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An $M^{[X]}/G(a,b)/1$ Queueing System with Breakdown and Second Optional Repair, Stand-by Server, Balking, Variant Arrival Rate and Multiple Vacation

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Abstract: In this paper, we discussed a Non-Markovian batch arrival general bulk service single server queueing system with server's breakdown and second optional repair, stand-by server, balking, variant arrival rate and multiple vacation. The main server's service time, vacation time, stand-by server's service time are all follows general distributions and breakdown and two types of repair time for main server follows exponential distributions. There is a stand-by server which is employed during the period for which the regular server remains under repair. The probability generating function of queue size at an arbitrary time and some performance measures of the system are derived. Extensive numerical results are also illustrated.

MSC: 60K25, 90B22, 68M20.

Keywords: Non-Markovian queue, General bulk service, Multiple vacation, Breakdown and optional Repair, Stand-by server, Balking, Variant arrival rate.

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1. Introduction

The concept of bulk arrivals and bulk services has picked up a huge importance in present situations. Bulk queueing systems have been studied by several authors including Neuts (1967) and Chaudhry and Templeton (1983). Recently, Sasikala and Indhira (2016) provided a survey of the bulk service queueing models. The general bulk service rule states that the server will start to provide services only when at least $'a'$ units are present in the queue, and maximum service capacity is $'b'$. When the server could not be repaired or reestablished by the principal basic repair, the next repairs are expected to reestablish the server. Ayyappan and shyamala (2013) investigated batch arrival queue with second optional repair. Balamani (2014) has discussed a two stage batch arrival queue with compulsory server vacation and second optional repair.

For balancing the real time system's efficiency and availability, the queueing models with stand-by's support have become worth mentioning as far as the analysis of queueing modelling study is concerned. The provision of stand-by's and repairmen support to the queueing system maintains smooth functioning of the system. In the field of computer and communication systems, distribution and service systems, production/manufacturing systems etc., the applications of queueing models with standby's support is essential. Mok et al. (1987) has studied a transient queueing model for business office with standby servers. Khalaf et al (2012), Preeti et al (2014), Kamlesh Kumar et al (2013) have discussed a standby server queueing models with a combination of vacation. In this work, we studied a batch arrival general bulk service queuing system with

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active breakdown and second optional repair, balking, variant arrival rate, multiple vacation with an additional significant assumption that the system employs a standby server during the repair period of the main server.

This paper is organised as follows. In section 2 the queuing problem is defined. The system equations have been developed in sections 3. The probability generating function (PGF) of the queue length distribution in steady state is obtained in section 4. Various performance measures of the queuing system are derived in section 5. A computational study is illustrated in section 6. Conclusions are given in section 7.

2. Model description

This paper deals with a queueing model whose arrival follows a compound Poisson process with intensity rate λ_1, λ_2, λ_3 during main server in the system, stand-by in the system, main server is in vacation respectively. The main server's service time, vacation time, stand-by server's service time are all follows general distributions, breakdown and two repair times of main server follows exponential distributions with rate α, η_1, and η_2 respectively. Customers balking during main server and stand-by server's busy period with probability ω, no balking with probability $(1-\omega)$. The main server may breakdown at any time during service with exponential rate α and in such a case the main server immediately goes for a repair which follows an exponential distribution with rate η_1 after the first repair completion the main server go to second repair which is optional with probability ϵ and the repair rate η_2. The interrupted batch service is exchanged to the stand-by server who starts service to that batch afresh. The stand-by server remain in the system until the main server's repair's completion. At the instant of repair completion, if the stand-by server is busy then the service to that batch of customers is interrupted and that batch of customers is transferred to the main server who starts service to that batch of customers afresh. At the instant of the completion of a service (served by main server) or the main server's repair completion, number of customers in the queue is less than a the server will avail a vacation of a random length. The server takes a sequence of vacation until the queue size reaches at least a.

2.1. Notations

Let X be the group size random variable of arrival, g_k be the probability of 'k' customers arrive in a batch and $X(z)$ be its PGF. $S_b(.)$, $S_s(.)$ and $V(.)$ represent the Cumulative Distribution Functions (CDF) of service time of main server, service time of stand-by server and vacation time of main server with corresponding probability density functions are $s_b(x)$, $s_s(x)$ and $v(x)$ respectively. $S_b^0(t)$, $S_s^0(t)$ and $V^{(0)}(t)$ represent the remaining service time of service given by main server, service given by stand-by server and remaining vacation time of main server at time 't' respectively. $\tilde{S}_b(\theta)$, $\tilde{S}_s(\theta)$ and $\tilde{V}(\theta)$ represent the Laplace Stieltjes transform (LST) of S_b, S_s, and V respectively. For the further development of the queueing system, let us define the following.

$\varepsilon(t) = 1, 2, 3,$ and 4 at time t the main server is in service, vacation and the stand-by server is in service, idle respectively.

$Z(t) = j$, if the server is on the j^{th} vacation.

$N_s(t)$=Number of customers in service station at time t.

$N_q(t)$=Number of customers in the queue at time t.

Define the probabilities

$$T_n(t)\Delta t = Pr\{N_q(t) = n,\ \varepsilon(t) = 4\},\ 0 \le n \le a-1,$$

$$P_{m,n}(x,t)\Delta t = Pr\{N_s(t) = m,\ N_q(t) = n,\ x \le S_b^0(t) \le x + \Delta t,\ \varepsilon(t) = 1\},\ a \le m \le b,\ n \ge 0,$$

$$B_{m,n}(x,t)\Delta t = Pr\{N_s(t) = m,\ N_q(t) = n,\ x \le S_s^0(t) \le x + \Delta t,\ \varepsilon(t) = 3\},\ a \le m \le b,\ n \ge 0,$$

$$Q_{l,j}(x,t)\Delta t = Pr\{Z(t)=l,\ N_q(t)=j,\ x\le V^0(t)\le x+\Delta t,\ \varepsilon(t)=2\},\ l\ge 1, j\ge 0.$$

3. Queue Size Distribution

From the above-defined probabilities we can easily construct the following steady state equations

$$(\lambda_2+\eta_1+\eta_2)T_0 = \sum_{m=a}^{b} B_{m,0}(0) + \epsilon\eta_1 T_0, \tag{1}$$

$$(\lambda_2+\eta_1+\eta_2)T_n = \sum_{m=a}^{b} B_{m,n}(0) + \epsilon\eta_1 T_0 + \sum_{k=1}^{n} T_{n-k}\lambda g_k,\ 1\le n\le a-1, \tag{2}$$

$$\begin{aligned} -P'_{i,0}(x) &= -(\lambda_1+\alpha)P_{i,0}(x) + \sum_{m=a}^{b} P_{m,i}(0)s_b(x) + \eta_2\int_0^\infty B_{i,0}(y)\ dys_b(x) \\ &\quad + \eta_1(1-\epsilon)\int_0^\infty B_{i,0}(y)\ dys_b(x) + \sum_{l=1}^{\infty} Q_{l,i}(0)s_b(x) + \omega\lambda_1 P_{i,0}(x),\ a\le i\le b, \end{aligned} \tag{3}$$

$$\begin{aligned} -P'_{i,j}(x) &= -(\lambda_1+\alpha)P_{i,j}(x) + \omega\lambda_1 P_{i,j}(x) + \eta_1(1-\epsilon)\int_0^\infty B_{i,j}(y)\ dys_b(x) \\ &\quad + \eta_2\int_0^\infty B_{i,j}(y)\ dys_b(x) + (1-\omega)\sum_{k=1}^{j} P_{i,j-k}(x)\lambda_1 g_k,\ j\ge 1,\ a\le i\le b-1, \end{aligned} \tag{4}$$

$$\begin{aligned} -P'_{b,j}(x) &= -(\lambda_1+\alpha)P_{b,j}(x) + \sum_{m=a}^{b} P_{m,b+j}(0)s_b(x) + \omega\lambda_1 P_{b,j}(x) \\ &\quad + \eta_1(1-\epsilon)\int_0^\infty B_{b,j}(y)\ dys_b(x) + \eta_2\int_0^\infty B_{b,j}(y)\ dys_b(x) \\ &\quad + \sum_{l=1}^{\infty} Q_{l,b+j}(0)s_b(x) + (1-\omega)\sum_{k=1}^{j} P_{b,j-k}(x)\lambda_1 g_k,\ j\ge 1, \end{aligned} \tag{5}$$

$$\begin{aligned} -B'_{i,0}(x) &= -(\lambda_2+\eta_1+\eta_2)B_{i,0}(x) + \sum_{m=a}^{b} B_{m,i}(0)s_s(x) + \epsilon\eta_1 B_{i,0}(x) \\ &\quad + \alpha\int_0^\infty P_{i,0}(y)\ dys_s(x) + \sum_{k=0}^{a-1} T_k\lambda_2 g_{i-k}s_s(x) + \omega\lambda_2 B_{i,0}(x),\ a\le i\le b, \end{aligned} \tag{6}$$

$$\begin{aligned} -B'_{i,j}(x) &= -(\lambda_2+\eta_1+\eta_2)B_{i,j}(x) + \alpha\int_0^\infty P_{i,j}(y)\ dys_s(x) + \epsilon\eta_1 B_{i,j}(x) \\ &\quad + (1-\omega)\sum_{k=1}^{j} B_{i,j-k}(x)\lambda_2 g_k + \omega\lambda_2 B_{i,j}(x),\ j\ge 1,\ a\le i\le b-1, \end{aligned} \tag{7}$$

$$\begin{aligned} -B'_{b,j}(x) &= -(\lambda_2+\eta_1+\eta_2)B_{b,j}(x) + \sum_{m=a}^{b} B_{m,b+j}(0)s_s(x) + \omega\lambda_2 B_{b,j}(x) \\ &\quad + (1-\omega)\sum_{k=1}^{j} B_{b,j-k}(x)\lambda_2 g_k + \alpha\int_0^\infty P_{b,j}(y)\ dy\ s_s(x) + \epsilon\eta_1 B_{b,j}(x) \\ &\quad + \sum_{k=0}^{a-1} T_k\lambda_2 g_{b+j-k}s_s(x),\ j\ge 1, \end{aligned} \tag{8}$$

$$-Q'_{1,0}(x) = -\lambda_3 Q_{1,0}(x) + \sum_{m=a}^{b} P_{m,0}(0)v(x) + \eta_2 T_0 v(x) + (1-\epsilon)\eta_1 T_0 v(x), \tag{9}$$

$$\begin{aligned} -Q'_{1,n}(x) &= -\lambda_3 Q_{1,n}(x) + \sum_{m=a}^{b} P_{m,n}(0)v(x) + \eta_2 T_n v(x) + (1-\epsilon)\eta_1 T_n v(x) \\ &\quad + \sum_{k=1}^{n} Q_{1,n-k}(x)\lambda_3 g_k,\quad 1\le n\le a-1, \end{aligned} \tag{10}$$

$$, -Q'_{1,n}(x) = -\lambda_3 Q_{1,n}(x) + \sum_{k=1}^{n} Q_{1,n-k}(x)\lambda_3 g_k,\ n\ge a, \tag{11}$$

$$-Q'_{j,0}(x) = -\lambda_3 Q_{j,0}(x) + Q_{j-1,0}(0)v(x),\ j\ge 2, \tag{12}$$

$$-Q_{j,n}^{'}(x) = -\lambda_3 Q_{j,n}(x) + Q_{j-1,n}(0)v(x) + \sum_{k=1}^{n} Q_{j,n-k}(x)\lambda_3 g_k,\ j \geq 2,\ 1 \leq n \leq a-1, \tag{13}$$

$$-Q_{j,n}^{'}(x) = -\lambda_3 Q_{j,n}(x) + \sum_{k=1}^{n} Q_{j,n-k}(x)\lambda_3 g_k,\ j \geq 2,\ n \geq a. \tag{14}$$

Taking LST on both sides of equations (3) to (14), we get,

$$\begin{aligned}\theta\tilde{P}_{i,0}(\theta) - P_{i,0}(0) &= (\lambda_1+\alpha)\tilde{P}_{i,0}(\theta) - \sum_{m=a}^{b} P_{m,i}(0)\tilde{S}_b(\theta) - \omega\lambda_1\tilde{P}_{i,0}(\theta) - \sum_{l=1}^{\infty} Q_{l,i}(0)\tilde{S}_b(\theta)\\ &- \eta_1(1-\epsilon)\int_0^{\infty} B_{i,0}(y)\ dy\tilde{S}_b(\theta) - \eta_2\int_0^{\infty} B_{i,0}(y)\ dy\tilde{S}_b(\theta),\ a \leq i \leq b,\end{aligned} \tag{15}$$

$$\begin{aligned}\theta\tilde{P}_{i,j}(\theta) - P_{i,j}(0) &= (\lambda_1+\alpha)\tilde{P}_{i,j}(\theta) - \omega\lambda_1\tilde{P}_{i,j}(\theta) - \eta_2\int_0^{\infty} B_{i,j}(y)\ dy\tilde{S}_b(\theta) - \eta_1(1-\epsilon)\int_0^{\infty} B_{i,j}(y)\ dy\tilde{S}_b(\theta)\\ &- (1-\omega)\sum_{k=1}^{j}\tilde{P}_{i,j-k}(\theta)\lambda_1 g_k,\quad j \geq 1,\quad a \leq i \leq b-1,\end{aligned} \tag{16}$$

$$\begin{aligned}\theta\tilde{P}_{b,j}(\theta) - P_{b,j}(0) &= -(\lambda_1+\alpha)\tilde{P}_{b,j}(\theta) - \sum_{m=a}^{b} P_{m,b+j}(0)\tilde{S}_b(\theta) - \omega\lambda_1\tilde{P}_{b,j}(\theta) - \eta_1(1-\epsilon)\int_0^{\infty} B_{b,j}(y)\ dy\tilde{S}_b(\theta)\\ &- \eta_2\int_0^{\infty} B_{b,j}(y)\ dy\tilde{S}_b(\theta) - \sum_{l=1}^{\infty} Q_{l,b+j}(0)\tilde{S}_b(\theta) - (1-\omega)\sum_{k=1}^{j}\tilde{P}_{b,j-k}(\theta)\lambda_1 g_k,\ j \geq 1,\end{aligned} \tag{17}$$

$$\begin{aligned}\theta\tilde{B}_{i,0}(\theta) - B_{i,0}(0) &= (\lambda_2+\eta_1+\eta_2)\tilde{B}_{i,0}(\theta) - \sum_{m=a}^{b} B_{m,i}(0)\tilde{S}_s(\theta) - \epsilon\eta_1\tilde{B}_{i,0}(\theta) - \alpha\int_0^{\infty} P_{i,0}(y)\ dy\tilde{S}_s(\theta)\\ &- \sum_{k=0}^{a-1} T_k\lambda_2 g_{i-k}\tilde{S}_s(\theta) - \omega\lambda_2\tilde{B}_{i,0}(\theta),\ a \leq i \leq b,\end{aligned} \tag{18}$$

$$\begin{aligned}\theta\tilde{B}_{i,j}(\theta) - B_{i,j}(0) &= (\lambda_2+\eta_1+\eta_2)\tilde{B}_{i,j}(\theta) - \alpha\int_0^{\infty} P_{i,j}(y)\ dy\ \tilde{S}_s(\theta) - \epsilon\eta_1\tilde{B}_{i,j}(\theta) - (1-\omega)\sum_{k=1}^{j}\tilde{B}_{i,j-k}(\theta)\lambda_2 g_k\\ &- \omega\lambda_2\tilde{B}_{i,j}(\theta),\ j \geq 1,\ a \leq i \leq b-1,\end{aligned} \tag{19}$$

$$\begin{aligned}\theta\tilde{B}_{b,j}(\theta) - B_{b,j}(0) &= (\lambda_2+\eta_1+\eta_2)\tilde{B}_{b,j}(\theta) - \sum_{m=a}^{b} B_{m,b+j}(0)\tilde{S}_s(\theta) - \omega\lambda_2\tilde{B}_{b,j}(\theta) - (1-\omega)\sum_{k=1}^{j}\tilde{B}_{b,j-k}(\theta)\lambda_2 g_k\\ &- \alpha\int_0^{\infty} P_{b,j}(y)\ dy\ \tilde{S}_s(\theta) - \epsilon\eta_1\tilde{B}_{b,j}(\theta) - \sum_{k=0}^{a-1} T_k\lambda_2 g_{b+j-k}\tilde{S}_s(\theta),\ j \geq 1,\end{aligned} \tag{20}$$

$$\theta\tilde{Q}_{1,0}(\theta) - Q_{1,0}(0) = \lambda_3\tilde{Q}_{1,0}(\theta) - \sum_{m=a}^{b} P_{m,0}(0)\tilde{V}(\theta) - (1-\epsilon)\eta_1 T_0\tilde{V}(\theta - \eta_2 T_0\tilde{V}(\theta), \tag{21}$$

$$\begin{aligned}\theta\tilde{Q}_{1,n}(\theta) - Q_{1,n}(0) &= \lambda_3\tilde{Q}_{1,n}(\theta) - \sum_{m=a}^{b} P_{m,n}(0)\tilde{V}(\theta) - (1-\epsilon)\eta_1 T_n\tilde{V}(\theta) - \eta_2 T_n\tilde{V}(\theta)\\ &- \sum_{k=1}^{n}\tilde{Q}_{1,n-k}(\theta)\lambda_3 g_k,\quad 1 \leq n \leq a-1,\end{aligned} \tag{22}$$

$$\theta\tilde{Q}_{1,n}(\theta) - Q_{1,n}(0) = \lambda_3\tilde{Q}_{1,n}(\theta) - \sum_{k=1}^{n}\tilde{Q}_{1,n-k}(\theta)\lambda_3 g_k,\ n \geq a, \tag{23}$$

$$\theta\tilde{Q}_{j,0}(\theta) - Q_{j,0}(0) = \lambda_3\tilde{Q}_{j,0}(\theta) - Q_{j-1,0}(0)\tilde{V}(\theta),\ j \geq 2, \tag{24}$$

$$\theta\tilde{Q}_{j,n}(\theta) - Q_{j,n}(0) = -\lambda_3\tilde{Q}_{j,n}(\theta) - Q_{j-1,n}(0)\tilde{V}(\theta) - \sum_{k=1}^{n}\tilde{Q}_{j,n-k}(\theta)\lambda_3 g_k, j \geq 2,\ 1 \leq n \leq a-1, \tag{25}$$

$$\theta\tilde{Q}_{j,n}(\theta) - Q_{j,n}(0) = \lambda_3\tilde{Q}_{j,n}(\theta) - \sum_{k=1}^{n}\tilde{Q}_{j,n-k}(\theta)\lambda_3 g_k,\ j \geq 2,\ n \geq a. \tag{26}$$

4. System Size Distribution

To find the Probability Generating Function (PGF) for queue size, we define the following PGF's

$$\begin{aligned}\tilde{P}_i(z,\theta) &= \sum_{j=0}^{\infty}\tilde{P}_{i,j}(\theta)z^j, \quad P_i(z,0)=\sum_{j=0}^{\infty}P_{i,j}(0)z^j,\ a\le i\le b,\\ \tilde{B}_i(z,\theta) &= \sum_{j=0}^{\infty}\tilde{B}_{i,j}(\theta)z^j, \quad B_i(z,0)=\sum_{j=0}^{\infty}B_{i,j}(0)z^j,\ a\le i\le b,\\ \tilde{Q}_l(z,\theta) &= \sum_{j=0}^{\infty}\tilde{Q}_{l,j}(\theta)z^j \quad Q_l(z,0)=\sum_{j=0}^{\infty}Q_{l,j}(0)z^j,\ l\ge 1.\end{aligned} \tag{27}$$

By multiplying Equations (15) to (26) with suitable power of z^j and summing over j ($j=0$ to ∞), and using Equation (27), we get,

$$(\theta-u(z))\tilde{P}_i(z,\theta)=P_i(z,0)-\tilde{S}_b(\theta)\Big[\sum_{m=a}^{b}P_{m,i}(0)+\sum_{j=0}^{\infty}Q_{l,i}(0)+\eta\tilde{B}_i(z,0)\Big],\ a\le i\le b-1, \tag{28}$$

$$\begin{aligned}z^b(\theta-u(z))\tilde{P}_b(z,\theta) &= (z^b-\tilde{S}_b(\theta))P_b(z,0)-\tilde{S}_b(\theta)\Big[\sum_{m=a}^{b-1}P_m(z,0)+\sum_{l=1}^{\infty}Q_l(z,0)+z^b\eta\tilde{B}_b(z,0)\\ &\quad -\sum_{j=0}^{b-1}\Big(\sum_{m=a}^{b}P_{m,j}(0)z^j+\sum_{l=1}^{\infty}Q_{l,j}(0)z^j\Big)\Big],\end{aligned} \tag{29}$$

$$(\theta-v(z))\tilde{B}_i(z,\theta)=B_i(z,0)-\tilde{S}_s(\theta)\Big[\alpha\tilde{P}_i(z,0)+\sum_{m=a}^{b}B_{m,i}(0)+\sum_{k=0}^{a-1}T_k\lambda_2 g_{i-k}\Big],\ a\le i\le b-1, \tag{30}$$

$$\begin{aligned}z^b(\theta-v(z))\tilde{B}_b(z,\theta) &= (z^b-\tilde{S}_s(\theta))B_b(z,0)-\tilde{S}_s(\theta)\Big[\sum_{m=a}^{b-1}B_m(z,0)+z^b\alpha\tilde{P}_b(z,0)-\sum_{j=0}^{b-1}\sum_{m=a}^{b}B_{m,j}(0)z^j\\ &\quad +\lambda_2\sum_{k=0}^{a-1}\sum_{j=b}^{\infty}T_k z^k g_{j-k}z^{j-k}\Big],\end{aligned} \tag{31}$$

$$(\theta-w(z))\tilde{Q}_1(z,\theta)=Q_1(z,0)-\tilde{V}(\theta)\sum_{n=0}^{a-1}\Big[\sum_{m=a}^{b}P_{m,n}(0)z^n+\eta T_n z^n\Big], \tag{32}$$

$$(\theta-w(z))\tilde{Q}_j(z,\theta)=Q_j(z,0)-\tilde{V}(\theta)\sum_{n=0}^{a-1}Q_{j-1,n}(0)z^n,\ j\ge 2, \tag{33}$$

where $u(z)=\alpha+\lambda_1(1-\omega)(1-X(z))$, $v(z)=\eta+\lambda_2(1-\omega)(1-X(z))$, $w(z)=\lambda_3-\lambda_3X(z)$, $\eta=((1-\epsilon)\eta_1+\eta_2)$. Substitute $\theta=u(z)$ in (28) and (29), we get,

$$P_i(z,0)=\tilde{S}_b(u(z))\Big[\sum_{m=a}^{b}P_{m,i}(0)+\sum_{l=1}^{\infty}Q_{l,i}(0)+\eta\tilde{B}_i(z,0)\Big],\ a\le i\le b-1, \tag{34}$$

$$P_b(z,0)=\frac{\tilde{S}_b(u(z))}{(z^b-\tilde{S}_b(u(z)))}\Big[\sum_{m=a}^{b-1}P_m(z,0)+\sum_{l=1}^{\infty}Q_l(z,0)+z^b\eta\tilde{B}_b(z,0)-\sum_{j=0}^{b-1}\Big(\sum_{m=a}^{b}P_{m,j}(0)z^j+\sum_{j=0}^{\infty}Q_{l,j}(0)z^j\Big)\Big], \tag{35}$$

Substitute $\theta=v(z)$ in (30) and (31), we get,

$$B_i(z,0)=\tilde{S}_s(v(z))\Big[\alpha\tilde{P}_i(z,0)+\sum_{m=a}^{b}B_{m,i}(0)+\sum_{k=0}^{a-1}T_k\lambda_2 g_{i-k}\Big],\ a\le i\le b-1, \tag{36}$$

$$B_b(z,0)=\frac{\tilde{S}_s(v(z))}{(z^b-\tilde{S}_s(v(z)))}\Big[\sum_{m=a}^{b-1}B_m(z,0)+z^b\alpha\tilde{P}_b(z,0)+\lambda_2\sum_{k=0}^{a-1}\sum_{j=b}^{\infty}T_k z^k g_{j-k}z^{j-k}-\sum_{j=0}^{b-1}\sum_{m=a}^{b}B_{m,j}(0)z^j\Big], \tag{37}$$

Substitute $\theta=w(z)$ in (32) and (33), we get,

$$Q_1(z,0)=\tilde{V}(w(z))\sum_{n=0}^{a-1}\Big[\sum_{m=a}^{b}P_{m,n}(0)z^n+\eta T_n z^n\Big], \tag{38}$$

$$Q_j(z,0) = \tilde{V}(w(z)) \sum_{n=0}^{a-1} Q_{j-1,n}(0) z^n, \ j \geq 2. \tag{39}$$

Substitute the equations (34) to (39) in equations (28) to (33) after simplification, we get,

$$(\theta - u(z))\tilde{P}_i(z,\theta) = (\tilde{S}_b(u(z)) - \tilde{S}_b(\theta))\Big[\sum_{m=a}^{b} P_{m,i}(0) + \sum_{j=0}^{\infty} Q_{l,i}(0) + \eta \tilde{B}_i(z,0)\Big], \ a \leq i \leq b-1, \tag{40}$$

$$(\theta - u(z))\tilde{P}_b(z,\theta) = \frac{(\tilde{S}_b(u(z)) - \tilde{S}_b(\theta))}{(z^b - \tilde{S}_b(u(z)))}\Big[\sum_{m=a}^{b-1} P_m(z,0) + \sum_{l=1}^{\infty} Q_l(z,0) + z^b \eta \tilde{B}_b(z,0) - \sum_{j=0}^{b-1}\Big(\sum_{m=a}^{b} P_{m,j}(0) z^j + \sum_{j=0}^{\infty} Q_{l,j}(0) z^j\Big)\Big], \tag{41}$$

$$(\theta - v(z))\tilde{B}_i(z,\theta) = (\tilde{S}_s(v(z)) - \tilde{S}_s(\theta))\Big[\alpha \tilde{P}_i(z,0) + \sum_{m=a}^{b} B_{m,i}(0) + \sum_{k=0}^{a-1} T_k \lambda_2 g_{i-k}\Big], \ a \leq i \leq b-1, \tag{42}$$

$$(\theta - v(z))\tilde{B}_b(z,\theta) = \frac{(\tilde{S}_s(v(z)) - \tilde{S}_s(\theta))}{(z^b - \tilde{S}_s(v(z)))}\Big[\sum_{m=a}^{b-1} B_m(z,0) + z^b \alpha \tilde{P}_b(z,0) + \lambda_2 \sum_{k=0}^{a-1}\sum_{j=b}^{\infty} T_k z^k g_{j-k} z^{j-k} - \sum_{j=0}^{b-1}\sum_{m=a}^{b} B_{m,j}(0) z^j\Big], \tag{43}$$

$$(\theta - w(z))\tilde{Q}_1(z,\theta) = (\tilde{V}(w(z)) - \tilde{V}(\theta)) \sum_{n=0}^{a-1}\Big[\sum_{m=a}^{b} P_{m,n}(0) z^n + \eta T_n z^n\Big], \tag{44}$$

$$(\theta - w(z))\tilde{Q}_j(z,\theta) = (\tilde{V}(w(z)) - \tilde{V}(\theta)) \sum_{n=0}^{a-1} Q_{j-1,n}(0) z^n, \ j \geq 2. \tag{45}$$

5. Probability Generating Function of Queue Size

5.1. The PGF of the queue size at an arbitrary time epoch

Let $P(z)$ be the PGF of the queue size at an arbitrary time epoch. Then,

$$P(z) = \sum_{i=a}^{b} \tilde{P}_i(z,0) + \sum_{i=a}^{b} \tilde{B}_i(z,0) + \sum_{l=1}^{\infty} \tilde{Q}_l(z,0) + T(z). \tag{46}$$

By substituting $\theta = 0$ in equations (40) to (45) then equation (46) becomes

$$P(z) = \frac{K_1(z)\sum_{i=a}^{b-1}(z^b - z^i)c_i + K_2(z)\sum_{i=a}^{b-1}(z^b - z^i)d_i + (1 - \tilde{V}(w(z)))K_3(z)\sum_{n=0}^{a-1} c_n z^n + \Big[\eta[Y_1(z) - \tilde{V}(w(z))K_3(z)] - v_1(z)K_2(z) + w(z)Y_1(z)\Big]\sum_{k=0}^{a-1} T_k z^k}{w(z)Y_1(z)} \tag{47}$$

where

$$p_i = \sum_{m=a}^{b} P_{m,i}(0), \ v_i = \sum_{l=1}^{\infty} Q_{l,i}(0), \ q_i = \sum_{m=a}^{b} B_{m,i}(0), \ c_i = p_i + v_i, \ \text{and} \ \ d_i = q_i + \sum_{k=0}^{a-1} T_k \lambda_2 g_{i-k}$$

and the expressions for $K_1(z)$, $K_2(z)$, $K_3(z)$, $v_1(z)$ and $Y_1(z)$ are defined in Appendix-I.

5.2. Steady state condition

The probability generating function has to satisfy $P(1) = 1$. In order to satisfy this condition applying L' Hopital's rule and evaluating $\lim_{z \to 1} P(z)$, then equating the expression to 1, we have, $H = (-\lambda_3 X_1)E_1$, where the expressions H and E_1 are defined in Appendix-II. Since c_i, d_i and T_i are probabilities of $'i'$ customers being in the queue, it follows that H must be positive. Thus $P(1) = 1$ is satisfied iff $(-\lambda_3 X_1)E_1 > 0$. If

$$\rho = \frac{(1-\omega)X_1(\alpha\lambda_2+\eta\lambda_1)(1-\tilde{S}_b(\alpha))(1-\tilde{S}_s(\eta))}{b\alpha\eta[\tilde{S}_b(\alpha)(1-\tilde{S}_s(\eta))+\tilde{S}_s(\eta)(1-\tilde{S}_b(\alpha))]}$$

then $\rho < 1$ is the condition for the existence of steady state for the model under consideration.

5.3. Computational aspects

Equation (47) has $2b$ unknowns $c_0, c_1, ..., c_{b-1}, d_a, ..., d_{b-1}$, and $T_0, T_1, ..., T_{a-1}$. Now equation (47) gives the PGF of the number of customers involving only $2b$ unknowns. By Rouche's theorem, it can be proved that $Y_1(z)$ has $2b-1$ zeros inside and one on the unit circle $|z| = 1$. Since $P(z)$ is analytic within and on the unit circle, the numerator must vanish at these points, which gives $2b$ equations in $2b$ unknowns. We can solve these equations by any suitable numerical technique.

5.4. Particular case

When there is no breakdown, no balking and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ then equation (47) reduces to

$$P(z) = \frac{(\tilde{S}_b(w(z))-1)\sum_{i=a}^{b-1}(z^b-z^i)c_i + (z^b-1)(\tilde{V}(w(z))-1)\sum_{n=0}^{a-1}c_n z^n}{(-w(z))(z^b-\tilde{S}_b(w(z)))} \tag{48}$$

which coincides with the PGF of Senthilnathan et al. (2012) without closedown.

PGF of queue size at main server's service completion epoch: The probability generating function of main server's service completion epoch $M(z)$ is obtained from the equations (40) and (41)

$$M(z) = \frac{\begin{aligned}&(1-\tilde{S}_b(u(z)))\Big[z^b v(z)(z^b-\tilde{S}_s(v(z)))\sum_{i=a}^{b-1}c_i + z^{2b}\eta(1-\tilde{S}_s(v(z)))\sum_{i=a}^{b-1}d_i\\&+v(z)(z^b-\tilde{S}_s(v(z)))\Big(\tilde{V}(w(z))\sum_{n=0}^{a-1}(p_n+\eta T_n+v_n)z^n - \sum_{i=0}^{b-1}c_j z^j\Big)\\&+z^b\eta(1-\tilde{S}_s(v(z)))\Big(\sum_{j=b}^{\infty}\sum_{k=0}^{a-1}T_k z^k\lambda_2 g_{j-k}z^{j-k} - \sum_{j=0}^{b-1}q_j z^j\Big)\Big]\end{aligned}}{Y_1(z)} \tag{49}$$

PGF of queue size at vacation completion epoch: The PGF of main server's vacation completion epoch $V(z)$ is obtained from the equations (44) and (45) we get,

$$V(z) = \frac{(1-\tilde{V}(w(z)))\sum_{n=0}^{a-1}(p_n+\eta T_n+v_n)z^n}{w(z)} \tag{50}$$

PGF of queue size at stand-by server's service completion epoch: The probability generating function of stand-by server's service completion epoch $N(z)$ is derived from the equations (42) and (43), we get,

$$N(z) = \frac{\begin{aligned}&(1-\tilde{S}_s(v(z)))\Big[z^b u(z)(z^b-\tilde{S}_b(u(z)))\sum_{i=a}^{b-1}d_i + z^{2b}\alpha(1-\tilde{S}_b(u(z)))\sum_{i=a}^{b-1}c_i\\&+z^b\alpha(1-\tilde{S}_b(u(z)))\Big(\tilde{V}(w(z))\sum_{n=0}^{a-1}(p_n+\eta T_n+v_n)z^n - \sum_{i=0}^{b-1}c_j z^j\Big)\\&+u(z)(z^b-\tilde{S}_b(u(z)))\Big(\sum_{j=b}^{\infty}\sum_{k=0}^{a-1}T_k z^k\lambda_2 g_{j-k}z^{j-k} - \sum_{j=0}^{b-1}q_j z^j\Big)\Big]\end{aligned}}{Y_1(z)} \tag{51}$$

6. Some Performance Measures

6.1. Main server's expected length of idle period

Let K is the random variable denoting the 'Idle period due to multiple vacation process'. Let Y be the random variable defined by

$$Y = \begin{cases} 0 & \text{if the server finds atleast '}a\text{' customers after the first vacation} \\ 1 & \text{if the server finds less than '}a\text{' customers after the first vacation} \end{cases}$$

Now

$$\begin{aligned} E(K) &= E(K/Y=0)P(Y=0) + E(K/Y=1)P(Y=1) \\ &= E(V)P(Y=0) + (E(V)+E(K))P(Y=1), \end{aligned}$$

solving for $E(K)$, we get

$$E(K) = \frac{E(V)}{(1-P(Y=1))} = \frac{E(V)}{\left(1 - \sum_{n=0}^{a-1}\sum_{i=0}^{n}\left[\gamma_i[p_{n-i} + \eta T_{n-i}]\right]\right)}$$

where γ_i are the probabilities of i customers arrive during main server's vacation time.

6.2. Expected queue length

The mean number of customers waiting in the queue $E(Q)$ at an arbitrary time epoch, is obtained by differentiating $P(z)$ at $z=1$ and is given by

$$E(Q) = \frac{\begin{aligned} & f_1(X,S_b,S_s)\Big[\sum_{i=a}^{b-1}[b(b-1)-i(i-1)]c_i\Big] + f_1(X,S_b,S_s)\Big[\sum_{i=a}^{b-1}(b(b-1)-i(i-1))d_i\Big] \\ & + f_2(X,S_b,S_s)\Big[\sum_{i=a}^{b-1}(b-i)c_i\Big] + f_3(X,S_b,S_s)\sum_{i=a}^{b-1}(b-i)d_i + f_4(X,S_b,S_s,V)\sum_{n=0}^{a-1}c_n \\ & + f_5(X,S_b,S_s,V)\sum_{n=0}^{a-1}nc_n + f_6(X,S_b,S_s,V)\sum_{n=0}^{a-1}T_n + f_7(X,S_b,S_s,V)\sum_{n=0}^{a-1}nT_n \end{aligned}}{3(E_{19})^2}, \tag{52}$$

the expressions for $f_i(i=1,2,...,7)$ are defined in Appendix-II.

6.3. Expected Waiting time

The expected waiting time is obtained using the Little's formula as

$$E(W) = \frac{E(Q)}{\lambda E(X)} \tag{53}$$

where $E(Q)$ is given in equation (52).

7. Numerical Example

A numerical example of our model is analysed for a particular case with the following assumptions:

(1). Batch size distribution of the arrival process is geometric distribution with mean 2.

(2). Take a=5, b=8 and service time distribution is Erlang-2 (both servers).

(3). Vacation time of main server follows exponential distribution with parameter $\xi = 5$.

(4). Let m_1 be the service rate for the main server.

(5). Let m_2 be the service rate for the stand-by server.

The unknown probabilities of the queue size distribution are computed using numerical techniques. The zeros of the function $Y_1(z)$ are obtained and simultaneous equations are solved by using MATLAB. The expected queue length $E(Q)$ and the expected waiting time $E(W)$ are calculated for various arrival rate and service rate and the results are tabulated. From **Tables 1 and 2** the following observations can be made.

(1). As arrival rate λ increases, the expected queue size and expected waiting time are also increasing.

(2). When the main server's service rate increases, the expected queue size and expected waiting time are decreasing.

λ_1	ρ	$E(Q)$	$E(W)$
8.00	0.237171	22.411092	0.659150
8.25	0.243581	26.612317	0.771372
8.50	0.249991	31.347449	0.895641
8.75	0.256401	36.664048	1.032790
9.00	0.262811	42.613069	1.183696
9.25	0.269221	49.248890	1.349285
9.50	0.275631	56.629509	1.530527
9.75	0.282041	64.816643	1.728444
10.00	0.288451	73.876750	1.944125
10.25	0.294861	83.880140	2.178705
10.50	0.301271	94.901974	2.433384
10.75	0.307681	107.022511	2.709431
11.00	0.314091	120.327418	3.008185

Table 1. Arrival rate vs expected queue length and expected waiting time for the values $\lambda_2 = 5,\ \lambda_3 = 4,\ m_1 = 7,\ m_2 = 5,\ \alpha = 1,\ \eta_1 = 2,\ \eta_2 = 3\ \omega = 0.2,$ and $\epsilon = 0.5$

m_1	ρ	$E(Q)$	$E(W)$
5.00	0.431507	254.868361	5.540617
5.25	0.412984	210.408446	4.574097
5.50	0.395983	174.830196	3.800656
5.75	0.380324	146.017683	3.174297
6.00	0.365854	122.435815	2.661648
6.25	0.352443	102.954901	2.238150
6.50	0.339979	86.727275	1.885376
6.75	0.328366	73.107729	1.589298
7.00	0.317518	61.600645	1.339144
7.25	0.307364	51.819086	1.126502
7.50	0.297838	43.458621	0.944753
7.75	0.288885	36.276967	0.788630
8.00	0.280453	30.079824	0.653909

Table 2. Main server's service rate vs expected queue length and expected waiting time for the values $\lambda_1 = 10,\ \lambda_2 = 8,\ \lambda_3 = 5,\ m_2 = 4,\ \alpha = 1,\ \eta_1 = 2,\ \eta_2 = 3\ \omega = 0.2,$ and $\epsilon = 0.5$

8. Conclusion

In this paper, a batch arrival general bulk service single server queueing system with server's breakdown and second optional repair, stand-by server, balking, variant arrival rate and multiple vacation is analysed. Probability generating function of queue size distribution at an arbitrary time is obtained. Some performance measures are calculated. Particular cases of the model are also deduced. From the numerical results, it is observed that when the arrival rate increases the expected queue length and waiting time of the customers are also increasing. It is also observed that if the main sever's service rate are increase, then the expected queue length and expected waiting time are decreased.

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Appendix I

The expressions used in equation (47) are defined as follows:

$$K_1(z) = w(z)(1-\tilde{S}_b(u(z)))A_1(z),$$

$$K_2(z) = w(z)(1-\tilde{S}_s(v(z)))A_2(z),$$

$$K_3(z) = Y_1(z) - K_1(z),$$

where

$$A_1(z) = v(z)(z^b - \tilde{S}_s(v(z))) + z^b\alpha(1-\tilde{S}_s(v(z))),$$

$$A_2(z) = u(z)(z^b - \tilde{S}_b(u(z))) + z^b\eta(1 - \tilde{S}_b(u(z))), \quad v_1(z) = \eta - \lambda_2(X-1),$$

$$Y_1(z) = u(z)v(z)(z^b - \tilde{S}_b(u(z)))(z^b - \tilde{S}_s(v(z))) - z^{2b}\alpha\eta(1 - \tilde{S}_b(u(z)))(1 - \tilde{S}_s(v(z))).$$

Appendix II

The expressions for f_i (i=1,2,...,7) in (52) are defined as follows:

$$f_1(X, S_b, S_s) = 3E_9E_{19},$$

$$f_2(X, S_b, S_s) = 3E_{10}E_{19} - 2E_9E_{20},$$

$$f_3(X, S_b, S_s) = 3E_{12}E_{19} - 2E_9E_{20},$$

$$f_4(X, S_b, S_s, V) = 2V_1E_{14}E_{20} - 3[V_2E_{14} + V_1E_{15}]E_{19},$$

$$f_5(X, S_b, S_s, V) = -6V_1E_{14}E_{19},$$

$$f_6(X, S_b, S_s, V) = E_{18}E_{19} - E_{17}E_{20},$$

$$f_7(X, S_b, S_s, V) = 3E_{17}E_{19},$$

where

$$E_1 = -(1-\omega)X_1(\lambda_2\alpha + \lambda_1\eta)(1 - \tilde{S}_b(\alpha))(1 - \tilde{S}_s(\eta)) - b\alpha\eta[\tilde{S}_b(\alpha)(\tilde{S}_s(\eta) - 1) + \tilde{S}_s(\eta)(\tilde{S}_b(\alpha) - 1)],$$

$$E_2 = [2\lambda_1\lambda_2(1-\omega)^2{X_1}^2 - (1-\omega)X_2(\lambda_2\alpha + \lambda_1\eta)](1 - \tilde{S}_b(\alpha))(1 - \tilde{S}_s(\eta))$$
$$- 2(1-\omega)X_1(\lambda_2\alpha + \lambda_1\eta)[(b - S_{b1})(1 - \tilde{S}_s(\eta)) + (b - S_{s1})(1 - \tilde{S}_b(\alpha))]$$
$$+ \alpha\eta[b(b-1) - S_{b2})(1 - \tilde{S}_s(\eta)) + 2(b - S_{b1})(b - S_{s1}) + (b(b-1) - S_{s2})(1 - \tilde{S}_b(\alpha))],$$

$$E_3 = [b\alpha - (1-\omega)\lambda_2X_1](1 - \tilde{S}_s(\eta)) - (\alpha + \eta)S_{s1} + b\eta,$$

$$E_4 = [\alpha b(b-1) - (1-\omega)\lambda_2X_2](1 - \tilde{S}_s(\eta)) - 2S_{s1}(b\alpha - (1-\omega)\lambda_2X_1) - (\alpha + \eta)S_{s2}$$
$$+ \eta b(b-1) - 2b(1-\omega)\lambda_1X_1,$$

$$E_5 = [\alpha b(b-1)(b-2) - (1-\omega)\lambda_2X_3](1 - \tilde{S}_s(\eta)) - 3S_{s1}(b(b-1)\alpha - (1-\omega)\lambda_2X_2)$$
$$- 3S_{s2}(b\alpha - (1-\omega)\lambda_2X_1) - (\alpha + \eta)S_{s3} - 3b(1-\omega)\lambda_2X_2$$
$$- 3b(b-1)(1-\omega)\lambda_2X_1 + \eta b(b-1)(b-2),$$

$$E_6 = [b\eta - (1-\omega)\lambda_1X_1](1 - \tilde{S}_b(\alpha)) - (\alpha + \eta)S_{b1} + b\alpha,$$

$$E_7 = [\eta b(b-1) - (1-\omega)\lambda_1X_2](1 - \tilde{S}_b(\alpha)) - 2S_{b1}(b\eta - (1-\omega)\lambda_1X_1) - (\alpha + \eta)S_{b2}$$
$$+ \alpha b(b-1) - 2b(1-\omega)\lambda_1X_1,$$

$$E_8 = [\eta b(b-1)(b-2) - (1-\omega)\lambda_1X_3](1 - \tilde{S}_b(\alpha)) - 3S_{b1}(b(b-1)\eta - (1-\omega)\lambda_1X_2)$$
$$- 3S_{b2}(b\eta - (1-\omega)\lambda_1X_1) - (\alpha + \eta)S_{b3} - 3b(1-\omega)\lambda_1X_2$$
$$- 3b(b-1)(1-\omega)\lambda_1X_1 + \alpha b(b-1)(b-2),$$

$$E_9 = -X_1\lambda_3(\alpha + \eta)(1 - \tilde{S}_b(\alpha))(1 - \tilde{S}_s(\eta)),$$

$$E_{10} = (\tilde{S}_b(\alpha) - 1)[\lambda_3X_2(\alpha + \eta)(1 - \tilde{S}_s(\eta)) + 2X_1\lambda_3E_3] + 2X_1\lambda_3(\alpha + \eta)S_{b1}(1 - \tilde{S}_s(\eta)),$$

$$E_{11} = (\tilde{S}_b(\alpha) - 1)[\lambda_3X_3(\alpha + \eta)(1 - \tilde{S}_s(\eta)) + 3X_2\lambda_3E_3 + 3X_1\lambda_3E_4]$$
$$+ 3S_{b1}[X_2\lambda_3(\alpha + \eta)(1 - \tilde{S}_s(\eta)) + 2X_1\lambda_3E_3] + 3S_{b2}X_1\lambda_3(\alpha + \eta)(1 - \tilde{S}_s(\eta)),$$

$$E_{12} = (\tilde{S}_s(\eta) - 1)[\lambda_3 X_2(\alpha+\eta)(1-\tilde{S}_b(\alpha)) + 2X_1\lambda_3 E_6] + 2X_1\lambda_3(\alpha+\eta)S_{s1}(1-\tilde{S}_b(\alpha)),$$

$$E_{13} = (\tilde{S}_s(\eta) - 1)[\lambda_3 X_3(\alpha+\eta)(1-\tilde{S}_b(\alpha)) + 3X_2\lambda_3 E_6 + 3X_1\lambda_3 E_7]$$
$$+ 3S_{s1}[X_2\lambda_3(\alpha+\eta)(1-\tilde{S}_b(\alpha)) + 2X_1\lambda_3 E_6] + 3S_{s2}X_1\lambda_3(\alpha+\eta)(1-\tilde{S}_b(\alpha)),$$

$$E_{14} = E_1 - E_9, \quad E_{15} = E_2 - E_{10},$$

$$E_{17} = \eta[E_2 - E_{15} - 2E_{14}V_1] + 2E_9\lambda_2 X_1 - \eta E_{12} - 2\lambda_3 X_1 E_1,$$

$$E_{18} = \eta[E_{11} - 3E_{15}V_1 - 3E_{14}V_2] + \lambda_2[X_2E_9 + X_1E_{12}] - \eta E_{13} - 3\lambda_3[X_2E_1 + X_1E_2],$$

$$E_{19} = -2X_1\lambda_3 E_1, \qquad E_{20} = -3\lambda_3[X_2E_1 + X_1E_2], \quad \lambda = \lambda_1 + \lambda_2 + \lambda_3.$$

$$H = \begin{cases} E_9 \sum_{i=a}^{b-1}(b-i)c_i + E_9 \sum_{i=a}^{b-1}(b-i)d_i - E_{14}V_1 \sum_{n=0}^{a-1} c_n \\ +(E_{17}/2)\sum_{n=0}^{a-1} T_n, \end{cases}$$

where

$$S_{b1} = -\lambda_1 X_1 S'_b(\alpha), \quad S_{b2} = (-\lambda_1 X_1)^2 S''_b(\alpha) - \lambda_1 X_2 S'_b(\alpha),$$

$$S_{s1} = -\lambda_2 X_1 S'_s(\eta), \quad S_{s2} = (-\lambda_2 X_1)^2 S''_s(\eta) - \lambda_2 X_2 S'_s(\eta),$$

$$V_1 = -\lambda_3 X_1 E(V), \qquad V_2 = (\lambda_3 X_1)^2 E(V^2) + \lambda_3 X_2 E(V).$$