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q-Steffensen's Inequality for Convex Functions

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Abstract: In this paper, q- Steffensen's Inequality for convex functions is presented with illustrative examples. Review of research works on Steffensen's Inequality and q-calculus is extensively carried out. Methods of q- differentiability and monotonicity of functions are employed to establish the results.

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1. Introduction

The Steffensen's inequality (1) was discovered in [16]

$$\int_{b-\lambda}^{b} g(x)dx \leq \int_{a}^{b} g(x)f(x)dx \leq \int_{a}^{a+\lambda} g(x)dx, \tag{1}$$

where $\lambda = \int_a^b f(x)dx$, f and g are integrable functions defined on (a,b), g is decreasing and $0 \leq f(x) \leq 1$ for each $x \in (a,b)$. This inequality was initially not popular in the research environment until its appearance again in [15]. Many research papers have been written on the inequality providing refinements, generalisations and numerous applications (see [8–11] and the references cited therein). The first generalisation of this inequality appeared in [1] and the result was later detected to be incorrect in [3]. About two decades later, Pečarić presented a corrected version of Bellman in [10] as

$$\left(\int_0^1 f(x)g(x)dx\right)^p \leq \int_0^{\lambda} g(x)^p dx \tag{2}$$

where $\lambda = \left(\int_0^1 f(x)dx\right)^p$, $g : [0,1] \longrightarrow \Re$ is a non-negative and non-increasing function and $f : [0,1] \longrightarrow \mathbb{R}$ is an integrable function such that $0 \leq f(x) \leq 1$ $(\forall x \in [0,1])$ for $p \geq 1$. Moreover, an analogous inequality to (2) was further given as

$$\frac{\int_0^1 f(x)g(x)dx}{\int_0^1 f(x)dx} \leq \frac{1}{\lambda}\int_0^{\lambda} g(x)dx. \tag{3}$$

Using the substitution $f(x) = \frac{\lambda F(x)}{\int_a^b F(x)dx}$, a further inequality was established in [11] as

$$\frac{1}{\lambda}\int_{b-\lambda}^{b} g(x)dx \leq \frac{\int_a^b g(x)F(x)dx}{\int_a^b F(x)dx} \leq \frac{1}{\lambda}\int_{a}^{a+\lambda} g(x)dx \tag{4}$$

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where $f(x)$ and $g(x)$are assumed to be integrable functions defined on $[a,b]$ and that $g(x)$ never increases and

$$0 \leq \lambda F(x) \leq \int_a^b F(x)dx, \qquad (\forall x \in [a,b]),$$

where λ is a positive number. Further generalisation of (1) appeared in [6], but this result was detected to be incorrect in [17] (see also [5] and [12]) and modified as

$$\int_a^b g(x)f(x)dx \leq \int_a^{a+\lambda} g(x)h(x)dx, \tag{5}$$

where λ is given by

$$\int_a^{a+\lambda} h(x)dx = \int_a^b f(x)dx,$$

with f, g and h being integrable functions on (a,b), g decreasing and $0 \leq f \leq h$. The second inequality of (1) was also modified as

$$\int_{b-\lambda}^b g(x)h(x)dx \leq \int_a^b g(x)f(x)dx, \tag{6}$$

where λ is given by

$$\int_{b-\lambda}^b h(x)dx = \int_a^b f(x)dx,$$

with f, g and h being integrable functions on (a,b), g decreasing and $0 \leq f \leq h$. The double inequality of (1) was thus re-established as

$$\int_{b-\lambda}^b g(x)h(x)dx \leq \int_a^b g(x)f(x)dx \leq \int_a^{a+\lambda} g(x)h(x)dx, \tag{7}$$

provided that there exists $\lambda \in [0, b-a]$ such that

$$\int_{b-\lambda}^b h(x)dx = \int_a^b f(x)dx = \int_a^{a+\lambda} h(x)dx,$$

with f, g and h being integrable functions on (a,b), g decreasing and $0 \leq f \leq h$. The study of q-analysis attracted the attention of many researchers as well as those working on Steffensen's inequality and this led to further re-establishment of (1) via q-calculus (See for example [4]).

This paper aims at presenting another generalisation of the Steffensen's inequality with the involvement of convex functions and q-calculus.

2. Preliminaries on q-calculus

The notion of q-calculus (an analogue of the usual calculus) is presented in this section. This q-analysis was earlier discovered in the eighteenth century by Euler, but the notion of the definite integral was introduced by Jackson in 1910 (see [4] and the references cited therein). Some definitions and facts on q-calculus for the understanding of this paper is discussed here. Throughout this paper, the real number q satisfies $0 < q < 1$.

Definition 2.1. *Let $f(x)$ be any arbitrary function. The q-differential is defined as*

$$(d_q f)(x) = f(qx) - f(x).$$

In particular,

$$d_q x = (q-1)x.$$

Definition 2.2. *Let $f(x)$ be any arbitrary function. The q-derivative is defined as*

$$(D_q f)(x) = \frac{(d_q f)(x)}{d_q x} = \frac{f(qx) - f(x)}{(q-1)x}.$$

It follows that $(D_q f)(x) \to \frac{(df)(x)}{dx}$ as $q \to 1$.

Remark 2.3. *The q-analogue of the Leibniz rule is given as (See [4, 14] and the references cited therein).*

$$(D_q fg)(x) = g(x) D_q f(x) + f(qx) D_q g(x)$$

Example 2.4. *Let $f(x) = x^\alpha$ where $\alpha \in \mathbb{C}$. Then*

$$D_q x^\alpha = \frac{(qx)^\alpha - x^\alpha}{(q-1)x} = \frac{q^\alpha - 1}{q-1} x^{\alpha-1} = [\alpha]_q x^{\alpha-1}$$

where $[\alpha]_q$ is the q-analogue of α given by

$$\begin{aligned}[\alpha]_q &= \frac{q^\alpha - 1}{q-1} \\ &= q^{\alpha-1} + \cdots + q + 1.\end{aligned}$$

Definition 2.5. *Let $0 < a < b$. The definite q-integral also known as the q-Jackson integral is defined as (see [2, 4, 14])*

$$\int_0^b f(x) d_q(x) = (1-q) b \sum_{j=0}^{\infty} q^j f(q^j b) \tag{8}$$

provided the series converges.

Note that

$$\int_a^b f(x) d_q(x) = \int_0^b f(x) d_q(x) - \int_0^a f(x) d_q(x). \tag{9}$$

The values of such defined q-integrals of the polynomials form have very similar form to those in the standard integral calculus. For example [7].

$$\int_a^b t^n d_q t = \frac{b^{n+1} - a^{n+1}}{[n+1]_q}. \tag{10}$$

Remark 2.6 ([4])**.** *If $f(x) \geq 0$, it is not necessarily true that $\int_a^b f(x) d_q(x) \geq 0$.*

Definition 2.7. *The q-integration by parts for suitable functions f and g is given as ([2, 4]).*

$$\int_a^b f(x)(D_q g)(x) d_q(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)(D_q f)(x) d_q(x). \tag{11}$$

Theorem 2.8 ([13])**.** *Let $f(x)$ be a continuous functions on a segment $[a, b]$. Then there exists $\zeta \in (a, b)$ such that*

$$\int_a^b f(t) d_q t = f(\zeta)(b-a) \tag{12}$$

for every $q \in (0, 1)$.

Theorem 2.9 ([13])**.** *Let $f(x)$ and $g(x)$ be some continuous functions on a segment $[a, b]$. Then there exists $\zeta \in (a, b)$ such that*

$$\int_a^b f(t) g(t) d_q t = f(\zeta) \int_a^b g(t) d_q t \tag{13}$$

for every $q \in (0, 1)$.

3. Results and Discussions

This section now presents the q-Steffensen's inequality for convex functions.

Lemma 3.1. *Let f, g be two q-integrable functions such that g is positive and q-decreasing defined on $[0,1]$ and $0 \leq f(t) \leq 1$ for every $t \in [0,1]$, then*

$$\left(\int_0^1 f(t)g(t)d_qt\right)^p \leq \int_0^\lambda g(t)^p d_qt \tag{14}$$

where $\lambda = \left(\int_0^1 f(t)d_qt\right)^p$ for $p \geq 1$.

Proof. Want to prove that

$$\int_0^\lambda g(t)^p d_qt - \left(\int_0^1 f(t)g(t)d_qt\right)^p \geq 0. \tag{15}$$

Since g is q-decreasing which implies $g(qt) \geq g(t)$ for every $t \in [0,1]$. Then using equations (8) and (13) and for each $\zeta \in (0,1)$ we have

$$\begin{aligned}\int_0^\lambda g(t)^p d_qt - \left(\int_0^1 f(t)g(t)d_qt\right)^p &= (1-q)\lambda\sum_{j=0}^\infty h(\lambda q^j)q^j - \left(f(\zeta)(1-q)\sum_{j=0}^\infty g(q^j)q^j\right)^p \\ &\geq 0\end{aligned}$$

where $(g(t))^p = h(t)$ and $\lambda = \left((1-q)\sum_{j=0}^\infty f(q^j)q^j\right)^p$ for $p \geq 1$. □

Remark 3.2. *A particular case of $p = 1$ reduces inequality (14) to the right side of the Steffensen Inequality (1) for $a = 0$ and $b = 1$.*

Theorem 3.3. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function with $0 \leq f(t) \leq 1$ for each $t \in [0,1]$. If $\Phi : [0,1] \to \mathbb{R}$ is a convex and q-differentiable function with $\Phi(0) = 0$, then*

$$\Phi\left(\int_0^1 f(t)d_qt\right) \leq \int_0^1 f(t)(D_q\Phi)(t)d_qt \tag{16}$$

for every $t \in [0,1]$.

Proof. Following Remark 3.2 we have

$$\int_0^1 f(t)g(t)d_qt \leq \int_0^\lambda g(t)d_qt \tag{17}$$

Since Φ is convex and $-(D_q\Phi)(t)$ is q-decreasing for all t, replacing g in (17) yields

$$\int_0^\lambda (D_q\Phi)(t)d_qt \leq \int_0^1 f(t)(D_q\Phi)(t)d_qt \tag{18}$$

This gives

$$\Phi(\lambda) - \Phi(0) \leq \int_0^1 f(t)(D_q\Phi)(t)d_qt \tag{19}$$

Since $\lambda = \int_0^1 f(t)d_qt$ and $\Phi(0) = 0$, thus

$$\Phi\left(\int_0^1 f(t)d_qt\right) \leq \int_0^1 f(t)(D_q\Phi)(t)d_qt.$$

□

Remark 3.4. *A special case of $\Phi(t) = t^k$ for $k \geq 1$ using (16) yields*

$$\left(\int_0^1 f(t)d_qt\right)^k \leq [k]\int_0^1 f(t)t^{k-1}d_qt$$

where $(D_q\Phi)(t) = [k]_q t^{k-1}$ and $[k]_q = \frac{q^k-1}{q-1}$.

Example 3.5. *Let $n > 1$ and*

$$f(t) = \begin{cases} t^n & \text{for} \quad 0 < t \leq 1 \\ 0 & \text{elsewhere} \quad . \end{cases}$$

Then

$$\left(\int_0^1 t^n d_qt\right)^k \leq [k]_q \int_0^1 t^{n+k-1}d_qt.$$

Applying (10) yields

$$\frac{1}{([n+1]_q)^k} \leq \frac{[k]_q}{[n+k]_q}.$$

Lemma 3.6. *Let f, g and h be q-integrable functions on $[0,1]$ with g decreasing and let $0 \leq f(t) \leq h(t)$, $t \in [0,1]$. Then*

$$\int_0^1 f(t)g(t)d_qt \leq \int_0^\lambda g(t)h(t)d_qt \tag{20}$$

where λ is given by

$$\int_0^\lambda h(t)d_qt = \int_0^1 f(t)d_qt \tag{21}$$

Proof. Following exactly the proof in [5] leads to the result in terms of q-calculus. □

Theorem 3.7. *Let f and h be q-integrable functions on $[0,1]$ with $0 \leq f(t) \leq h(t)$, $t \in [0,1]$. If Φ is convex, then*

$$\int_0^\lambda (D_q\Phi)(t)h(t)d_qt \leq \int_0^1 (D_q\Phi)(t)f(t)d_qt \tag{22}$$

Proof. Replace $g(t)$ with $-(D_q\Phi)(t)$ in (20) and the result follows immediately after simplification. □

4. Conclusion

A review of the well-known Steffensen's Inequality and q-calculus was presented together with some extensions and generalisations. Using q-calculus, new results were established for the Steffensen's Inequality for convex functions. Moreover, the results were supported with remarks and illustrative examples.

References

[1] R. Bellman, *On inequalities with alternating signs.*, proc. Amer. Math. Soc., 10(1959), 807-809.

[2] A. Fitouhi, K. Brahim and N. Bettaibi, *Asymptotic Approximations in Quantum Calculus*, Journal of Nonlinear Mathemataical Physics 12(4)(2005), 586-606

[3] E.K. Godunova and V.I. Levin, *A general class of inequalities containing Steffensens inequality*, Mat. Zametki, 3(1968), 339-344

[4] H. Gauchman, *Integral Inequalities in q-Calculus*, Computers and Mathematics with Applications, 47(2004), 281-300.

[5] Z. Liu, *On Extensions of Steffensens Inequality*, J. Math. Anal. and Approx. Theory, 2(2)(2007), 132-139.

[6] P.R. Mercer, *Extensions of Steffensens Inequality*, J. Math. Anal. Appl., 246(2000), 325-329.

[7] S. Marinkovic, P. Rajkovic and M. Stankovic, *The inequalities for some types of q-integrals*, Computers and Mathematics with Applications, 56(2008), 2490-2498.

[8] D.S. Mitrinovic, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, (1993).

[9] D.S. Mitrinovic and J.E. Pečarić, *On the Bellman Generalization of Steffensen's Inequality III*, J. Math. Anal. and Appl., 135(1988), 342-345.

[10] J.E. Pečarić, *On the Bellman Generalization of Steffensen's Inequality*, J. Math. Anal. and Appl., 88(1982), 505-507.

[11] J.E. Pečarić, *On the Bellman Generalization of Steffensen's Inequality II*, J. Math. Anal. and Appl., 104(1984), 432-434.

[12] J.E. Pečarić, A. Perusic and K. Smoljak, *Mercer and Wu–Srivastava generalisations of Steffensens inequality*, Appl. Math. Comput., 219(2013), 10548-10558.

[13] P.M. Rajković, M.S. Stanković and S.D. Marinković, *Mean Value Theorems in q-Calculus*, Matematnykn Bechnk, 54(2002), 171-178.

[14] A.D. Sole and V.G. Kac, *On integral representations of q-gamma and q-beta functions*, Rend. Mat. Acc. Lincei, 16(1)(2005), 11-29.

[15] J. Steffensen, *Bounds of certain trigonometric integrals*, Tenth Scan-dinavian Mathematical congress, Copenhagan, Hellerup, 1946(1947), 181-186.

[16] J. Steffensen, *On certain inequalities between mean values and their application to actuarial problems*, Skandinavisk Aktuarietidskrift, (1918), 82-97.

[17] S.H. Wu and H.M. Srivastava, *Some improvements and generalizations of Steffensen's Integral Inequality*, Appl. Math. Comput., 192(2007), 422-428.