Int. J. Math. And Appl., 6(2–A)(2018), 119–124
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Independent and Upper Steiner Domination Number of Graphs

K. Ramalakshmi[1,*] and K. Palani[1]

1 Department of Mathematics, Sri Sarada College for Women(Affiliated to Manonmaniam Sundaranar University), Tamilnadu, India.

2 Department of Mathematics, A.P.C Mahalakshmi College(Affiliated to Manonmaniam Sundaranar University), Tamilnadu, India.

Abstract: In this paper, independent and upper steiner domination number of graphs are introduced. Also, these numbers were found for some standard graphs.

MSC: Primary: 05C.

Keywords: Steiner number, steiner domination number, independent steiner domination number, upper steiner domination number.

Accepted on: 01.05.2018

1. Introduction

The concept of domination in graphs was introduced by Ore and Berge [4]. Throughout this paper $G = (V, E)$ denotes a finite undirected simple graph with vertex set V and edge set E. A subset D of $V(G)$ is a dominating set of G if every vertex in $V - D$ is adjacent to at least one vertex in D. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. The concept of Steiner number of a graph was introduced by G. Chatrand and P. Zhang [1]. For a nonempty set W of vertices in a connected graph G, the Steiner distance $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily each such subgraph is a tree and is called a Steiner tree with respect to W or a Steiner W-tree. The set of all vertices of G that lie in some Steiner W-tree is denoted by $S(W)$. If $S(W) = V$, then W is called a Steiner set for G. A Steiner set with minimum cardinality is the Steiner number of G and is denoted by $s(G)$. The concept of Steiner domination number of a graph was introduced by J. John et al., [3]. For a connected graph G, a set of vertices W in G is called a Steiner dominating set if W is both a Steiner set and a dominating set. The minimum cardinality of a Steiner dominating set of G is its Steiner domination number and is denoted by $\gamma_s(G)$. A steiner dominating set of cardinality $\gamma_s(G)$ is said to be a γ_s-set.

The minimum of $\{degv : v \in V(G)\}$ is denoted by $\delta(G)$. A k-dominating set is a subset S of $V(G)$ such that every vertex $v \in V - S$ is adjacent to atleast k vertices in S. A vertex and an edge are said to cover each other if they are incident. A set of vertices which covers all the edges of a graph G is called a vertex cover for G. The smallest number of vertices in any vertex cover for G is called its vertex covering number and is denoted by $\alpha_0(G)$ or α_0. A set S of vertices in a graph G is independent if no two of its vertices are adjacent in G. The largest number of vertices in such a set is called the vertex independence number of G and is denoted by $\beta_0(G)$ or β_0. If G is a graph with p vertices, then $\alpha_0(G) + \beta_0(G) = p$. A

* *E-mail: krams1983@gmail.com*

dominating set S of a graph G which is also independent is called an independent dominating set of G. The minimum cardinality of all independent dominating sets of G is called its independent domination number and is denoted by $i(G)$.

Theorem 1.1 ([3]). *For the complete bipartite graph $G = K_{m,n}$,*

$$s(G) = \gamma_s(G) = \begin{cases} 2 & \text{if } m = n = 1; \\ n & \text{if } n \geq 2, m = 1; \\ min\{m, n\} & \text{if } m, n \geq 2 \end{cases}$$

Theorem 1.2 ([7]). *For a Wheel graph $W_{1,n}, n \geq 5, \gamma_s(W_{1,n}) = n - 2$.*

Theorem 1.3 ([3]). *Each extreme vertex of a connected graph G belongs to every minimum Steiner dominating set of G.*

Theorem 1.4 ([3]). *For the complete graph $K_p (p \geq 2), \gamma_s(K_p) = p$.*

Lemma 1.5 ([5]). *Let $n \geq 6$. Then, $\lceil \frac{n}{3} \rceil = \lfloor \frac{n}{2} \rfloor$ if and only if $n = 7$.*

Theorem 1.6 ([6]). $\gamma_s(P_n) = \begin{cases} \lceil \frac{n-4}{3} \rceil + 2 & \text{if } n \geq 5; \\ 2 & \text{if } n = 2, 3 \text{ or } 4. \end{cases}$

Theorem 1.7 ([6]). *For $n > 5, \gamma_s(C_n) = \lceil \frac{n}{3} \rceil$.*

2. Independent Steiner Domination Number

Definition 2.1. *A Steiner dominating set W of G is said to be an independent steiner dominating set of G if the subgraph induced by W is independent.*

Definition 2.2. *Let ζ denote the collection of all graphs having atleast one independent steiner dominating set. Let $G \in \zeta$. Then, the minimum cardinality among all independent steiner dominating set of G is called the independent steiner domination number of G. It is denoted by $I\gamma_s(G)$. An independent steiner dominating set of cardinality $I\gamma_s(G)$ is called an $I\gamma_s$-set of G.*

Example 2.3. *Consider the graph G in Figure 1*

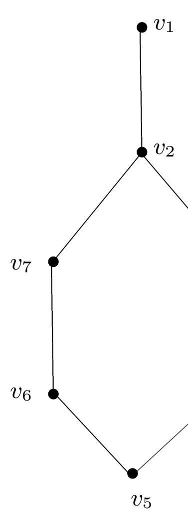

Figure 1

Here, $W = \{v_1, v_2, v_5\}$ is the minimum steiner dominating set of G. Therefore, $\gamma_s(G) = 3$. But, the subgraph induced by W is not independent. Here, $\{v_1, v_3, v_5, v_7\}$ is a minimum independent steiner dominating set of G and hence $I\gamma_s(G) = 4$.

Observation 2.4. *Let G be a connected graph. Then,*

(i). All graphs do not possess independent steiner dominating sets.

(ii). For a complete graph on n vertices, the vertex set $V(G)$ is the unique steiner dominating set. But it is not independent and so complete graphs have no independent steiner dominating sets.

(iii). C_3, C_5 and C_7 have no independent steiner dominating set.

Observation 2.5. *Let $G \in \zeta$. Then, the following are observed.*

(i). Every independent steiner dominating set is a steiner dominating set of G. Therefore, $2 \leq \gamma_s(G) \leq I\gamma_s(G) \leq p$.

(ii). Every extreme vertex of G belongs to every independent steiner dominating set of G.

(iii). Let W be the set of all extreme vertices of G. If it is an independent steiner dominating set of G, then W is the unique minimum independent steiner dominating set of G.

(iv). If G contains atleast two adjacent extreme vertices, then G has no independent steiner dominating set.

Remark 2.6. *Let G be a connected graph $G \in \zeta$. Clearly, every independent steiner dominating set of G is an independent dominating set of G. Therefore, the following are true:*

(i). $I\gamma_s(G) \geq i(G)$.

(ii). If W is a minimum independent steiner dominating set of G, then $V - W$ is a dominating set of G.

Theorem 2.7. *Let G be a connected graph with p vertices. Let $G \in \zeta$. Then, $I\gamma_s(G) \leq p - \gamma(G)$.*

Proof. Let W be a minimum independent steiner dominating set of G. By Remark 2.6 (ii), $\gamma(G) \leq |V - W|$. Therefore, $\gamma(G) \leq |V| - |W| = p - I\gamma_s(G)$. Therefore, $I\gamma_s(G) \leq p - \gamma(G)$. □

Theorem 2.8. *Let $G \in \zeta$ and let W be an independent steiner dominating set of G. If $\delta(G) \geq k$, then $V - W$ is a k-dominating set of G. Further, $I\gamma_s(G) \leq p - \gamma_k(G)$.*

Proof. Let $v \in W$. Then, W is independent and $\delta(G) \geq k$ imply that v is adjacent to atleast k vertices of $V - W$. Therefore, $V - W$ is a k-dominating set of G and $\gamma_k(G) \leq |V - W| = |V| - |W| \leq p - I\gamma_s(G)$. Therefore, $I\gamma_s(G) \leq p - \gamma_k(G)$. □

Remark 2.9. *Let G be a connected graph with $p(\geq 3)$ vertices. Then, $I\gamma_s(G) \leq \beta_0(G) = p - \alpha_0(G)$.*

Theorem 2.10. *Let $G \in \zeta$ be a connected graph with $p(\geq 3)$ vertices. Then, $I\gamma_s(G) = 2$ if and only if $\gamma_s(G) = 2$.*

Proof. Suppose $I\gamma_s(G) = 2$. Then, by Observation 2.5 (i), $2 \leq \gamma_s(G) \leq I\gamma_s(G) = 2$. Therefore, $\gamma_s(G) = 2$. Conversely, if $p \geq 3$ and $\gamma_s(G) = 2$, then every minimum steiner dominating set is independent and so an independent steiner dominating set. Hence, $I\gamma_s(G) \leq \gamma_s(G) = 2$. Therefore, $I\gamma_s(G) = 2$. □

Theorem 2.11. *For $n \geq 3$, $I\gamma_s(P_n) = \gamma_s(P_n)$.*

Proof. Let $n \geq 3$ and $P_n = (v_1, v_2, ..., v_n)$. If $n \equiv 0(mod 3)$ or $n \equiv 1(mod 3)$ or $n \equiv 2(mod 3)$, then $W = \{v_1, v_4, ..., v_{n-2}, v_n\}$ or $W = \{v_1, v_4, ..., v_{n-3}, v_n\}$ or $W = \{v_1, v_4, ..., v_{n-2}, v_n\}$ is a minimum steiner dominating set of P_n. Also, W is independent. Therefore, $I\gamma_s(P_n) \leq \gamma_s(P_n)$. Hence, by Observation 2.5(i), $I\gamma_s(P_n) = \gamma_s(P_n)$. □

Theorem 2.12. *For $n \geq 4(n \neq 5, 7), I\gamma_s(C_n) = \gamma_s(C_n) = \lceil \frac{n}{3} \rceil$.*

Proof. Let $C_n = (v_1, v_2, ..., v_n, v_1)$.

Case (i): $n = 4$. In this case, every steiner dominating set is independent. Hence, $I\gamma_s(C_4) = \gamma_s(C_4) = 2$.

Case (ii): $n > 4$. By observation 2.4 (iii), it is enough to prove the theorem for $n \neq 5$ and 7. If $n \equiv 0(mod\ 3)$ or $n \equiv 1(mod\ 3)$ or $n \equiv 2(mod\ 3)$, then $W = \{v_1, v_4, ..., v_{n-2}\}$ or $W = \{v_1, v_3, v_6, v_9, ..., v_{n-4}, v_{n-2}\}$ or $W = \{v_1, v_4, ..., v_{n-4}, v_{n-1}\}$ is a minimum steiner dominating set of C_n. Also, W is independent implies, $I\gamma_s(C_n) \leq \gamma_s(C_n)$. Therefore, by Observation 2.5 (i), $I\gamma_s(C_n) = \gamma_s(C_n)$. Hence, by Theorem 1.7, for $n \geq 4(n \neq 5, 7)$, $I\gamma_s(C_n) = \gamma_s(C_n) = \lceil \frac{n}{3} \rceil$. □

Theorem 2.13. *The Wheel graph $W_{1,p}(p > 4)$, has no independent steiner dominating set.*

Proof. Let $p > 4$, then $W = \{v_1, v_3, v_4, v_5, ..., v_{p-1}\}$ is the minimum steiner dominating set of $W_{1,p}$ and is not independent. Hence, the wheel graph $W_{1,p}$ where $p > 4$ has no independent steiner dominating set. □

Observation 2.14. *For the wheel graph $W_{1,p}$ where $p = 4$, $I\gamma_s(W_{1,p}) = \gamma_s(W_{1,p}) = 2$.*

Theorem 2.15. *Let $m, n \geq 2$. Then, $I\gamma_s(K_{m,n}) = min\{m, n\}$.*

Proof. Let S and T be the bi partitions of $K_{m,n}$ with $|S| = m$ and $|T| = n$. Let W be a minimum steiner dominating set of $K_{m,n}$. Then, by Theorem 1.1 , $W = S$ or T. Clearly, W is independent and so W is an independent steiner dominating set of $K_{m,n}$. Therefore, $I\gamma_s(K_{m,n}) = \gamma_s(K_{m,n}) = min\{m, n\}$. □

Theorem 2.16. *Let G be a connected graph on p vertices. Then, $G^+ \in \zeta$ and $I\gamma_s(G^+) = p$.*

Proof. Let $V(G) = (v_1, v_2, ..., v_p)$ and $w_1, w_2, ..., w_p$ be the end vertices attached to $v_1, v_2, ..., v_p$ respectively in G^+. Then, $W = \{w_1, w_2, ..., w_p\}$ is the unique minimum independent steiner dominating set of G^+ and so $I\gamma_s(G^+) = p$. □

3. Upper Steiner Domination Number

Definition 3.1. *A steiner dominating set W is said to be minimal steiner dominating set of G if no proper subset of W is a steiner dominating set of G. The maximum cardinality of all minimal steiner dominating sets of G is called the upper steiner domination number of G and is denoted by $\Gamma_s(G)$. A steiner dominating set of cardinality $\Gamma_s(G)$ is called a Γ_s -set of G.*

Example 3.2. *Consider the graph G in Figure 2. Here, $W = \{v_1, v_2, v_3, v_4, v_{11}, v_{12}, v_{13}\}$, $W_1 = \{v_1, v_2, v_3, v_5, v_6, v_7, v_{11}, v_{12}, v_{13}\}$ and $W_2 = \{v_1, v_2, v_3, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}\}$ are the minimal γ_s-sets of G. Also, no subset T of $V(G)$ is a γ_s-set of G if $|T| < |W|$ or $|T| > |W_1| = |W_2|$. Therefore, W is a γ_s-set of G and W_1 and W_2 are Γ_s-sets of G. Hence, $\gamma_s(G) = 7$ and $\Gamma_s(G) = 9$.*

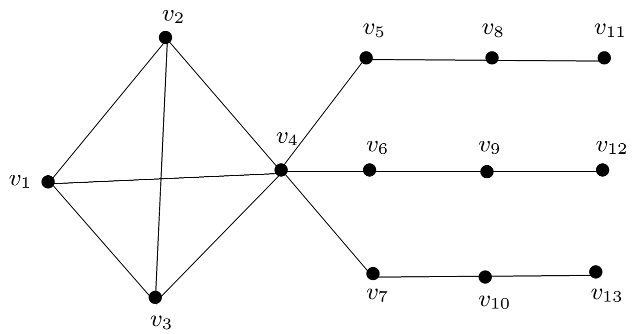

Figure 2

Example 3.3. *Consider the graph G in Figure 3.*

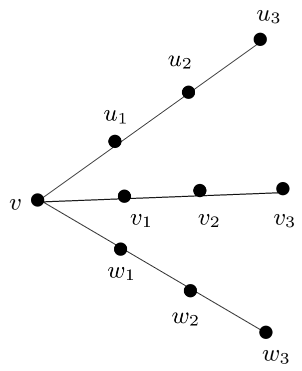

Figure 3

Here, $W = \{v, u_3, v_3, w_3\}, W_1 = \{u_1, u_2, v_2, w_2, u_3, v_3, w_3\}, W_2 = \{v_1, u_2, v_2, w_2, u_3, v_3, w_3\}, W_3 = \{w_1, u_2, v_2, w_2, u_3, v_3, w_3\}$ are the minimal γ_s-sets of G. Also, no subset T of $V(G)$ is a γ_s-set of G if $|T| < |W|$ or $|T| > |W_1| = |W_2| = |W_3|$. Therefore, W is a γ_s-set of G and W_1, W_2 and W_3 are Γ_s-sets of G. Hence, $\gamma_s(G) = 4$ and $\Gamma_s(G) = 7$.

Example 3.4. *Consider the graph $G = C_6$ in Figure 4. Here, $\{v_1, v_4\}, \{v_2, v_5\}$and $\{v_3, v_6\}$ are the only γ_s-sets of G and are all minimum. Therefore, $\gamma_s(G) = \Gamma_s(G) = 2$.*

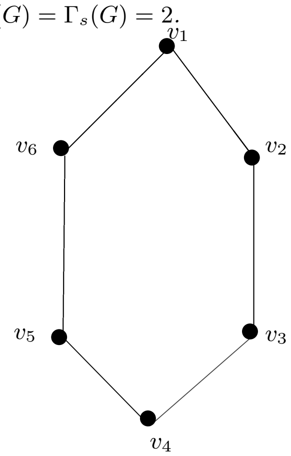

Figure 4

Remark 3.5. *Let G be a (p, q) connected graph. Then, $2 \leq \gamma_s(G) \leq \Gamma_s(G) \leq p$. The above bounds are sharp. For, $\gamma_s(C_6) = \Gamma_s(C_6) = 2$ and $\gamma_s(K_p) = \Gamma_s(K_p) = p$. The above bounds are also strict. For example, consider the graph G in Example 3.2. We observe that, $V(G) = 13, \gamma_s(G) = 7$ and $\Gamma_s(G) = 9$. Hence, $2 < \gamma_s(G) < \Gamma_s(G) \leq p$.*

Remark 3.6. *For a complete graph K_p on p vertices, $\gamma_s(K_p) = \Gamma_s(K_p) = p$.*

Theorem 3.7. *For the path $P_n, n \geq 3, \Gamma_s(P_n) = \lceil \frac{n}{2} \rceil$.*

Proof. Let $V(P_n) = \{v_1, v_2, ..., v_n\}$. Let $W = \{v_1, v_3, ..., v_{n-2}, v_n\}$ when n is odd and $W' = \{v_1, v_3, ..., v_{n-3}, v_n\}$ when n is even. Clearly, W and W' are upper steiner dominating sets of P_n, in the corresponding cases. Therefore,

$$\Gamma_s(P_n) = \begin{cases} |W| & \text{if n is odd} \\ |W'| & \text{if n is even} \end{cases} = \lceil \frac{n}{2} \rceil.$$

□

Theorem 3.8. *For the cycle $C_n, \Gamma_s(C_n) = \lfloor \frac{n}{2} \rfloor, n \geq 6$. Also, $\Gamma_s(C_n) = \gamma_s(Cn)$ if and only if $n = 3, 4, 5$ or 7.*

Proof. Let $n \geq 6$ and $C_n = (v_1, v_2, ..., v_n, v_1)$. By Theorem 1.7, $\gamma_s(C_n) = \lceil \frac{n}{3} \rceil$. Now, $W = \{v_1, v_3, ..., v_{n-2}\}$ and $W' = \{v_1, v_3, ..., v_{n-1}\}$ are the upper steiner dominating sets of C_n according as n is odd or n is even and so

$$\Gamma_s(C_n) = \begin{cases} |W| & \text{if n is odd} \\ |W'| & \text{if n is even} \end{cases} = \lfloor \frac{n}{2} \rfloor.$$

It is easy to observe that, $\Gamma_s(C_3) = \gamma_s(C_3) = 2, \Gamma_s(C_4) = \gamma_s(C_4) = 2$ and $\Gamma_s(C_5) = \gamma_s(C_5) = 3$. Also by Lemma 1.5, for $n \geq 6, \Gamma_s(C_n) = \gamma_s(C_n)$ if and only if $\lceil \frac{n}{3} \rceil = \lfloor \frac{n}{2} \rfloor$ if and only if $n = 7$. Hence, $\Gamma_s(C_n) = \gamma_s(C_n)$ if and only if $n = 3, 4, 5$ or 7. □

References

[1] G. Chatrand and P. Zhang, *The Steiner number of a graph*, Discrete Math., 242(2002), 41-54.

[2] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Domination in graphs*, Marcel Decker, Inc., New York, (1998).

[3] J. John, G. Edwin and P. Paul Sudhahar, *The Steiner domination number of a graph*, International Journal of Mathematics and Computer Applications Researchs, 3(3)(2013), 37-42.

[4] Ore and Berge, *Theory of Graphs*, American Mathematical Society, Colloquium Publications, XXXVIII(1962).

[5] K. Palani, A. Nagarajan and A. Subramanian, *Upper (G,D) - Number of Some Standard Graphs*, Outreach - A Multi Disciplinary Refereed Journal, IV(2)(2010-2011), 37-40.

[6] K. Ramalakshmi, K. Palani and T. Tamil Chelvam, *Steiner Domination Number of Graphs*, Proceedings of International Conference on Recent Trends in Mathematical Modelling, ISBN 13-978-93-82592-00-06, 128-134.

[7] S.K. Vaidya and R.N. Mehta, *Steiner domination number of some wheel related graphs*, International journal of Mathematics and Soft Computing, 5(2)(2015), 15-19.