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On (0,1,2) Trigonometric Interpolation

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Abstract: The aim of this paper is to discuss the case of $(0,1,2)$ interpolation by trigonometric polynomial on the zeros of $\sin mx$ at the point $x_k=\frac{2\pi k}{n}$, where $k=0,1,2,\ldots,n-1$, where n is even $(n=2m)$ and the convergence behavior of this trigonometric polynomial. Let $R_n(x)$ be a trigonometric polynomial of order such that

$$R_n(x_k)=a_k$$
$$R_n'(x_k)=b_k$$
$$R_n''(x_k)=c_k$$

where $x_k=\frac{2\pi k}{n}$, $k=0,1,2,\ldots,n-1$ and n is even $(n=2m)$.

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1. Introduction

This paper is about to consider the case of $(0,1,2)$ interpolation by trigonometric polynomials when nodes are taken to be $x_k=\frac{2\pi k}{n}$, $k=0,1,2,\ldots,n-1$. Here we find the explicit form of the trigonometric polynomial $R_n(x)$ of order n for which

$$R_n(x_k)=a_k,\qquad R_n'(x_k)=b_k,\qquad R_n''(x_k)=c_k,$$

where a_k, b_k, c_k are arbitrary numbers and $x_k=\frac{2\pi k}{n}$, $k=0,1,2,\ldots,n-1$ are prescribed at the given nodes. In view of previous works chronically in 1960 O.Kiš [7] discuss the simple case of $(0,2)$ trigonometric interpolation. In 1965 A.Sharma and A.K.Varma [3] generalized this case as $(0,M)$ trigonometric interpolation. A.Sharma and A.K.Varma have done the plausible work to consider the $(0,M,N)$ case $M<N$. For the justification of this case A.Sharma and A.K.Varma [4] consider in 1968 a very simple case $(0,2,3)$. Many more convincing works have done by A.K.Varma [1, 2] as Some remarks on trigonometric interpolation consider the $(0,1,2,4)$ in 1969. Hermite-Birkhoff trigonometric interpolation the (0,1,2,M) case in 1973. In the series of these work several mathematician have considered different cases. There is a difference between the case $(0,2)$ studied by O.Kiš [7] and $(0,2,3)$ case studied by A.Sharma and A.K.Varma [4] that in the $(0,2,3)$ case interpolatory polynomials exist and unique for both n even and n odd. Another distinction between these two cases the interpolatory polynomials of case $(0,2,3)$ converges uniformly. These cases motivated us to consider $(0,1,2)$ trigonometric interpolation . Here we are interested to determine the explicit forms and convergence of the trigonometric polynomial $R_n(x)$ for n even$(=2m)$.

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2. Statement of the Main Theorem

We are interested in the trigonometric polynomial $R_n(x)$ of suitable order such that

$$R_n(x_k) = a_k, \qquad R'_n(x_k) = b_k, \qquad R''_n(x_k) = c_k, \tag{1}$$

where $x_k = \frac{2\pi k}{n}$, $k = 0, 1, 2.......n-1$. This is called the case of $(0, 1, 2)$ interpolation, when n is even $(= 2m)$. We require the trigonometric polynomial $R_n(x)$ to have the form

$$d_0 + \sum_{k=1}^{3m-1} (d_k \cos kx + e_k \sin kx) + d_{3m} \cos 3mx \tag{2}$$

by A.Zygmund [5]. We shall prove the following.

Theorem 2.1. *The trigonometric polynomial $R_n(x)$ satisfying (1) having form (2) is given by*

$$R_n(x) = \sum_{k=0}^{n-1} a_k U(x - x_k) + \sum_{k=0}^{n-1} b_k V(x - x_k) + \sum_{k=0}^{n-1} c_k W(x - x_k) \tag{3}$$

where

$$W(x) = \frac{1}{n^3}\left[1 + 2\sum_{j=1}^{m-1} \cos jx - \sum_{j=m+1}^{3m-1} \cos jx + \frac{cosmx}{2} - \frac{\cos 3mx}{2}\right] \tag{4}$$

$$V(x) = \frac{1}{n^2}\left[4\sin 2mx + \frac{4}{n}\sum_{j=1}^{m-1} j\sin jx + 2\sum_{j=m}^{2m-1} \sin jx - \sum_{j=m+1}^{3m-1}\left(1 + \frac{2j}{n}\right)\sin jx\right] \tag{5}$$

$$U(x) = \frac{1}{n}\left[1 + \sum_{j=1}^{m-1}\left(\frac{3n-4j}{2n}\right) - \frac{1}{2}\sum_{j=1}^{m-1} \sin jx \sin mx + \frac{1}{n^2}\sum_{j=m+1}^{3m-1}(j^2 - 3nj + 2n^2)\cos jx + \frac{1}{8}(5\cos mx - \cos 3mx)\right] \tag{6}$$

Let us consider

$$R_n(x) = \sum_{k=0}^{n-1} f(x_k) U(x - x_k) + \sum_{k=0}^{n-1} b_k V(x - x_k) + \sum_{k=0}^{n-1} c_k W(x - x_k) \tag{7}$$

$f(x)$ is a 2π-periodic continuous function and b_k , c_k are arbitrary numbers. We have to prove the following

Theorem 2.2. *Let $f(x)$ be 2π-periodic continuous function with $f(x) \in Lip\alpha, \alpha > 0$ and if*

$$\mid b_k \mid = o\left(\frac{n}{\log n}\right), \mid c_k \mid = o\left(\frac{n^2}{\log n}\right), k = 0, 1, 2, \ldots, n-1 \tag{8}$$

then $R_n(x)$ as given by (7) converges uniformly to $f(x)$ on every closed finite interval on the x-axis.

3. Proof of Theorem 2.1

Here we shall discuss the method of finding $U(x)$, $V(x)$, $W(x)$ which satisfies the given conditions respectively.

$$U(x_k) = \begin{cases} 1, & \text{for } k = 0 \\ 0, & \text{for } 1 \le |k| \le n-1 \end{cases}, \qquad U'(x_k) = 0, \qquad U''(x_k) = 0 \tag{9}$$

$$V(x_k) = 0, \qquad V'(x_k) = \begin{cases} 1, & \text{for } k = 0 \\ 0, & \text{for } 1 \le |k| \le n-1 \end{cases}, \qquad V''(x_k) = 0 \tag{10}$$

$$W(x_k) = 0, \qquad W'(x_k) = 0, \qquad W''(x_k) = \begin{cases} 1, & \text{for } k = 0 \\ 0, & \text{for } 1 \leq |k| \leq n-1 \end{cases} \tag{11}$$

We shall find the $U(x)$. Let

$$U(x) = \frac{1}{2m} \sin mx \cot \frac{x}{2} + \sin mx h(x) \tag{12}$$

where $h(x)$ is a trigonometric polynomial of order $2m$. Let $U'(x) = \sin mx g(x)$, where $g(x)$ is a trigonometric polynomial of order $\leq 2m$. Then $U''(x_k) = 0$ gives $g(x_k) = 0,\ k = 0, 1, 2, \ldots, n-1$.

Hence

$$g(x) = r(x) \sin mx$$

where $r(x)$ is a trigonometric polynomial of order $\leq m$, which gives us

$$U'(x) = r(x) \sin^2 mx \tag{13}$$

On differentiating the (12) and equalize with (13), which gives us

$$h(x_k) = \frac{1}{m^2} \sum_{j=1}^{m-1} (j-m) \sin jx_k \tag{14}$$

for $k = 0, 1, 2, \ldots, n-1$. Hence

$$h(x) = \frac{1}{m^2} \sum_{j=1}^{m-1} (j-m) \sin jx + p(x) \sin mx \tag{15}$$

where $p(x)$ is trigonometric polynomial of order m. Using (15) in (12), we get

$$p(x) = \frac{1}{2m^3} \sum_{j=1}^{m-1} j^2 \cos jx + \frac{1}{m^3} \sum_{j=1}^{m-1} j(m-j) \cos jx + \frac{\cos mx}{4m} + a \sin mx \tag{16}$$

where a is arbitrary constant. Using (15) and (16) in (12), which gives us $U(x)$. We have $a = 0$, because of $U(x - x_k)$ does not contain the term of $\sin 3mx$. Similarly one can obtain the explicit forms of $V(x)$ and $W(x)$ owing to condition (10), (11) respectively. Hence the theorem follows.

4. Estimates of the Fundamental Polynomials

Lemma 4.1. *The $W(x)$ defined in (4). Then*

$$\sum_{k=0}^{n-1} |W(x - x_k)| \leq C_1 n^{-2} \log n, \tag{17}$$

where C_1 is a numerical constant.

Proof. In order to prove (17) we can write $W(x)$ in this form

$$W(x) = \frac{\sin mx}{4m^3} \left\{ \sum_{j=1}^{2m-1} \sin jx + \frac{1}{2} \sin 2mx \right\} \tag{18}$$

Since

$$\sum_{k=0}^{n-1} \mid W(x-x_k) \mid \leq \sum_{k=0}^{n-1} \max_{1\leq p\leq 2m-1} \frac{1}{4m^3}\left|\sin m(x-x_k)\left\{\sum_{j=1}^{p}\sin j(x-x_k)+\frac{1}{2}\sin 2m(x-x_k)\right\}\right| \tag{19}$$

By using the well known inequality (Jackson [6], page 120)

$$\sum_{k=0}^{n-1} \max_{p}\left|\sum_{j=o}^{p}\sin j(x-x_k)\right| \leq 4\log n \tag{20}$$

the lemma follows. □

Lemma 4.2. *The* $V(x)$ *defined in* (5). *Then*

$$\sum_{k=0}^{n-1} |V(x-x_k)| \leq C_2 n^{-1}\log n, \tag{21}$$

where C_2 *is a numerical constant.*

Proof. We have $V(x)$in (5). Then

$$\begin{aligned}\sum_{k=0}^{n-1} \mid V(x-x_k) \mid \leq & \sum_{k=0}^{n-1}\left|\frac{4\sin 2m(x-x_k)}{n^2}\right| + \sum_{k=0}^{n-1}\left|\sum_{j=1}^{m-1}\frac{4j}{n^3}\sin j(x-x_k)\right| \\ & + \sum_{k=0}^{n-1}\left|\sum_{j=m}^{2m-1}\frac{2}{n^2}\sin j(x-x_k)\right| + \sum_{k=0}^{n-1}\left|\sum_{j=m+1}^{3m-1}\left(\frac{n+2j}{n^3}\right)\sin j(x-x_k)\right|\end{aligned} \tag{22}$$

Since

$$\sum_{k=0}^{n-1} \mid V(x-x_k) \mid \leq \mid S_1 \mid + \mid S_2 \mid + \mid S_3 \mid + \mid S_4 \mid$$

We have

$$\mid S_2 \mid \leq \left|\sum_{j=1}^{m-1}\frac{4j}{n^3}\sin j(x-x_k)\right|$$

Now we use Abel's inequality on the series as the coefficient in the above sum is increasing function of j for $1 \leq j \leq m-1$. Thus

$$\mid S_2 \mid \leq \frac{1}{2m^2}\max_{1\leq p\leq m-1}\left|\sum_{j=0}^{p}\sin j(x-x_k)\right|$$

and now

$$\mid S_4 \mid \leq \left|\sum_{j=m+1}^{3m-1}\left(\frac{n+2j}{n^3}\right)\sin j(x-x_k)\right|$$

Now again using the Abel's inequality $\left(\frac{n+2j}{n^3}\right)$ is an increasing function of j for $1 \leq j \leq 3m-1$. Thus

$$\mid S_4 \mid \leq \left|\frac{1}{m^2}\max_{1\leq p\leq 3m-1}\sum_{j=1}^{p}\sin j(x-x_k)\right|$$

Similarly we can find the estimate of S_1 and S_3. Now combining the estimates of S_1, S_2, S_3, and S_4 and using (20) we have lemma (18). □

Lemma 4.3. *The $U(x)$ defined in (6). Then*

$$\sum_{k=0}^{n-1} |U(x-x_k)| \le C_3 \log n, \tag{23}$$

where C_3 is a numerical constant.

Proof. The fundamental polynomial $U(x)$ is given in (6). Thus

$$\begin{aligned}\sum_{k=o}^{n-1} \mid U(x-x_k)\mid \le & \sum_{k=0}^{n-1}\left|\frac{1}{2m}\right| + \sum_{k=0}^{n-1}\left|\frac{5}{16m}\cos m(x-x_k)\right| + \sum_{k=0}^{n-1}\left|\frac{1}{16m}\cos 3m(x-x_k)\right| \\ & + \sum_{k=0}^{n-1}\left|\frac{1}{4m}\sum_{j=1}^{m-1}\sin j(x-x_k)\sin m(x-x_k)\right| + \sum_{k=0}^{n-1}\left|\sum_{j=1}^{m-1} a_j \cos j(x-x_k)\right| \\ & + \sum_{k=0}^{n-1}\left|\sum_{j=m+1}^{3m-1} b_j \cos j(x-x_k)\right| \end{aligned} \tag{24}$$

where $a_j = \frac{3m-2j}{4m^2}$; $b_j = \frac{j^2-6mj+8m^2}{8m^3}$. Since, a_j is a decreasing function of j for $1 \le j \le m-1$ while b_j is an increasing function as $0 \le b_j \le \frac{1}{4m}$ for $m+1 \le j \le 3m-1$. using Abel's inequality and using (20) we get the required result. □

5. Proof of Theorem 2.2

In order to prove that

Lemma 5.1. *If $f(x)$ is a continuous 2π-periodic function and satisfying $f(x) \in Lip\alpha, 0 < \alpha \le 1$, Then there exist a trigonometric polynomial $T_n(x)$ of order $\le n$ such that*

$$\mid f(x) - T_n(x)\mid = O(n^{-\alpha}) \tag{25}$$

$$\mid T_n^{(p)}(x)\mid = O(n^{p-\alpha}), p = 1, 2 \tag{26}$$

The formula (25) is well know due to Jackson. The proof of (26) is exactly similar to a corresponding lemma of O.Kiš.(P.270−271). A trigonometric polynomial $T_n(x)$ of order n which satisfies (25), (26) By the uniqueness theorem we have

$$\begin{aligned}\mid f(x) - R_n(x)\mid &= \mid f(x) - T_n(x) + T_n(x) - R_n(x)\mid \\ &\le \mid f(x) - T_n(x)\mid + \mid T_n(x) - R_n(x)\mid \\ &= \mid f(x) - T_n(x)\mid + \left|\sum_{k=0}^{n-1}(T_n(x_k) - f(x_k))U(x-x_k)\right. \\ &\left. + \sum_{k=0}^{n-1}(T_n'(x_k) - b_k)V(x-x_k) + \sum_{k=0}^{n-1}(T_n''(x_k) - c_k)W(x-x_k)\right| \\ &= \sum_{r=1}^{3} S_r + \mid f(x) - T_n(x)\mid\end{aligned}$$

By using the (23) and (25) we have

$$S_1 = C_3 \log n o(n^{-\alpha}) = o(1),$$

as $0 < \alpha \le +1$. By using (26), (21), (8) we get

$$S_2 = C_2 n^{-1} \log n o(n^{1-\alpha}) - o\left(\frac{n}{\log n}\right) C_2 n^{(-1)} \log n = o(1),$$

as $0 < \alpha \leq 1$ and $| b_k |= o\left(\frac{n}{\log n}\right)$. Now at last use (26), (8), (17) we have

$$S_3 = C_1 n^{-2} \log n o(n^{2-\alpha}) - C_1 n^{-2} \log n o\left(\frac{n^{-2}}{\log n}\right) = o(1),$$

as $\alpha > 0$ and $| c_k |= o\left(\frac{n^2}{\log n}\right)$. By using (25). Therefore

$$| f(x) - R_n(x) | \leq o(1)+ | f(x) - T_n(x) |$$

$$| f(x) - R_n | = o(1)$$

The theorem as follows.

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