). f is $\alpha - \beta-$admissible,

(3). there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,

(4). there exist $C_\alpha > 0$, $C_\beta \ge 0$ such that $\alpha(fx_0, x_0) \ge C_\alpha$, $\beta(fx_0, x_0) \le C_\beta$.

Then f has a fixed point.

Theorem 1.6 ([4]). *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $f : X \to X$ be a nondecreasing, $\alpha-\beta-\psi-$ contractive mapping satisfying the following conditions:*

(1). f is $\alpha - \beta-$ admissible map,

(2). there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,

(3). there exist $C_\alpha > 0$, $C_\beta \ge 0$ such that $\alpha(x_0, fx_0) \ge C_\alpha$, $\beta(x_0, fx_0) \le C_\beta$,

(4). if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge C_\alpha$, $\beta(x_n, x_{n+1}) \le C_\beta$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \ge C_\alpha$ and $\beta(x_n, x) \le C_\beta$,

(5). if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then f has a fixed point in X.

In this paper, we introduce generalized $\alpha - \beta - \psi-$ contractive mappings in partially ordered sets and prove the existence and uniqueness of fixed points in partially ordered complete metric spaces for such maps. We draw some corollaries and provide examples in support of our main results. our results generalize the results of Asgari and Badehian [4].

2. Main Results

In the following we introduce the notion of generalized $\alpha-\beta-\psi-$contractive mappings in partially ordered metric spaces.

Definition 2.1. *Let $(X,\preceq,d)$ be a partially ordered metric space. We say that $f:X\to X$ is a generalized $\alpha-\beta-\psi-$ contractive mapping if there exist three functions $\alpha,\beta:X\times X\to[0,\infty),\psi\in\Psi$ such that for all $x,y\in X$ with $x\preceq y$,*

$$\alpha(x,y)d(f(x),f(y))\leq\beta(x,y)\psi(M(x,y)), \tag{2}$$

where $M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\}$.

Example 2.2. *Let $X=[0,\infty)$ with partial order $\preceq$ defined by $x\preceq y$ if and only if $x\leq y$. We define $f:X\to X$ by*

$$f(x)=\begin{cases}\frac{x}{2} & \text{if } x\in[0,1],\\ e^x & \text{if } x>1.\end{cases}$$

Then clearly f is a nondecreasing map. Now, we define functions $\alpha,\beta:X\times X\to[0,\infty)$ and $\psi:[0,\infty)\to[0,\infty)$ by

$$\alpha(x,y)=\begin{cases}1 & \text{if } x,y\in[0,1] \ \textit{with}\ \ x\preceq y\\ 0 & \text{otherwise,}\end{cases}$$
$$\beta(x,y)=\begin{cases}\frac{3}{4} & \text{if } x,y\in[0,1] \ \textit{with}\ x\preceq y\\ 0 & \text{otherwise,}\end{cases} \quad \text{and}$$
$$\psi(t)=\frac{6}{7}t \text{ for } t\geq 0.$$

Then f is $\alpha-\beta-$ admissible map with $C_\alpha=C_\beta=1$. Now we show that f is a generalized $\alpha-\beta-\psi-$ contractive mapping. Let $x,y\in[0,1]$ with $x\preceq y$. Now

$$\begin{aligned}\alpha(x,y)d(fx,fy)=1.\frac{|x-y|}{2}&\leq\frac{18}{28}\psi(|x-y|)\\ &=\frac{3}{4}.\frac{6}{7}|x-y|=\frac{3}{4}\psi(d(x,y))\\ &\leq\frac{3}{4}\psi(M(x,y))=\beta(x,y)\psi(M(x,y)).\end{aligned}$$

Hence f is a generalized $\alpha-\beta-\psi-$ contractive map.

Theorem 2.3. *Let $(X,\preceq,d)$ be a partially ordered complete metric space. Let $f:X\to X$ be a nondecreasing selfmap of X. Assume that f is a generalized $\alpha-\beta-\psi-$ contractive mapping satisfying the following conditions:*

(i). f is continuous,

(ii). f is $\alpha-\beta-$admissible,

(iii). there exists $x_0\in X$ such that $x_0\preceq fx_0$,

(iv). there exist $C_\alpha>0$, $C_\beta\geq 0$ such that $\alpha(x_0,fx_0)\geq C_\alpha$, $\beta(x_0,fx_0)\leq C_\beta$.

Then f has a fixed point.

Proof. By condition (iii), there exists $x_0 \in X$ such that $x_0 \preceq fx_0 = x_1$ (say). Since f is nondecreasing, we have $fx_0 \preceq fx_1 = x_2$ (say), i.e., $x_1 \preceq x_2$. On continuing this process we get a sequence, defined by $x_{n+1} = fx_n$ for $n = 0, 1, 2, \ldots$, and $x_0 \preceq x_1 \preceq x_3 \preceq x_n \preceq \ldots$ is an increasing sequence. Since f is $\alpha - \beta$-admissible, by (iv), we have $\alpha(x_0, x_1) \geq C_\alpha \Rightarrow \alpha(fx_0, fx_1) \geq C_\alpha$ and $\beta(x_0, x_1) \leq C_\beta \Rightarrow \beta(fx_0, fx_1) \leq C_\beta$, i.e., $\alpha(x_1, x_2) \geq C_\alpha$ and $\beta(x_1, x_2) \leq C_\beta$. Again, since f is $\alpha - \beta-$admissible so that $\alpha(x_1, x_2) \geq C_\alpha \implies \alpha(fx_1, fx_2) \geq C_\alpha$ and $\beta(x_1, x_2) \leq C_\beta \Rightarrow \beta(fx_1, fx_2) \leq C_\beta$, i.e., $\alpha(x_2, x_3) \geq C_\alpha$ and $\beta(x_2, x_3) \leq C_\beta$. In a similar manner, on continuing this process we get that $\alpha(x_n, x_{n+1}) \geq C_\alpha$ and $\beta(x_n, x_{n+1}) \leq C_\beta$ for $n = 1, 2, 3, \ldots$. We consider

$$\begin{aligned} C_\alpha d(x_1, x_2) = C_\alpha d(fx_0, fx_1) &\leq \alpha(x_0, x_1) d(fx_0, fx_1) &(3)\\ &\leq \beta(x_0, x_1)\psi(M(x_0, x_1)) &(4)\\ &\leq C_\beta \psi(M(x_0, x_1). &(5)\end{aligned}$$

It follows that

$$d(x_1, x_2) \leq \frac{C_\beta}{C_\alpha}\psi(M(x_0, x_1)) \leq \psi(M(x_0, x_1)), \tag{6}$$

where

$$\begin{aligned} M(x_0, x_1) &= \max\{d(x_0, x_1), d(x_0, fx_0), d(x_1, fx_1), \frac{d(x_0, fx_1) + d(x_1, fx_0)}{2}\}\\ &= \max\{d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_2) + d(x_1, x_1)}{2}\}\\ &= \max\{d(x_0, x_1), d(x_1, x_2), \frac{d(x_0, x_2)}{2}\}\\ &\leq \max\{d(x_0, x_1), d(x_1, x_2)\}.\end{aligned}$$

Suppose $M(x_0, x_1) = d(x_1, x_2)$. Then from (6), we have

$$d(x_1, x_2) \leq \frac{C_\beta}{C_\alpha}\psi(M(x_0, x_1)) \leq \psi(d(x_1, x_2) < d(x_1, x_2),$$

a contradiction. Therefore

$$d(x_1, x_2) \leq \frac{C_\beta}{C_\alpha}\psi(M(x_0, x_1)) \leq \psi(M(x_0, x_1) \leq \psi(d(x_0, x_1)) < d(x_0, x_1).$$

By induction, it is easy to see that

$$d(x_{n+1}, x_{n+2}) \leq \frac{C_\beta}{C_\alpha}\psi(M(x_n, x_{n+1})) \leq \psi(M(x_n, x_{n+1})) \leq \psi(d(x_n, x_{n+1})) < d(x_n, x_{n+1}),$$

for $n = 0, 1, 2, \ldots$. Therefore $\{d(x_n, x_{n+1})\}$ is a decreasing sequence which is bounded below by zero. So there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. Suppose $r > 0$. Now

$$\begin{aligned} C_\alpha d(x_{n+1}, x_{n+2}) = C_\alpha d(fx_n, fx_{n+1}) &\leq \alpha(x_n, x_{n+1}) d(fx_n, fx_{n+1}) &(7)\\ &\leq \beta(x_n, x_{n+1})\psi(M(x_n, x_{n+1})) &(8)\\ &\leq C_\beta \psi(M(x_n, x_{n+1})), &(9)\end{aligned}$$

which implies that

$$d(x_{n+1}, x_{n+2}) \leq \frac{C_\beta}{C_\alpha}\psi(M(x_n, x_{n+1})) \leq \psi(M(x_n, x_{n+1})) \leq \psi^n(d(x_0, x_1)).$$

Now on letting $n \to \infty$, we have

$$r = \lim_{n\to\infty} d(x_{n+1}, x_{n+2}) \leq \lim_{n\to\infty} \psi^n(d(x_0, x_1)) = 0$$

so that $r = 0$, a contradiction. Hence $r = 0$, i.e., $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. We now show that the sequence $\{x_n\}$ is Cauchy. We fix $\epsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that $\sum_{n\geq n_0}^{\infty} \psi^n(d(x_0, x_1)) < \epsilon$. Let $m, n \in \mathbb{N}$ with $m > n > n_0$. Therefore by applying triangle inequality, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq \psi^n\big(d(x_0, x_1)\big) + \psi^{n+1}\big(d(x_0, x_1)\big) + ... + \psi^{m-1}\big(d(x_0, x_1)\big) \\ &= \sum_{n\geq n_0}^{m-1} \psi^n(d(x_0, x_1)) \leq \sum_{n\geq n_0}^{\infty} \psi^n(d(x_0, x_1)) < \epsilon. \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space so that there exists $x \in X$ such that $x_n \to x$. Now by the continuity of f, we have $fx = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$. Therefore x is a fixed point of f. □

Remark 2.4. *Theorem 1.5 follows as corollary to Theorem2.3, since every $\alpha - \beta - \psi-$ contractive mapping is a generalized $\alpha - \beta - \psi-$ contractive mapping.*

By relaxing the continuity assumption on f from Theorem 2.3 and by imposing sequential convergence of a sequence in X, we prove the following result.

Theorem 2.5. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $f : X \to X$ be a nondecreasing, generalized $\alpha - \beta - \psi-$ contractive mapping that satisfies the following conditions:*

(i). f is $\alpha - \beta-$ admissible map,

(ii). there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,

(iii). there exist $C_\alpha > 0$, $C_\beta \geq 0$ such that $\alpha(x_0, fx_0) \geq C_\alpha$, $\beta(x_0, fx_0) \leq C_\beta$,

(iv). if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq C_\alpha$, $\beta(x_n, x_{n+1}) \leq C_\beta$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq C_\alpha$ and $\beta(x_n, x) \leq C_\beta$,

(v). if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then f has a fixed point in X, provided ψ is continuous.

Proof. As in the proof of Theorem 2.3, the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n, n = 0, 1, 2, \ldots$ is a Cauchy sequence in X and is convergent to $x \in X$. Further, we obtained that $\alpha(x_n, x_{n+1}) \geq C_\alpha$ and $\beta(x_n, x_{n+1}) \leq C_\beta$. Therefore, by condition (iv), it follows that $\alpha(x_n, x) \geq C_\alpha$ and $\beta(x_n, x) \leq C_\beta$. Suppose that $fx \neq x$. From the condition (v) and by the fact that f is an $\alpha - \beta - \psi-$ contractive mapping, we have

$$\begin{aligned} C_\alpha d(x_{n+1}, fx) = C_\alpha d(fx_n, fx) &\leq \alpha(x_n, x) d(fx_n, fx) \\ &\leq \beta(x_n, x)\psi(M(x_n, x)) \\ &\leq C_\beta \psi(M(x_n, x)), \end{aligned}$$

which implies that

$$d(x_{n+1}, fx) \leq \frac{C_\beta}{C_\alpha}\psi(M(x_n, x)), \tag{10}$$

where

$$M(x_n, x) = \max\{d(x_n, x), d(x_n, x_{n+1}), d(x, fx), \frac{d(x_n, fx) + d(x, x_{n+1})}{2}\}. \tag{11}$$

Now on letting $n \to \infty$ in (11), we get

$$\lim_{n\to\infty} M(x_n, x) = d(x, fx). \tag{12}$$

Again on letting $n \to \infty$ in (10) and using (12), we get

$$d(x, fx) \leq \frac{C_\beta}{C_\alpha}\psi(\lim_{n\to\infty} M(x_n, x)) \leq \psi(\lim_{n\to\infty} M(x_n, x)) = \psi(d(x, fx)) < d(x, fx), \tag{13}$$

a contradiction. Hence x is a fixed point of f. □

Theorem 2.6. *Assume that the hypotheses of Theorem 2.3 hold. (Theorem 2.5),*

(*i*) *If (H1): for each $x, y \in X$ with $x \preceq y$, $\alpha(x, y) \geq C_\alpha$ and $\beta(x, y) \leq C_\beta$ holds then f has a unique fixed point in X.*

(*ii*) *If $\psi : [0, \infty) \to [0.\infty)$ is continuous, x and y in X are not comparable and* (H2): *hold, where*

(H2): there exists $z \in X$ such that

$$\begin{cases} z \preceq fz, z \preceq x, z \preceq y, \\ \alpha(z, fz) \geq C_\alpha \quad \text{and} \quad \beta(z, fz) \leq C_\beta, \\ \alpha(z, x) \geq C_\alpha \quad \text{and} \quad \beta(z, x) \leq C_\beta, \\ \alpha(z, y) \geq C_\alpha \quad \text{and} \quad \beta(z, y) \leq C_\beta, \end{cases} \tag{14}$$

then f has a unique fixed point in X.

Proof. **Case (i):** Suppose that x and y are two fixed points. Since f is a generalized $\alpha-\beta-\psi-$contractive mapping and from (H1), we have

$$\begin{aligned} C_\alpha d(x, y) = C_\alpha d(fx, fy) &\leq \alpha(x, y)d(fx, fy) \\ &\leq \beta(x, y)\psi(M(x, y)) \\ &\leq C_\beta\psi(M(x, y)), \end{aligned}$$

which implies that

$$d(x, y) \leq \frac{C_\beta}{C_\alpha}\psi(M(x, y)) \leq \psi(M(x, y)), \tag{15}$$

where $M(x, y) = \max\{d(x, y), d(x, fx), d(y, fy), \frac{d(x,fy)+d(y,fx)}{2}\} = d(x, y)$. Now, we substitute the value of $M(x, y)$ in (15), we get

$$d(x, y) \leq \frac{C_\beta}{C_\alpha}\psi(M(x, y)) \leq \psi(M(x, y)) \leq \psi(d(x, y)) < d(x, y),$$

a contradiction. Hence $x = y$.

Case (ii): Suppose that x and y are two fixed points of f which are not comparable. By (H2), there exist $z \in X$ such that

$$z \preceq x, \alpha(z,x) \geq C_\alpha \text{ and } \beta(z,x) \leq C_\beta. \tag{16}$$

Since f is $\alpha - \beta-$ admissible map and increasing, we get $fz \preceq fx, \alpha(fz, fx) \geq C_\alpha$ and $\beta(fz, fx) \leq C_\beta$, which implies that $fz \preceq x, \alpha(fz,x) \geq C_\alpha$ and $\beta(fz,x) \leq C_\beta$. On continuing this process, we have $f^n(z) \preceq x, \alpha(f^n(z), x) \geq C_\alpha$ and $\beta(f^n(z), x) \leq C_\beta$ for all $n \in \mathbb{N}$. Since f is a generalized $\alpha - \beta - \psi-$ contractive mapping, we get

$$\begin{aligned} C_\alpha d(f^{n+1}(z), x) = C_\alpha d(f(f^n(z)), fx) &\leq \alpha(f^n(z), x) d(f(f^n(z)), fx) \\ &\leq \beta(f^n(z), x) \\ &\leq C_\beta \psi(M(f^n(z), x)), \end{aligned}$$

which implies that

$$d(f^{n+1}(z), x) \leq \frac{C_\beta}{C_\alpha}\psi(M(f^n(z), x)) \tag{17}$$

where

$$M(f^n(z), x) = \max\{d(f^n(z), x), d(f^n(z), f^{n+1}(z)), d(x, fx), \frac{d(f^n(z), fx) + d(x, f^{n+1}(z))}{2}\}.$$

From (H2) and Theorem 2.3, $f^n(z)$ converges to some point $z' \in X$. Now on letting $n \to \infty$ in (17), we have

$$d(z', x) \leq \frac{C_\beta}{C_\alpha}\psi(\lim_{n\to\infty} M(f^n(z), x)) \leq \psi(\lim_{n\to\infty} M(f^n(z), x)) = \psi(z', x) \tag{18}$$

where $\lim_{n\to\infty} M(f^n(z), x) = d(z', x)$. If $d(z', x) > 0$. Then

$$d(z', x) \leq \psi(z', x) < d(z', x), \tag{19}$$

a contradiction. Therefore $z' = x$. Similarly, we can prove that $z' = y$. Hence we have $y = x$, and uniqueness of fixed point of f follows. □

3. Corollaries and Examples

Choose $C_\alpha = C_\beta = 1$ and $\beta(x, y) = 1$ for all $x, y \in X$ with $x \preceq y$ in Theorem 2.3, we get the following theorem as a corollary of Theorem 2.3.

Theorem 3.1 ([3]). *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $f : X \to X$ be a nondecreasing, $\alpha - \psi-$ contractive mapping satisfying the following conditions:*

(i). f is ordered $\alpha-$ admissible map,

(ii). there exists $x_0 \in X$ such that $x_0 \preceq fx_0$ and $\alpha(x_0, fx_0) \geq 1$

Then f has a fixed point in X.

Choose $C_\alpha = C_\beta = 1$ and $\beta(x,y) = 1 = \alpha(x,y)$ for all$x, y \in X$ with $x \preceq y$ in Theorem 2.3, we get the following Theorem as a corollary of Theorem 2.3.

Theorem 3.2. *[7] Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $f : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exist $\psi \in \Psi$ such that*

$$d(fx, fy) \le \psi(M(x, y))$$

for all $x, y \in X$ with $x \preceq y$. Suppose also that the following conditions hold:

(i). there exists $x_0 \in X$ such that $x_0 \preceq fx_0$ and

(ii). f is continuous.

Then f has a fixed point in X.

Choose $C_\beta = 1$ and $\beta(x, y) = 1$ for all $x, y \in X$ with $x \preceq y$ in Theorem 2.3 and Theorem 2.5, we get the following corollaries.

Corollary 3.3. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $f : X \to X$ be a nondecreasing, generalized $\alpha - \psi-$ contractive mapping satisfying the following conditions:*

(i). f is continuous,

(ii). f is $\alpha-$admissible,

(iii). there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,

(iv). there exist $C_\alpha > 0$ such that $\alpha(x_0, fx_0) \ge C_\alpha$.

Then f has a fixed point.

Corollary 3.4. *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $f : X \to X$ be a nondecreasing, generalized $\alpha - \psi-$ contractive mapping that satisfies the following conditions:*

(i). f is $\alpha-$ admissible map,

(ii). there exists $x_0 \in X$ such that $x_0 \preceq fx_0$,

(iii). there exist $C_\alpha > 0$ such that $\alpha(x_0, fx_0) \ge C_\alpha$,

(iv). if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \ge C_\alpha$, for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \ge C_\alpha$

(v). if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then f has a fixed point in X, provided ψ is continuous.

The following is an example in support of Theorem 2.3.

Example 3.5. *Let $X = [0, \infty)$, with the usual metric d. We define a partial order $\preceq$ on X by $\preceq := \{(x, y) \in X \times X : x = y\} \cup \{(0, \frac{1}{2}), (\frac{3}{2}, 2)\}$ with $x \preceq y$ if and only if $x \le y$ in the usual sense. We define $f : X \to X$ by*

$$f(x) = \begin{cases} x + \frac{3}{2} & \text{if } x \in [0, \frac{1}{2}), \\ 2 & \text{if } x \ge \frac{1}{2}. \end{cases}$$

Clearly f is continuous and nondecreasing on X. Now, we define functions $\alpha, \beta : X \times X \to [0,\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} \frac{3}{2} & \text{if } x \preceq y \\ 0 & \text{otherwise,} \end{cases} \qquad \beta(x,y) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \psi(t) = \frac{t^2}{1+t} \text{ for all } t \geq 0.$$

let $C_\alpha = \frac{5}{4}$ and $C_\beta = 1$ then $\frac{C_\beta}{C_\alpha} = \frac{1}{\frac{5}{4}} = \frac{4}{5} < 1$. Clearly f is $\alpha - \beta-$admissible map. Also $\psi(t) < t$ and ψ is nondecreasing so that $\psi^{n+1}(t) \leq \psi^n(t)$ for $n = 0,1,2,\ldots$. Therefore $\{\psi^n(t)\}$ is a decreasing sequence of non-negative real numbers. Therefore there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} \psi^n(t) = l$. We write $u_n = \psi^n(t)$ so that $u_{n+1} = \frac{(u_n)^2}{1+u_n}$. Therefore $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{u_n}{1+u_n} = \frac{l}{1+l} < 1$. Hence by D'Alembert's ratio test, we have $\sum\limits_{n=1}^{\infty} u_n < \infty$. Therefore ψ satisfies all the properties of Ψ. Now we show that f is a generalized $\alpha - \beta - \psi-$ contractive mapping. Now, let $(x,y) = (0,\frac{1}{2}) \in X$, with $0 \preceq \frac{1}{2}$. Then $\alpha(0,\frac{1}{2}) = \frac{3}{2}$, $\beta((0,\frac{1}{2}) = 1$, $d(f0, f\frac{1}{2}) = d(\frac{3}{2}, 2) = \frac{1}{2}$ and $M(0,\frac{1}{2}) = \frac{3}{2}$. Now $\alpha((0,\frac{1}{2})d((f0,f\frac{1}{2}) = \frac{3}{2}.\frac{1}{2} \leq 1.\frac{9}{10} = \beta((0,\frac{1}{2})\psi(M(0,\frac{1}{2}))$.

In the remaining possible cases, the inequality (2) holds trivially. Therefore f is a generalized $\alpha-\beta-\psi-$ contractive mapping and all the hypotheses of Theorem 2.3 and (H1) are satisfied and hence the hypotheses of Theorem 2.6 (i) hold and 2 is the unique fixed point of f. But $\alpha(x,y)d(fx,fy) = \alpha((0,\frac{1}{2})d((f0,f\frac{1}{2}) = \frac{3}{2}.\frac{1}{2} \nleq 1.\frac{1}{3} = \beta((0,\frac{1}{2})\psi(d(0,\frac{1}{2})) = \beta((x,y)\psi(\frac{1}{2})$, fails to hold for any α, β and ψ. Hence (1) fails to hold so that f is not a $\alpha - \beta - \psi-$ contractive map so that Theorem 1.5 is not applicable.

The following example is in support of Theorem 2.5 in which f is not continuous.

Example 3.6. *Let $X = [0,\infty)$, with the usual metric d. We define a partial order on X by $\preceq := \{(x,y) \in X \times X : x = y\} \cup \{(\frac{1}{2^n}, 0)/n = 1,2,\ldots\}$ with $x \preceq y$ if and only if $x \geq y$ in the usual sense. We define $f : X \to X$ by*

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0,1), \\ \frac{2x+1}{3} & \text{if } x \geq 1. \end{cases}$$

Clearly f is not continuous and nondecreasing on X. Now, we define functions $\alpha, \beta : X \times X \to [0,\infty)$ and $\psi : [0,\infty) \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} \frac{1}{2} & \text{if } x \preceq y \\ 0 & \text{otherwise,} \end{cases} \qquad \beta(x,y) = \begin{cases} \frac{1}{3} & \text{if } x \preceq y \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \psi(t) = \begin{cases} \frac{4t}{4+t} & \text{if } t \geq 1 \\ \frac{4t}{5} & \text{if } t \leq 1. \end{cases}$$

let $C_\alpha = \frac{2}{5}$ and $C_\beta = \frac{1}{3}$ then $\frac{C_\beta}{C_\alpha} = \frac{\frac{1}{3}}{\frac{5}{4}} = \frac{5}{6} < 1$. Clearly f is $\alpha - \beta-$admissible map. Also $\psi(t) < t$, ψ is nondecreasing and $\sum\limits_{n=0}^{\infty} \psi^n(t) < \infty$. For

Case (i): *Let $t \leq 1$ so that $\psi^n(t) = (\frac{4}{5})^n(t)$ for $n = 1,2,3\ldots$. Therefore, $\sum\limits_{n=0}^{\infty} \psi^n(t) = \sum\limits_{n=0}^{\infty} (\frac{4}{5})^n(t). < \infty$.*

Case (ii): *Let $t \geq 1$ so that $\psi(t) = \frac{4t}{4+t}$. If $\psi(t) \geq 1$ then $\psi^2(t) = (\frac{2t}{2+t})$. If $\psi^2(t) \geq 1$ then $\psi^3(t) = (\frac{4t}{4+3t})$. If $\psi^3(t) \geq 1$ then $\psi^4(t) = (\frac{t}{1+t})$. Since $\psi^4(t) < 1$, $\psi^{n+4}(t) = (\frac{4}{5}^n)\psi^4(t)$ for $n = 0,1,2\ldots$. Therefore by comparison test, we have $\sum\limits_{n=0}^{\infty} \psi^n(t) < \infty$. Hence ψ satisfies all the properties of Ψ with ψ continuous on $[0,\infty)$. We now show that f is a generalized $\alpha - \beta - \psi-$ contractive mapping. Now, let $(x,y) = (\frac{1}{2^n}, 0) \in X$, with $\frac{1}{2^n} \preceq 0$. Then $\alpha(\frac{1}{2^n}, 0) = \frac{1}{2}$, $\beta(\frac{1}{2^n}, 0) = \frac{1}{3}$, $d(f(\frac{1}{2^n}), f0) = d(\frac{1}{2^{n+1}}, 0) = \frac{1}{2^{n+1}}$ and $M(\frac{1}{2^n}, 0) = \frac{1}{2^n}$. Consider $\alpha(\frac{1}{2^n}, 0)d(f(\frac{1}{2^n}), f0) = \frac{1}{2}\frac{1}{2^{n+1}} = \frac{1}{2^{n+2}} \leq \frac{1}{3}\frac{4}{5}\frac{1}{2^n} = \beta(\frac{1}{2^n}, 0)\psi(M(\frac{1}{2^n}, 0))$. In the remaining possible cases, the inequality (2) holds trivially. Now, we verify condition (iv) of Theorem 2.5. The sequences $\{x_n\}_{n=0}^{\infty}$ in X satisfying $\alpha(x_n, x_{n+1}) \geq \frac{2}{5}$ and $\beta(x_n, x_{n+1}) \leq \frac{1}{3}$ for $n = 0,1,2,\ldots$ are of the following forms.*

(i). $x_n = x$ *for* $n = 0, 1, 2 \ldots$ *so that* $\lim_{n\to\infty} x_n = x$.

(ii). There exists positive integers N, k *such that*

$$x_n = \begin{cases} \frac{1}{2^k} & \text{for } n = 0, 1, 2, \ldots, N \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

so that $\lim_{n\to\infty} x_n = 0 = x$. *It is easy to see that* $\alpha(x_n, x) \geq \frac{2}{5}$ *and* $\beta(x_n, x) \leq \frac{1}{3}$ *for* $n = 0, 1, 2, \ldots$.

Hence all the hypotheses of Theorem 2.5 are satisfied and $x = 0$ *and* $y = 1$ *are two fixed points of* f. *Here we observe that* 0 *and* 1 *are not comparable and (H2) fails to hold for any* $z \in X$.

References

[1] R. P. Agarwal, M. A. El-Gebeily and D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal., 87(2008), 109-116.

[2] I. Altun and H. Simsek, *Some fixed points theorems on ordered metric spaces and application*, Fixed point Theory Appl., 2010(2010), 17 pages.

[3] M. S. Asgari and Z. Badehian, *Fixed Point theorems for* $\alpha - \psi -$ *contractive mappings in partially ordered sets and application to ordinary differential equations*, Bull. Iranian Math. Soc., 41(6)(2015), 1375-1386.

[4] M. S. Asgari and Z. Badehian, *Fixed Point theorems for* $\alpha - \beta - \psi -$ *Contractive Mappings in Ordered set*, J. Nonlinear Sci. Appl., 8(2015), 518-528.

[5] S. Banach, *Surles operations dans les ensembles abstraits et leur applications aux equations integrales*, Fund. Math., 3(1922), 133-181.

[6] Lj. Ciric, N. Cakic, M. Rajovic and J. Ume, *Monotone generalized nonlinear contractions in partially ordered metric spaces*, Fixed point theory Appl., 2008(2008), 11 pages.

[7] K. Karapinar and B. Samet, *Generalized* $\alpha - \psi$ *contractive type mappings and related fixed point theorems with applications*, Abstr. Appl. Anal., 2012(2012), 17 pages.

[8] H. K. Nashine and B. Samet, *Fixed point results for mappings satisfying* $\alpha - \psi$*-weakly contractive conditions in partially ordered metric spaces*, Nonlinear Anal., 74(2011), 2201-2209.

[9] J. J. Nieto and R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order, 22(2005), 223-239.

[10] A. C. M. Ran and M. C. B Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc., 132(2003), 1435-1443.

[11] B. Samat, C. Vetro and P. Vetro, *Fixed point theorems for* $\alpha - \psi -$ *contractive type mappings*, Nonlinear Anal., 75(2012), 2154-2165.