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Asymptotic Attractivity and Stability for Random Functional Differential Equation

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Abstract: The random differential equations appears quite naturally in the study of changes or the rate of change of any processes like diffusion process or Brownian motion. The random differential equations have been the subject of rather extensive research area since long time. It have been utilized as models for a wide variety of random problems that have been encountered time and in the areas of all areas like Sciences and Environmental Sciences. In this paper, we have discussed an existence results for local asymptotic attractivity and asymptotic stability of random solution for nonlinear random functional differential equations in Banach algebra through hybrid random fixed point theorem. The results obtained, is generalize and extend the asymptotic attractivity and stability of random solutions for concerning random functional differential equations.

MSC: 60H25, 47H40.

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1. Statement of Problem

Let R denote the real line and let $I_0 = [-r, 0]$ and $I = [0, a]$ be two closed and bounded intervals in R. Let $J = I_0 \bigcup I$, then J is a closed and bounded intervals in R. Let C denote the Banach space of all random continuous real-valued functions ϕ on I_0 with the supremum norm $\|.\|_c$ defined by

$$\|\phi\|_c = \sup_{t\in I_0} |\phi(t)|$$

Clearly C is a Banach algebras with respect to this norm and the multiplication " . " defined by

$$(x \cdot y)(t, \omega) = x(t, \omega) \cdot y(t, \omega), \quad t \in I_0$$

Consider the first order random functional differential equation (RFDE)

$$\frac{d}{dt}\left(\frac{x(t, \omega)}{f(t, x(t, \omega), \omega)}\right) = g(t, x_{(t,\omega)}, \omega) \quad a.e. \quad t \in I, \omega \in \Omega, \tag{1}$$

$$x\,(t, \omega) = \phi\,(t, \omega)\,, \quad t \in I_0 \tag{2}$$

Where $f : I \times R \times \Omega \to R - \{0\}$ is random continuous, $g : I \times C \times \Omega \to R$ and function $x_{(t,\omega)}(\theta) = x(t + \theta, \omega)$ for all $\theta \in I_0$ and for all$\omega \in \Omega$. By a solution of RFDE (1) we means, a function $x \in C(J, R, \Omega) \bigcap AC(I_0, R, \Omega)$ that satisfies the

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equations in (1), where $AC(J, R)$ is the space of all absolutely random continuous real-valued functions on J. The study of random functional differential equations in Banach algebras is very rare. The study has been initiated via random fixed point theorems Dhage and Regan [4] and Dhage[3] and Palimkar [5, 6]. In this paper, we prove the local asymptotic attractivity results via a hybrid random fixed point theorem of Dhage [5] which gives the asymptotic stability of the random solutions for the RFDE (1).

2. Auxiliary Results

Let $X = BC(J, R)$ be the space of random continuous and bounded real-valued functions on J. Let Ω be a subset of X. Let $Q : X \to X$ be a random operator and consider the following random operator equation in X,

$$x\,(t, \omega) = Q\,x\,(t, \omega)\,, \quad for \quad t \in J. \tag{3}$$

Definition 2.1. *We say that random solutions of equation (3) are locally attractively if there exists a closed ball $\overline{B}_r(x_0)$ in the space $BC(J, R)$, for some $x_0 \in BC(J, R)$ and $r > 0$ such that for arbitrary random solutions $x = x(t, \omega)$ and $y = y(t, \omega)$ of equation (3) belonging to $\overline{B}_r(x_0)$ we have*

$$\lim_{t\to\infty} [x(t, \omega) - y(t, \omega)] = 0 \tag{4}$$

In the case when this limit (4) is uniform with respect to the set $B(x_0, r) \bigcap \Omega$ i.e. when for each $\varepsilon > 0$, $\exists\, T > 0$ such that

$$|x(t, \omega) - y(t, \omega)| \leq \varepsilon \tag{5}$$

For all $x, y \in B(x_0, r) \bigcap \Omega$ being random solution of (3) and $t \geq T$, we will say that random solutions of equation (3) are uniformly local attractive on J.

Definition 2.2. *A random solution $x_0 \in BC(J, R)$ of equation (3) is said to be asymptotic if there exists a real number α such that $\lim\limits_{t\to\infty} x(t, \omega) = \alpha$ and we say that the random solution x is asymptotic to the number α on J.*

Definition 2.3. *The random solutions of the equation (3) are said to be locally asymptotically attractive if there exists $x_0 \in BC(J, R)$ and $r > 0$ such that for all asymptotic random solutions $x = x(t, \omega)$, $y = y(t, \omega)$ of the equation (3) belonging to $B(x_0, r) \bigcap \Omega$, we have that condition (4) is satisfied. In the case when the condition (4) is satisfied uniformly with respect to the set $B(x_0, r) \bigcap \Omega$, we will say that random solutions of the equation (3) are uniformly locally asymptotically attractive.*

Definition 2.4. *Let X be a Banach space with norm $\|.\|$. An random operator $Q : X \to X$ is called Lipschitz if there exists a constant $K > 0$ such that*

$$\|Qx - Qy\| \leq K\,\|x - y\| \quad \forall \quad x, y \in X.$$

The constant K is called the Lipschitz constant of Q on X.

We use hybrid random fixed point theorem of Dhage for proving the main existence result for uniform local asymptotic attracivity of the random solutions for the RFDE (1). Also, we will need following definition.

Theorem 2.5 ([5])**.** *Let S be a closed convex and bounded subset of the Banach algebra X and let $A, B : S \times \Omega \to X$ be two operators such that*

(a). $A(\omega)$ is Lipschitz with Lipschit's constant K.

(b). $B(\omega)$ is completely continuous.

(c). $A(\omega)xB(\omega)x \in S$ for all $x \in S$ and $\omega \in \Omega$.

(d). $MK < 1$, where $M = \|B(S)\| = \sup\{\|Bx(\omega)\| : x \in J, \omega \in \Omega\}$.

Then the operator equation $Ax(\omega)Bx(\omega) = x(\omega)$ has a solution and the set of all solutions is compact in S.

We consider the following of hypotheses.

(A_1) The function $f : I \times R \to R - \{0\}$ is random continuous and there exists a bounded function $\ell : R_+ \to R_+$, with bound L satisfying $|f(t,x,\omega) - f(t,y,\omega)| \leq \ell(t,\omega)\,|x-y|$ for all $t \in J$ and $x, y \in R$.

(A_2) The function $F : I \to R$ defined by $F(t,\omega) = |f(t,\omega,0)|$ is bounded with $F_0 = \sup\limits_{t\geq 0} F(t,\omega)$.

(A_3) The function $\beta : R_+ \to R_+$ is random continuous.

(A_4) The function $q : R_+ \to R_+$ is random continuous and $\lim\limits_{t\to\infty} \phi(t,\omega)$.

(A_5) The function $g : I \times C \to R$ is random continuous and there exists random continuous function $a, b : R_+ \to R_+$ satisfying $|g(t,x,\omega)| \leq a(t,\omega)b(s,\omega)$, for all $t, s \in R_+ = I$ and $x \in C$, where, $\lim\limits_{t\to\infty} a(t,\omega) \int\limits_0^{\beta(t,\omega)} b(t,\omega)ds = 0$.

3. Main Result

Theorem 3.1. *Assume that the hypotheses (A_1), (A_2) through (A_5) hold. Furthermore, if $L(K_1 = K_2) < 1$, then the RFDE (1) has at least one random solution in the space $BC(J,R)$ and are random solutions are uniformly locally asymptotically attractive on J, where constants $K_1 > 0$, $K_2 > 0$ are*

$$K_1 = \lim_{t\to\infty} q(t,\omega) \tag{6}$$

And

$$K_2 = \sup_{t\geq 0} \vartheta(t,\omega) = \sup_{t\geq 0}\left[a(t,\omega)\int_0^{\beta(t,\omega)} b(s,\omega)ds\right]. \tag{7}$$

Proof. Set $X = BC(J,R)$. Consider the closed ball $\overline{B}_r(0)$ in X centered at origin O and radius r, where $r = \frac{F_0(K_1+K_2)}{1-L(K_1+K_2)} > 0$. Now the RFDE (1) is equivalent to the random functional integral equation (RFIE)

$$x(t,\omega) = [f(t,x(t,\omega),\omega]\,(\phi(0,\omega) + \int_0^{\beta(t,\omega)} g(s,x_s,\omega)ds), \quad t \in I \tag{8}$$

and $x(t,\omega) = \phi(t,\omega)$, $t \in I_0$. Define the two mapping A and B on $\overline{B}_r(0)$ by

$$Ax(t,\omega) = \begin{cases} f(t,x(t,\omega),\omega), & \text{if } t \in I; \\ 1, & \text{if } t \in I_0. \end{cases} \tag{9}$$

and

$$Bx(t,\omega) = \begin{cases} (\phi(0,\omega) + \int_0^{\beta(t,\omega)} g(s,x_{s,\omega},\omega)ds, & \text{if } t \in I; \\ \phi(t,\omega), & \text{if } t \in I_0. \end{cases} \tag{10}$$

Obviously A and B define the random operators $A, B : \overline{B}_r(0) \times \Omega \to X$. We shall show that A and B satisfy all the conditions of Theorem 3.1 on $\overline{B}_r(0)$. We first show that A is a Lipschitzian on $\overline{B}_r(0)$. Let $x, y \in \overline{B}_r(0)$. Then by (H_1),

$$\begin{aligned}|Ax(t,\omega) - Ay(t,\omega)| &\leq |f(t,x(t,\omega),\omega) - f(t,y(t,\omega),\omega)| \\ &\leq K(t,\omega)\,|x(t,\omega) - y(t,\omega)| \qquad \leq K(t,\omega)\,\|x-y\|\,, \quad \forall \ t \in J.\end{aligned}$$

Taking the supremum over t, we obtain $\|Ax - Ay\| \leq \|K\|\,\|x-y\|$ for all $x, y \in \overline{B}_r(0)$. So A is a Lipschitzian on $\overline{B}_r(0)$ with a Lipschitz constant $\|K\|$. Next we show that B is random continuous and random compact operator on $\overline{B}_r(0)$. First we show that B is random continuous on $\overline{B}_r(0)$, for this $\varepsilon > 0$ and $x, y \in \overline{B}_r(0)$ such that $\|x - y\| \leq \varepsilon$. Then we get

$$\begin{aligned}|Bx(t,\omega) - By(t,\omega)| &\leq \left|\phi(0,\omega) + \int_0^{\beta(t,\omega)} g(s,x_s,\omega)ds - \phi(0,\omega) - \int_0^{\beta(t,\omega)} g(s,y_s,\omega)ds\right| \\ &\leq \int_0^{\beta(t,\omega)} |g(s,x_s,\omega) - g(s,y_s,\omega)|\,ds \\ &\leq \int_0^{\beta(t,\omega)} \{|g(s,x_s,\omega)| + |g(s,y_s,\omega)|\}\,ds \\ &\leq \int_0^{\beta(t,\omega)} a(t,\omega)b(s,\omega)ds + \int_0^{\beta(t,\omega)} a(t,\omega)b(s,\omega)ds \\ &\leq 2\int_0^{\beta(t,\omega)} a(t,\omega)b(s,\omega)ds \\ &\leq 2\vartheta(t,\omega) \end{aligned} \tag{11}$$

Hence, in virtue of hypothesis(A_5), we infer that there exists $T > 0$ such that $\vartheta(t,\omega) \leq \varepsilon$ for $t \geq T$. Thus, for $t \geq T$ from (12), we obtain that

$$|Bx(t,\omega) - By(t,\omega)| \leq 2\varepsilon$$

Furthermore, let us assume that $t \in [0,T]$. Then evaluating similarly as above we obtain the estimate:

$$\begin{aligned}|Bx(t,\omega) - By(t,\omega)| &\leq \int_0^{\beta(t,\omega)} |g(s,x_{s,\omega},\omega) - g(s,y_{s,\omega},\omega)|\,ds \\ &\leq \int_0^{\beta(t,\omega)} w_r^T(g,\varepsilon,\omega)ds \\ &\leq \beta_T w_r^T(g,\varepsilon,\omega)\end{aligned} \tag{12}$$

Where we denoted $\beta_T = \sup\{\beta(t : t \in [o,T])\}$, and

$$w_r^T(g,\varepsilon,\omega) = \sup\{|g(s,x_s,\omega) - g(s,y_s,\omega)| : s \in [0,T], x, y \in [-r,r], |x-y| \leq \varepsilon\}.$$

Obviously, we have in view of continuity of β that $\beta_r < \infty$.

Moreover; from the uniform continuity of the function $g(t,x_t,\omega)$ on the set $I \times C$, we derive that $w_r^T(g,\varepsilon,\omega) \to o$ as $\varepsilon \to o$. Now, linking (12) and the above fact we conclude that the random operator B maps continuously the ball $\overline{B}_r(0)$ into itself. Next, we show that B is random compact on $\overline{B}_r(0)$. To finish, it is enough to show that every sequence $\{Bx_n\}$ in $B(\overline{B}_r(0))$ has Cauchy subsequence. Now by (A_4) and (A_5),

$$|Bx_n(t,\omega)| \leq |\phi(0)| + \int_0^{\beta(t,\omega)} |g(s,x_s,\omega)|\,ds \leq K_1 + \vartheta(t,\omega) \leq K_1 + K_2 \tag{13}$$

For all $t \in I$. Taking supremum over t, we obtain $\|Bx_n\| \leq K_1 + K_2$ for all $n \in N$. This shows that $\{Bx_n\}$ is a uniformly bounded sequence in $B(\overline{B}_r(0))$. We show that it is also equicontinuous. Let $\varepsilon > 0$ be given, since $\lim_{t\to\infty} \phi(t,\omega) = 0$ and $\lim_{t\to\infty} \vartheta(t,\omega) = 0$ there are constants $T_1 > 0$ and $T_2 > 0$ such that $|\phi(t,\omega)| < \frac{\varepsilon}{2}$ for all $t > T_1$ and $|\vartheta(t,\omega)| < \frac{\varepsilon}{2}$ for all $t > T_2$. Let $T = \max\{T_1, T_2\}$. Let $t, \lambda \in J$ be arbitrary. If $t, \lambda \in J$ then we have

$$\begin{aligned}|Bx_n(t,\omega) - Bx_n(\lambda,\omega)| &\leq |\phi(0) - \phi(0)| + \left|\int_0^{\beta(t,\omega)} g(s, x_{s_n}, \omega)ds - \int_0^{\beta(\lambda,\omega)} g(s, x_{s_n}, \omega)ds\right| \\ &\leq \left|\int_{\beta(\lambda,\omega)}^{\beta(t,\omega)} g(s, x_{s_n}, \omega)ds\right| \quad for \ \ t \in I.\end{aligned}$$

And $|Bx_n(t,\omega) - Bx_n(\lambda,\omega)| \leq |\phi(t) - \phi(\lambda)|$ for $t \in I_0$. By the uniform continuity of the function ϕ and J and the function g in$I \times C$, we obtain $|Bx_n(t,\omega) - Bx_n(\lambda,\omega)| \to 0$ as $t \to \lambda$. Hence $\{Bx_n\}$ is an equicontinuous sequence of functions in X. Now an application of Arzela-Ascoli theorem yields that $\{Bx_n\}$ has a uniformly convergent subsequence on the random compact subset $[o, T]$ of R. Without loss of generality, call the $\{Bx_n\}$ is Cauchy in X. Now $|Bx_n(t,\omega) - Bx(t,\omega)| \to 0$ as $x \to \infty$ for all $t \in [o, T]$. Then for given $\varepsilon > o \ \ \exists \ \ x_0 \in N$ such that

$$\sup_{0\leq p\leq T} \int_0^{\beta(p,\omega)} |g(s, x_{s_m}, \omega) - g(s, x_{s_n}, \omega)|\, ds < \frac{\varepsilon}{2}$$

For all $m, n \geq n_0$. Therefore, if $m, n \geq n_0$ then we have

$$\begin{aligned}\|Bx_m - Bx_n\| &= \sup_{0\leq t\leq\infty} \left|\int_0^{\beta(t,\omega)} |g(s, x_{s_m}, \omega) - g(s, x_{s_n}, \omega)|\, ds\right| \\ &\leq \sup_{0\leq p\leq T} \left|\int_0^{\beta(t,\omega)} |g(s, x_{s_m}, \omega) - g(s, x_{s_n}, \omega)|\, ds\right| + \sup_{p\leq T} \int_0^{\beta(p,\omega)} [|g(s, x_{s_n}, \omega)| + |g(s, x_{s_n}, \omega)|]\, ds \\ &< \varepsilon\end{aligned}$$

This shows that $\{Bx_n\} \subset B(\overline{B}_r(0)) \subset X$ is Cauchy. Since X is random complete, $\{Bx_n\}$ converges to a point in X. As $B(\overline{B}_r(0))$ is closed $\{Bx_n\}$ converges to a point in $B(\overline{B}_r(0))$. Hence $B(\overline{B}_r(0))$ is relatively random compact and consequently B is random continuous and random compact operator on $\overline{B}_r(0)$. Next, we show that $AxBx \in \overline{B}_r(0)$, for all $x \in \overline{B}_r(0)$. Let $x \in \overline{B}_r(0)$ be arbitrary. Then

$$\begin{aligned}|Ax(t,\omega)Bx(t,\omega)| &\leq |Ax(t,\omega)|\, |Bx(t,\omega)| \\ &\leq |f(t, x(t,\omega), \omega)| \left(|\phi(0)| + \int_0^{\beta(t,\omega)} |g(s, x_s, \omega)|\, ds\right) \\ &\leq [|f(t, x(t,\omega), \omega) - f(t, 0, \omega)| + |f(t, 0, \omega)|] \left(|\phi(0)| + \int_0^{\beta(t,\omega)} a(s,\omega) b(s,\omega) ds\right) \\ &\leq [\ell(t,\omega)\, |x(t,\omega)| + F(t,\omega)]\, (|\phi(0)| + \vartheta(t,\omega)) \\ &\leq [L\, |x(t,\omega)| + F_0]\, (K_1 + K_2) \\ &\leq L\, (K_1 + K_2)\, \|x\| + F_0\, (K_1 + K_2) \\ &= \frac{F_0\, (K_1 + K_2)}{1 - L\, (K_1 + K_2)} \\ &= r \quad for \ all \ t \in R_+.\end{aligned}$$

Taking the supremum over t, we obtain $\|AxBx\| \leq r$, for all $x \in \overline{B}_r(0)$. Hence hypothesis (c) of Theorem 2.1 holds. Here, one has

$$M = \left\|B(\overline{B}_r(0))\right\| = \sup\{\|Bx\| : x \in \overline{B}_r(0)\}$$

$$= \sup\{\sup_{t\geq 0}\{|\phi(0)| + \int_0^{\beta(t,\omega)} |g(s, x_s, \omega)|\, ds\} : x \in \overline{B}_r(0)\}$$

$$\leq \sup_{t\geq 0} |\phi(t,\omega)| + \sup_{t\geq 0} \vartheta(t,\omega)$$

$$\leq K_1 + K_2$$

And therefore, $M_K = L\,(K_1 + K_2) < \infty$. Now we apply Theorem (3) to conclude that the RFDE (1) has a random solution on J. Finally, we show the uniform locally asymptotic attractivity of the random solution for RFDE (1). Let x, y be any two solutions of the RFDE in $\overline{B}_r(0)$ defined on I. Then we have

$$\begin{aligned}
|x(t,\omega) - y(t,\omega)| &\leq \left| f(t, x(t,\omega), \omega) \left(\phi(0,\omega) + \int_0^{\beta(t,\omega)} |g(s, x_s, \omega)|\, ds \right) \right| \\
&+ \left| f(t, y(t,\omega), \omega) \left(\phi(0,\omega) + \int_0^{\beta(t,\omega)} |g(s, y_s, \omega)|\, ds \right) \right| \\
&\leq |f(t, x(t,\omega), \omega)| \left(|\phi(0,\omega)| + \int_0^{\beta(t,\omega)} |g(s, x_s, \omega)|\, ds \right) \\
&+ |f(t, y(t,\omega), \omega)| \left(|\phi(0,\omega)| + \int_0^{\beta(t,\omega)} |g(s, y_s, \omega)|\, ds \right) \\
&\leq 2\,(Lr + F_0)\,(|\phi(0,\omega) + \vartheta(t,\omega)|) \quad for\ all \quad t \in I.
\end{aligned}$$

Since $\sup_{t\to\infty} \vartheta(t,\omega) = 0$, for $\varepsilon > 0$, there are real numbers $' > 0$ and $T > 0$ such that for all $t \geq T'\, |\phi(0,\omega)| < \frac{\varepsilon}{2(Lr+F_0)}$ and $\vartheta(t,\omega) < \frac{\varepsilon}{2(Lr+F_0)}$ for all $t \geq T$. If we choose $T^* = \max\{T', T\}$, then from the above inequality it follows that $|x(t,\omega) - y(t,\omega)| \leq \varepsilon$ for all $t \in T^*$. It is easy to prove that every random solution of the RFDE (1) is asymptotic to zero on I. Consequently, the RFDE (1) has a random solution and all the random solutions are uniformly locally asymptotically attractive on I. The proof is complete. □

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