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On a q-analogue of the Nielsen's β-Function

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Abstract: Motivated by the Jackson's q-integral and the q-digamma function, we introduce a q-analogue of the Nielsen's β-function. We further study some identities, monotonicity and convexity properties, and some inequalities involving the new function.

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1. Introduction

The Nielsen's β-function, which first appeared in [15] is defined as

$$\beta(x) = \frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\}, \quad x > 0, \tag{1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+x}, \quad x > 0, \tag{2}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ is the digamma or psi function and $\Gamma(x)$ is the Euler's Gamma function. It has the integral representations

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}\,dt, \quad x > 0, \tag{3}$$

$$= \int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\,dt, \quad x > 0, \tag{4}$$

and satisfies the functional equation

$$\beta(x+1) = \frac{1}{x} - \beta(x), \quad x > 0. \tag{5}$$

Its derivatives are given as [11]

$$\beta^{(m)}(x) = \frac{1}{2^{m+1}}\left\{\psi^{(m)}\left(\frac{x+1}{2}\right) - \psi^{(m)}\left(\frac{x}{2}\right)\right\}, \quad x > 0, \tag{6}$$

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$$= \int_0^1 \frac{(\ln t)^m t^{x-1}}{1+t}\,dt, \quad x>0, \tag{7}$$

$$= (-1)^m \int_0^\infty \frac{t^m e^{-xt}}{1+e^{-t}}\,dt, \quad x>0, \tag{8}$$

where $m \in \mathbb{N}_0$ and $\beta^{(0)}(x) = \beta(x)$. For further information about this special function, one could refer to [1, 2, 4, 9, 11–13] and the recent paper [14], where a p-generalization of the function was given. As shown in [1] and [4], the Nielsen's β-function is very useful in evaluating and estimating certain integrals and mathematical constants. Motivated by Jackson's q-integral and the q-digamma function, our objective in this paper is to introduce a q-analogue of the Nielsen's β-function and further study some of its properties. We begin with the following auxiliary definitions.

2. Preliminary Definitions

The q-derivative of a function $f(x)$ is defined as [7]

$$D_q f(x) = \frac{d_q f(x)}{d_q x} = \frac{f(x)-f(qx)}{(1-q)x}, \quad x \neq 0, q \in (0,1).$$

If $f'(0)$ exist, then $D_q f(0) = f'(0)$. Also, $D_q f(x) \to f'(x)$ as $q \to 1^-$. The q-derivative satisfies the algebraic properties

$$D_q\left(af(x)+bg(x)\right) = aD_q f(x) + bD_q g(x),$$

$$D_q\left(f(x)g(x)\right) = g(x)D_q f(x) + f(qx)D_q g(x) = f(x)D_q g(x) + g(qx)D_q f(x),$$

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(qx)}.$$

The Jackson's q-integral from 0 to a, 0 to ∞ and a to ∞ are respectively defined as [7]

$$\int_0^a f(t)\,d_q t = (1-q)a\sum_{k=0}^{\infty} f(aq^k)q^k, \tag{9}$$

$$\int_0^\infty f(t)\,d_q t = (1-q)\sum_{k=-\infty}^{\infty} f(q^k)q^k, \tag{10}$$

and

$$\int_a^\infty f(t)\,d_q t = (1-q)\sum_{k=1}^{\infty} f(q^{-k})q^{-k}, \tag{11}$$

provided that the sums in (9), (10) and (11) converge absolutely. In a generic interval $[a,b]$, the q-integral takes the form

$$\int_a^b f(t)\,d_q t = \int_0^b f(t)\,d_q t - \int_0^a f(t)\,d_q t.$$

A function $F(t)$ is called a q-antiderivative of $f(t)$ if $D_q F(t) = f(t)$. That is, if

$$F(t) = \int f(t)\,d_q t + C,$$

where C is a constant of integration. The q-analogue of the Gamma function is defined as [6, 7]

$$\Gamma_q(x) = (1-q)^{1-x}\prod_{n=0}^{\infty}\frac{1-q^{n+1}}{1-q^{n+x}},$$

where $q \in (0,1)$ and $x > 0$. Based on this, the q-analogue of the beta function is given by (see also [3])

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}, \quad x,y>0.$$

The q-digamma function, $\psi_q(x)$ and the q-polygamma functions, $\psi_q^{(m)}(x)$ are defined as [5, 16]

$$\psi_q(x) = \frac{d}{dx}\ln\Gamma_q(x) = -\ln(1-q) + \frac{\ln q}{1-q}\int_0^q \frac{t^{x-1}}{1-t}\,d_qt \tag{12}$$

$$= -\ln(1-q) + \ln q \sum_{n=0}^{\infty} \frac{q^{n+x}}{1-q^{n+x}}$$

$$= -\ln(1-q) + \ln q \sum_{n=1}^{\infty} \frac{q^{nx}}{1-q^{n}}, \tag{13}$$

$$\psi_q^{(m)}(x) = \frac{d^m}{dx^m}\psi_q(x) = (\ln q)^{m+1}\sum_{n=1}^{\infty}\frac{n^m q^{nx}}{1-q^n}, \quad m \in \mathbb{N}_0. \tag{14}$$

Definition 2.1. *A function $f: I \to \mathbb{R}$ is said to be convex if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y),$$

holds for all $x,y \in I$ and $\lambda \in [0,1]$. If f is twice differentiable, then it is said to be convex if and only if $f''(x) \geq 0$ for every $x \in I$.

Definition 2.2. *A function $f: I \to \mathbb{R}^+$ is said to be logarithmically convex or in short log-convex if $\ln f$ is convex on I. That is if*

$$\ln f(\lambda x + (1-\lambda)y) \leq \lambda \ln f(x) + (1-\lambda)\ln f(y)$$

or equivalently

$$f(\lambda x + (1-\lambda)y) \leq (f(x))^{\lambda}(f(y))^{1-\lambda}$$

for each $x,y \in I$ and $\lambda \in [0,1]$.

Definition 2.3. *A function $f: I \to \mathbb{R}$ is said to be completely monotonic on I if f has derivatives of all order on I and*

$$(-1)^k f^{(k)}(x) \geq 0$$

for $x \in I$ and $k \in \mathbb{N}$ [17].

3. Main Results

In this section, we introduce a q-analogue of the Nielsen's β-function and further study some properties and inequalities involving the new function.

Proposition 3.1. *The q-analogue of the Nielsen's β-function is given by the following equivalent definitions.*

$$\beta_q(x) = \frac{1}{[2]_q}\left\{\psi_q\left(\frac{x+1}{2}\right) - \psi_q\left(\frac{x}{2}\right)\right\}, \tag{15}$$

$$= -\frac{\ln q}{1-q}\int_0^{\sqrt{q}} \frac{t^{x-1}}{1+t}\,d_qt, \tag{16}$$

$$= -\frac{\ln q}{1-q}\int_{-\ln\sqrt{q}}^{\infty} \frac{e^{-xt}}{1+e^{-t}}\left(\frac{e^{t-qt}-1}{t-qt}\right)d_qt, \tag{17}$$

where $x > 0$, $q \in (0,1)$ and $[x]_q = \frac{1-q^x}{1-q}$. It follows that, $\beta_q(x) \to \beta(x)$ as $q \to 1$.

Proof. By using relation (12), we obtain

$$\begin{aligned}\psi_q\left(\frac{x+1}{2}\right)-\psi_q\left(\frac{x}{2}\right)&=\frac{\ln q}{1-q}\left\{\int_0^q\frac{t^{\frac{x}{2}-\frac{1}{2}}}{1-t}\,d_qt-\int_0^q\frac{t^{\frac{x}{2}-1}}{1-t}\,d_qt\right\}\\&=\frac{\ln q}{1-q}\int_0^q\frac{t^{\frac{x}{2}}}{1-t}\left(\frac{1}{\sqrt{t}}-\frac{1}{t}\right)d_qt\\&=-\frac{\ln q}{1-q}\int_0^q\frac{t^{\frac{x}{2}-1}}{1-t}(1-\sqrt{t})\,d_qt\\&=-\frac{\ln q}{1-q}\int_0^q\frac{t^{\frac{x}{2}-1}}{1+\sqrt{t}}\,d_qt\\&=-[2]_q\frac{\ln q}{1-q}\int_0^{\sqrt{q}}\frac{u^{x-1}}{1+u}\,d_qu.\end{aligned}$$

Hence (15) implies (16). In this proof, we made the substitution $t=u^2$ which implies that, $d_qt=(1+q)ud_qu=[2]_qud_qu$. Similarly, by replacing t by e^{-t} in (16), we obtain (17). □

Proposition 3.2. *The function $\beta_q(x)$ satisfies the following series representations.*

$$\beta_q(x)=-\frac{\ln q}{[2]_q}\sum_{n=1}^{\infty}\frac{q^{\frac{nx}{2}}}{1+q^{\frac{n}{2}}},\tag{18}$$

$$=-\ln q\sum_{k=0}^{\infty}\frac{q^{(k+\frac{1}{2})x}}{1+q^{k+\frac{1}{2}}}.\tag{19}$$

Proof. By using (15) with (13), we obtain

$$\begin{aligned}\beta_q(x)=\frac{1}{[2]_q}\left\{\psi_q\left(\frac{x+1}{2}\right)-\psi_q\left(\frac{x}{2}\right)\right\}&=\frac{\ln q}{[2]_q}\sum_{n=1}^{\infty}\left[\frac{q^{\frac{nx}{2}+\frac{n}{2}}}{1-q^n}-\frac{q^{\frac{nx}{2}}}{1-q^n}\right]\\&=\frac{\ln q}{[2]_q}\sum_{n=1}^{\infty}\frac{q^{\frac{nx}{2}}\left(q^{\frac{n}{2}}-1\right)}{1-q^n}\\&=-\frac{\ln q}{[2]_q}\sum_{n=1}^{\infty}\frac{q^{\frac{nx}{2}}\left(1-q^{\frac{n}{2}}\right)}{1-q^n}\\&=-\frac{\ln q}{[2]_q}\sum_{n=1}^{\infty}\frac{q^{\frac{nx}{2}}\left(1-q^{\frac{n}{2}}\right)}{(1-q^{\frac{n}{2}})(1+q^{\frac{n}{2}})}\\&=-\frac{\ln q}{[2]_q}\sum_{n=1}^{\infty}\frac{q^{\frac{nx}{2}}}{1+q^{\frac{n}{2}}},\end{aligned}$$

which gives (18). Also, by using (16) with (9), we obtain

$$\begin{aligned}\beta_q(x)&=-\frac{\ln q}{1-q}\int_0^{\sqrt{q}}\frac{t^{x-1}}{1+t}\,d_qt\\&=-\frac{\ln q}{1-q}(1-q)q^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{\left(q^{k+\frac{1}{2}}\right)^{x-1}}{1+q^{k+\frac{1}{2}}}q^k\\&=-\ln q\sum_{k=0}^{\infty}\frac{q^{(k+\frac{1}{2})x}}{1+q^{k+\frac{1}{2}}},\end{aligned}$$

which gives (19). □

Proposition 3.3. *The function $\beta_q(x)$ is connected to the q-beta function, $B_q(x,y)$ by the identity*

$$\beta_q(x)=-\frac{2}{[2]_q}\frac{d}{dx}\left\{\ln B_q\left(\frac{x}{2},\frac{1}{2}\right)\right\}.\tag{20}$$

Proof. We have

$$-\frac{2}{[2]_q}\frac{d}{dx}\left\{\ln B_q\left(\frac{x}{2},\frac{1}{2}\right)\right\} = -\frac{2}{[2]_q}\left\{\frac{1}{2}\frac{\Gamma_q'\left(\frac{x}{2}\right)}{\Gamma_q\left(\frac{x}{2}\right)} - \frac{1}{2}\frac{\Gamma_q'\left(\frac{x+1}{2}\right)}{\Gamma_q\left(\frac{x+1}{2}\right)}\right\}$$
$$= \frac{1}{[2]_q}\left\{\psi_q\left(\frac{x+1}{2}\right) - \psi_q\left(\frac{x}{2}\right)\right\}$$
$$= \beta_q(x)$$

which gives the desired result. □

Proposition 3.4. *The function $\beta_q(x)$ satisfies the functional equation*

$$\beta_q(x+1) = -\frac{(\ln q)q^{\frac{x}{2}}}{1-q^x} - \beta_q(x). \tag{21}$$

Proof. By (16) and (9), we obtain

$$\beta_q(x+1) + \beta_q(x) = -\frac{\ln q}{1-q}\int_0^{\sqrt{q}} \frac{t^x + t^{x-1}}{1+t}\, d_q t$$
$$= -\frac{\ln q}{1-q}\int_0^{\sqrt{q}} t^{x-1}\, d_q t$$
$$= -\frac{\ln q}{1-q}\left\{(1-q)q^{\frac{x}{2}}\sum_{n=0}^{\infty} q^{nx}\right\}$$
$$= -\frac{(\ln q)q^{\frac{x}{2}}}{1-q^x}.$$

Notice that, since $q \in (0,1)$, then $q^x < 1$ for $x > 0$. Hence $\sum_{n=0}^{\infty} q^{nx} = \frac{1}{1-q^x}$. □

Remark 3.5. *By repeated applications of* (21)*, we obtain the generalized form*

$$\beta_q(x+n) = (\ln q)\sum_{k=0}^{n-1} \frac{(-1)^{n+k} q^{\frac{x+k}{2}}}{1-q^{x+k}} + (-1)^n \beta_q(x), \tag{22}$$

where $n \in \mathbb{N}$.

Also, by repeated differentiations of (15), (16) and (17), we obtain respectively

$$\beta_q^{(m)}(x) = \frac{1}{[2]_q 2^m}\left\{\psi_q^{(m)}\left(\frac{x+1}{2}\right) - \psi_q^{(m)}\left(\frac{x}{2}\right)\right\}, \tag{23}$$
$$= -\frac{\ln q}{1-q}\int_0^{\sqrt{q}} \frac{(\ln t)^m t^{x-1}}{1+t}\, d_q t, \tag{24}$$
$$= (-1)^{m+1}\frac{\ln q}{1-q}\int_{-\ln\sqrt{q}}^{\infty} \frac{t^m e^{-xt}}{1+e^{-t}}\left(\frac{e^{t-qt}-1}{t-qt}\right) d_q t, \tag{25}$$

where $m \in \mathbb{N}_0$, $\beta_q^{(0)}(x) = \beta_q(x)$ and $\beta_q^{(m)}(x) \to \beta^{(m)}(x)$ as $q \to 1$. It can easily be deduced from above that:

(a). $\beta_q(x)$ is positive and decreasing,

(b). $\beta_q^{(m)}(x)$ is positive and decreasing if $m \in \mathbb{N}_0$ is even,

(c). $\beta_q^{(m)}(x)$ is negative and increasing if $m \in \mathbb{N}_0$ is odd.

Remark 3.6. *Furthermore, differentiating m-times of (21) implies*

$$\beta_q^{(m)}(x+1) = -\frac{d^m}{dx^m}\left\{\frac{(\ln q)q^{\frac{x}{2}}}{1-q^x}\right\} - \beta_q^{(m)}(x), \quad m \in \mathbb{N}_0, \tag{26}$$

and in particular,

$$\beta_q'(x+1) = -\frac{(\ln q)^2 q^{\frac{x}{2}}}{(1-q^x)^2}(1+q^x) - \beta_q'(x). \tag{27}$$

As an immediate consequence of (21) and (27), we obtain the bounds

$$\beta_q(x) < -\frac{(\ln q)q^{\frac{x}{2}}}{1-q^x}, \tag{28}$$

$$-\frac{(\ln q)^2 q^{\frac{x}{2}}}{2(1-q^x)^2}(1+q^x) < \beta_q'(x). \tag{29}$$

Theorem 3.7. *The function $\beta_q(x)$ is logarithmically convex on $(0,\infty)$.*

Proof. Let $x, y \in (0,\infty)$, $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$. Then by (16) and the q-Hölder's inequality (see [8], [10]), we obtain

$$\begin{aligned}
\beta_q\left(\frac{x}{a}+\frac{y}{b}\right) &= -\frac{\ln q}{1-q}\int_0^{\sqrt{q}} \frac{t^{\frac{x}{a}+\frac{y}{b}-1}}{1+t}\, d_qt \\
&= -\frac{\ln q}{1-q}\int_0^{\sqrt{q}} \frac{t^{\frac{x-1}{a}}}{(1+t)^{\frac{1}{a}}}\frac{t^{\frac{y-1}{b}}}{(1+t)^{\frac{1}{b}}}\, d_qt \\
&\le -\frac{\ln q}{1-q}\left(\int_0^{\sqrt{q}} \frac{t^{x-1}}{1+t}\, d_qt\right)^{\frac{1}{a}}\left(\int_0^{\sqrt{q}} \frac{t^{y-1}}{1+t}\, d_qt\right)^{\frac{1}{b}} \\
&= \left(-\frac{\ln q}{1-q}\int_0^{\sqrt{q}} \frac{t^{x-1}}{1+t}\, d_qt\right)^{\frac{1}{a}}\left(-\frac{\ln q}{1-q}\int_0^{\sqrt{q}} \frac{t^{y-1}}{1+t}\, d_qt\right)^{\frac{1}{b}} \\
&= [\beta_q(x)]^{\frac{1}{a}}\,[\beta_q(y)]^{\frac{1}{b}},
\end{aligned}$$

which completes the proof. □

Remark 3.8. *Log-convexity of $\beta_q(x)$ implies that:*

(a). $\beta_q(x)\beta_q''(x) - \left(\beta_q'(x)\right)^2 \ge 0$, for $x \in (0,\infty)$,

(b). the function $\frac{\beta_q'(x)}{\beta_q(x)}$ is increasing on $(0,\infty)$.

Theorem 3.9. *For $x, y \in (0,\infty)$, the function $\beta_q(x)$ satisfies the inequality*

$$\beta_q(x+1)\beta_q(y+1) \le M_q \cdot \beta_q(x+y+1), \tag{30}$$

where $M_q = -\frac{\ln q}{[2]_q}\sum_{n=1}^{\infty}\frac{q^{\frac{n}{2}}}{1+q^{\frac{n}{2}}}$.

Proof. Let f be defined for $x, y \in (0,\infty)$ by

$$f(x,y) = \frac{\beta_q(x+1)\beta_q(y+1)}{\beta_q(x+y+1)},$$

and

$$g(x,y) = \ln f(x,y) = \ln\beta_q(x+1) + \ln\beta_q(y+1) - \ln\beta_q(x+y+1).$$

Then for a fixed y, we obtain

$$g'(x,y) = \frac{\beta_q'(x+1)}{\beta_q(x+1)} - \frac{\beta_q'(x+y+1)}{\beta_q(x+y+1)} \le 0,$$

since $\frac{\beta_q'(x)}{\beta_q(x)}$ is increasing. Thus, $g(x,y)$ as well as $f(x,y)$ are decreasing. Then for $x>0$, we obtain $f(x,y)\leq f(0,y)$ yielding

$$\frac{\beta_q(x+1)\beta_q(y+1)}{\beta_q(x+y+1)} \leq \beta_q(1) = -\frac{\ln q}{[2]_q}\sum_{n=1}^{\infty}\frac{q^{\frac{n}{2}}}{1+q^{\frac{n}{2}}} = M_q$$

which completes the proof. □

Theorem 3.10. *The inequality*

$$\beta_q(x)\beta_q(x+y+z) - \beta_q(x+y)\beta_q(x+z) \geq 0, \tag{31}$$

holds for $x,y,z \in (0,\infty)$.

Proof. Let $x,y,z\in(0,\infty)$. Then it is enough to show that the function

$$f(x) = \frac{\beta_q(x+z)}{\beta_q(x)},$$

is increasing on $(0,\infty)$. Let $u(x)=\ln f(x)$. Then

$$u'(x) = \frac{\beta_q'(x+z)}{\beta_q(x+z)} - \frac{\beta_q'(x)}{\beta_q(x)} \geq 0.$$

Thus, $f(x)$ is increasing. Hence $f(x+y)\geq f(x)$ which gives the result (31). □

Theorem 3.11. *The function $\beta_q(x)$ is completely monotonic on $(0,\infty)$.*

Proof. It is easily deduced from (14) that, $\psi_q^{(m)}(x)$ is increasing if m is even and decreasing if m is odd. Then it follows from (23) that

$$(-1)^m\beta_q^{(m)}(x) = \frac{(-1)^m}{[2]_q 2^m}\left\{\psi_q^{(m)}\left(\frac{x+1}{2}\right) - \psi_q^{(m)}\left(\frac{x}{2}\right)\right\} \geq 0,$$

for all $m\in\mathbb{N}$. This completes the proof. □

Theorem 3.12. *The function $\beta_q^{(m)}(x)$ is subadditive on $(0,\infty)$ if m is even, and superadditive on $(0,\infty)$ if m is odd. That is, for $x,y\in(0,\infty)$,*

$$\beta_q^{(m)}(x+y) \leq \beta_q^{(m)}(x) + \beta_q^{(m)}(y), \quad \textit{if } m \textit{ is even}, \tag{32}$$

and

$$\beta_q^{(m)}(x+y) \geq \beta_q^{(m)}(x) + \beta_q^{(m)}(y), \quad \textit{if } m \textit{ is odd}. \tag{33}$$

Proof. Suppose that m is even and let $G_q(x,y)=\beta_q^{(m)}(x+y)-\beta_q^{(m)}(x)-\beta_q^{(m)}(y)$ for $x,y\in(0,\infty)$. Without any loss of generality, let y be fixed. Then,

$$\begin{aligned} G_q'(x,y) &= \beta_q^{(m+1)}(x+y) - \beta_q^{(m+1)}(x) \\ &\geq 0, \end{aligned}$$

since $\beta_q^{(n)}(x)$ is increasing for odd n. Hence, $G_q(x,y)$ is increasing. Furthermore,

$$\begin{aligned} \lim_{x\to\infty} G_q(x,y) &= \lim_{x\to\infty}\left\{\beta_q^{(m)}(x+y) - \beta_q^{(m)}(x) - \beta_q^{(m)}(y)\right\} \\ &= -\beta_q^{(m)}(y) \\ &\leq 0. \end{aligned}$$

Therefore, $G_q(x,y)\leq \lim_{x\to\infty}G_q(x,y)\leq 0$ which gives the result (32). Likewise, if m is odd, we obtain $G_q'(x,y)\leq 0$ and $G_q(x,y)\geq \lim_{x\to\infty}G_q(x,y)\geq 0$ which gives the result (33). □

Theorem 3.13. *For $x \in (0,\infty)$ and odd m, the function $\beta_q^{(m)}(x)$ satisfies the following inequalities.*

$$\beta_q^{(m)}(\lambda x) \leq \lambda \beta_q^{(m)}(x), \quad \text{if} \quad 0 < \lambda \leq 1, \tag{34}$$

and

$$\beta_q^{(m)}(\lambda x) \geq \lambda \beta_q^{(m)}(x), \quad \text{if} \quad \lambda \geq 1. \tag{35}$$

Proof. Let $A_q(x) = \beta_q^{(m)}(\lambda x) - \lambda \beta_q^{(m)}(x)$ for odd m, $x \in (0,\infty)$ and $0 < \lambda \leq 1$. Then,

$$\begin{aligned} A_q'(x) &= \lambda \left\{ \beta_q^{(m+1)}(\lambda x) - \beta_q^{(m+1)}(x) \right\} \\ &\geq 0, \end{aligned}$$

since $\beta_q^{(n)}(x)$ is decreasing for even n. Thus, $A_q(x)$ is increasing. Moreover,

$$\begin{aligned} \lim_{x\to\infty} A_q(x) &= \lim_{x\to\infty} \left[\beta_q^{(m)}(\lambda x) - \lambda \beta_q^{(m)}(x) \right] \\ &= 0. \end{aligned}$$

Therefore, $A_q(x) \leq \lim_{x\to\infty} A_q(x) = 0$ which gives the result (34). Similarly, if $\lambda \geq 1$, we obtain $A_q'(x) \leq 0$ and $A_q(x) \geq \lim_{x\to\infty} A_q(x) = 0$ yielding the result (35). □

Remark 3.14. *Inequality (34) is another way of saying that the function $\beta_q^{(m)}(x)$ is star-shaped if m is odd. Moreover, if m is even, then the inequalities (34) and (35) are reversed.*

Theorem 3.15. *Let $m, k \in \mathbb{N}_0$ such that m and k are even and $m \geq k$. Then the Turan-type inequality*

$$\exp\left\{\beta_q^{(m-k)}(x)\right\} \cdot \exp\left\{\beta_q^{(m+k)}(x)\right\} \geq \left[\exp\left\{\beta_q^{(m)}(x)\right\}\right]^2 \tag{36}$$

holds for $x \in (0,\infty)$.

Proof. By applying (24), we obtain

$$\begin{aligned} \frac{\beta_q^{(m-k)}(x)}{2} + \frac{\beta_q^{(m+k)}(x)}{2} - \beta_q^{(m)}(x) &= -\frac{\ln q}{1-q} \int_0^{\sqrt{q}} \left[\frac{1}{2} \frac{(\ln t)^{m-k} t^{x-1}}{1+t} + \frac{1}{2} \frac{(\ln t)^{m+k} t^{x-1}}{1+t} - \frac{(\ln t)^m t^{x-1}}{1+t} \right] d_q t \\ &= -\frac{\ln q}{2(1-q)} \int_0^{\sqrt{q}} \frac{(\ln t)^{m-k} t^{x-1}}{1+t} \left[1 + (\ln t)^{2k} - 2(\ln t)^k \right] d_q t \\ &= -\frac{\ln q}{2(1-q)} \int_0^{\sqrt{q}} \frac{(\ln t)^{m-k} t^{x-1}}{1+t} \left[1 - (\ln t)^k \right]^2 d_q t \\ &\geq 0. \end{aligned}$$

Thus,

$$\frac{\beta_q^{(m-k)}(x)}{2} + \frac{\beta_q^{(m+k)}(x)}{2} \geq \beta_q^{(m)}(x),$$

and by taking exponents, we obtain the inequality (36). □

4. Concluding Remarks

We have introduced a q-analogue of the classical Nielsen's β-function and further studied some identities, monotonicity and convexity properties, and some inequalities involving the new function. By letting $q \to 1$ in the present results, we obtain the corresponding results for the Nielsen's β-function.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] K. N. Boyadzhiev, L. A. Medina, and V. H. Moll, *The integrals in Gradshteyn and Ryzhik. Part 11: The incomplete beta function*, Scientia, Ser. A, Math. Sci., 18(2009), 61-75.

[2] D. F. Connon, *On an integral involving the digamma function*, arXiv:1212.1432 [math.GM].

[3] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics and Its Applications, Cambridge University Press, (1990).

[4] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Edited by D. Zwillinger and V. Moll. Academic Press, New York, 8th Edition, (2014).

[5] M. E. H. Ismail and M. E. Muldoon, *Inequalities and monotonicity properties for gamma and q-gamma functions*, arXiv:1301.1749v1 [math.CA].

[6] F. H. Jackson, *A generalization of the functions* $\Gamma(n)$ *and* x^n, Proc. Roy. Soc. London., 74 (1904), 64-72.

[7] F. H. Jackson, *On a q-definite integrals*, Quart. J. Pure and Applied Math., (41)(1910), 193-203.

[8] V. Krasniqi, T. Mansour and A. Sh. Shabani, *Some inequalities for q-polygamma function and* ζ_q*-Riemann zeta functions*, Ann. Math. Inform., 37(2010), 95-100.

[9] L. Medina and V. Moll, *The integrals in Gradshteyn and Ryzhik. Part 10: The digamma function*, Scientia, Ser. A, Math. Sci., 17(2009), 45-66.

[10] K. Nantomah, *Generalized Hölders and Minkowskis Inequalities for Jacksons q-Integral and Some Applications to the Incomplete q-Gamma Function*, Abstr. Appl. Anal., 2017(2017), Art ID: 9796873.

[11] K. Nantomah, *On Some Properties and Inequalities for the Nielsen's β-Function*, arXiv:1708.06604v1 [math.CA], 12 pages.

[12] K. Nantomah, *Monotonicity and Convexity Properties of the Nielsen's β-Function*, Probl. Anal. Issues Anal., 6(24)(2)(2017), 81-93.

[13] K. Nantomah, *Monotonicity and convexity properties and some inequalities involving a generalized form of the Wallis' cosine formula*, Asian Research Journal of Mathematics, 6(3)(2017), 1-10.

[14] K. Nantomah, *A generalization of the Nielsen's β-function*, Int. J. Open Problems Compt. Math., 11(2)(2018), 16-26.

[15] N. Nielsen, *Handbuch der Theorie der Gammafunktion*, First Edition, Leipzig : B. G. Teubner, (1906).

[16] F. Qi, *A completely monotonic function related to the q-trigamma function*, U.P.B. Sci. Bull., Series A, 76(1)(2014), 107-114.

[17] D. V. Widder, *The Laplace Transform*, Princeton Mathematical Series 6, Princeton University Press, (1946).