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Binary Linear Topological Spaces

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Abstract: In this paper we define and study the concept of binary linear topological spaces (BLTS) and their properties. Here we prove that the binary product of two linear topological spaces is a BLTS. Also we have the main result that the binary product preserve metrizability and normability. Finally we construct a BLTS from a family of binary seminorms on a pair of vector spaces.

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1. Introduction

P. Thangavelu and Nithanantha Jothi introduced the concept of binary topology in [4]. It is a single topological structure that carries the subsets of a set X as well as the subsets of another set Y for studying the information about the ordered pair (A, B) of subsets of X and Y. A linear topological space is a linear space endowed with a topology such that the vector addition and scalar multiplication are both continuous. The theory of linear topological spaces provide a remarkable economy in discussion of many classical mathematical problems. We introduce the concept of binary topology to linear topological spaces and form the theory of binary linear topology. Section 2 contains the prerequisites for the paper. In section 3 we define the concept of binary linear topological spaces (BLTS). We prove that the binary product of two linear topological spaces is a BLTS. Also we discuss the concept of locally convex BLTS and locally bounded BLTS and prove some of their properties. In section 4 we define binary metric and binary norm. The main result of this section is that the binary product preserve metrizability and normability. Section 5 deals with the construction of a BLTS using a family of binary seminorms.

2. Preliminaries

Definition 2.1 ([4])**.** *Let X and Y be any two non-empty sets and $\wp(X)$ and $\wp(Y)$ be their power sets respectively. A binary topology from X to Y is a binary structure $M \subseteq \wp(X) \times \wp(Y)$ that satisfies the following axioms.*

(1). (ϕ, ϕ) and $(X, Y) \in M$

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(2). If (A_1, B_1) and $(A_2, B_2) \in M$, then $(A_1 \cap A_2, B_1 \cap B_2) \in M$.

(3). If $\{(A_\alpha, B_\alpha) : \alpha \in \Delta\}$ is a family of members of M, then $(\cup_{\alpha\in\Delta} A_\alpha, \cup_{\alpha\in\Delta} B_\alpha) \in M$.

If M is a binary topology from X to Y then the triplet (X, Y, M) is called a binary topological space and the members of M are called binary open sets. (C, D) is called binary closed if $(X \setminus C, Y \setminus D)$ is binary open. The elements of $X \times Y$ are called the binary points of the binary topological space (X, Y, M). Let (X, Y, M) be a binary topological space and let $(x, y) \in X \times Y$. The binary open set (A, B) is called a binary neighbourhood of (x, y) if $x \in A$ and $y \in B$. If $X = Y$ then M is called a binary topology on X and we write (X, M) as a binary space.

Proposition 2.2 ([4])**.** *Let (X, Y, M) be a binary topological space. Then*

(1). $\tau(M) = \{A \subseteq X : (A, B) \in M$ for some $B \subseteq Y\}$ is a topology on X.

(2). $\tau'(M) = \{B \subseteq Y : (A, B) \in M$ for some $A \subseteq X\}$ is a topology on Y.

Proposition 2.3 ([4])**.** *Suppose (X, ρ) and (Y, σ) are two topological spaces. Then $\rho \times \sigma$ is a binary topology from X to Y such that $\tau(\rho \times \sigma) = \rho$ and $\tau'(\rho \times \sigma) = \sigma$.*

Definition 2.4 ([5])**.** *A linear topological space is a linear space E with a topology such that addition and scalar multiplication are both continuous. That is for every elements $x, y \in E$ and for every neighbourhood V of $x + y$ there exists neighbourhoods V_1 of x and V_2 of y such that $V_1 + V_2 \subseteq V$ and also for every neighbourhood W of λx there exists neighbourhoods K of λ and U of x such that $KU \subseteq W$. A base for the neighbourhood system of 0 in E is called a local base.*

Throughout this paper we consider vector spaces over the same field K.

Definition 2.5 ([1])**.** *Let $\{\rho_\alpha\}_{\alpha\in J}$ be a family of seminorms on a vector space X. Then the αth open strip of radius r centered at $x \in X$ is $B_r^\alpha(x) = \{y \in X : \rho_\alpha(x - y) < r\}$. Let ε be the collection of all open strips in X : $\varepsilon = \{B_r^\alpha(x) : \alpha \in J, r > 0, x \in X\}$. The topology $\tau(\varepsilon)$ generated by ε is called the topology induced by $\{\rho_\alpha\}_{\alpha\in J}$.*

Proposition 2.6 ([1])**.** *Let $\{\rho_\alpha\}_{\alpha\in J}$ be a family of seminorms on a vector space X. Then $\mathcal{B} = \{\cap_{j=1}^n B_r^{\alpha_j}(x) : n \in \mathbb{N}, \alpha_j \in J, r > 0, x \in X\}$ forms a base for the topology induced from these seminorms. In fact if U is open and $x \in U$, then there exists an $r > 0$ and $\alpha_1, ..., \alpha_n \in J$ such that $\cap_{j=1}^n B_r^{\alpha_j}(x) \subseteq U$. Further every element of $\mathcal{B}$ is convex.*

Theorem 2.7 ([1])**.** *If X is a vector space whose topology is induced from a family of seminorms $\{\rho_\alpha\}_{\alpha\in J}$, then X is a locally convex topological vector space.*

3. Binary Linear Topology

Definition 3.1. *A binary topology between two vector spaces is said to be binary linear if the two operations are continuous i.e. if V_1 and V_2 are two vector spaces over the same field K and for every neighbourhoods U of $(x_1 + x_2, y_1 + y_2) \in V_1 \times V_2$, $\exists$ two neighbourhoods U_1 and U_2 of (x_1, y_1) and (x_2, y_2) respectively such that $U_1 + U_2 \subseteq U$. Similarly for every neighbourhood W of $(\lambda x, \lambda y) \in V_1 \times V_2$ there exists a neighbourhood W' of (x, y) such that $\lambda W' \subseteq W$. If M is a binary linear topology between two vector spaces V_1 and V_2, then the triplet (V_1, V_2, M) is called a binary linear topological space (BLTS).*

Definition 3.2. *Suppose (X_1, τ_1) and (X_2, τ_2) are two linear topological spaces. Then $(X_1, X_2, \tau_1 \times \tau_2)$ is called the binary product of the given spaces.*

Proposition 3.3. *If (V_1, τ_1) and (V_2, τ_2) are two linear topological spaces, then $(V_1, V_2, \tau_1 \times \tau_2)$ is a binary linear topological space.*

Proof. By proposition 2.3, $(V_1, V_2, \tau_1 \times \tau_2)$ is a binary topological space. It remains to show that $\tau_1 \times \tau_2$ is a binary linear topology. Let $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ and (A_1, A_2) be a neighbourhood of $[(x_1, x_2) + (y_1, y_2)]$. Then $x_1 + y_1 \in A_1$ and $x_2 + y_2 \in A_2$. Since $A_1 \in \tau_1$ and $A_2 \in \tau_2$, and τ_1 and τ_2 are linear topologies, there exist neighbourhoods B_1 and C_1 of x_1 and y_1 respectively in τ_1 such that $B_1 + C_1 \subseteq A_1$ and neighbourhoods B_2 and C_2 of x_2 and y_2 respectively in τ_2 such that $B_2 + C_2 \subseteq A_2$. Then in $\tau_1 \times \tau_2, (B_1, B_2)$ is a neighbourhood of (x_1, x_2) and (C_1, C_2) is a neighbourhood of (y_1, y_2) such that $(B_1, B_2) + (C_1, C_2) = (B_1 + C_1, B_2 + C_2) \subseteq (A_1, A_2)$. Now let (A_1, A_2) be a neighbourhood of $\lambda(x_1, x_2)$ in $\tau_1 \times \tau_2$. Then A_1 is a neighbourhood of λx_1 in τ_1 and A_2 is a neighbourhood of λx_2 in τ_2. So there exists two neighbourhoods B_1 and B_2 of x_1 and x_2 respectively such that $\lambda B_1 \subseteq A_1$ and $\lambda B_2 \subseteq A_2$. This implies that (B_1, B_2) is a neighbourhood of (x_1, x_2) such that $\lambda(B_1, B_2) \subseteq (A_1, A_2)$. Thus $\tau_1 \times \tau_2$ is a binary linear topology. □

Proposition 3.4. *If (V_1, V_2, M) is a BLTS, then $\tau(M) = \{A \subseteq V_1 : (A, B) \in M$ for some $B \subseteq V_2\}$ is a linear topology on V_1 and $\tau'(M) = \{B \subseteq V_2 : (A, B) \in M$ for some $A \subseteq V_1\}$ is a linear topology on V_2.*

Proof. By Proposition 2.2 $\tau(M)$ and $\tau'(M)$ are both topologies in V_1 and V_2 respectively. Let $x_1, y_1 \in V_1$ and $A \in \tau(M)$ contains $x_1 + y_1$. Then for some $x_2, y_2 \in V_2$ there exists $B \subseteq V_2$ such that $(x_1 + y_1, x_2 + y_2) \in (A, B)$ where $(A, B) \in M$. Since M is a binary linear topology, there exists (E_1, E_2) and (F_1, F_2) in M such that $(x_1, x_2) \in (E_1, E_2), (y_1, y_2) \in (F_1, F_2)$ and $(E_1, E_2) + (F_1, F_2) \subseteq (A, B)$. Then $x_1 \in E_1, y_1 \in F_1$, and $E_1 + F_1 \subseteq A$ by the definition of binary sets. Also E_1 and $F_1 \in \tau(M)$ by the construction of $\tau(M)$. Similarly for $\lambda x \in A$, where $A \in \tau(M)$ we can find a neighbourhood of x say U such that $\lambda U \subseteq A$. Thus $\tau(M)$ is a linear topology. In the same way we can prove that $\tau'(M)$ is also a linear topology. □

Definition 3.5. *A local base of a binary linear topology (V_1, V_2, M) is the base consisting of the neighbourhood of a binary point (x, y).*

Definition 3.6. *A set $(A, B) \in \wp(V_1) \times \wp(V_2)$ is convex if for all pairs $(x_1, x_2), (y_1, y_2) \in (A, B), \lambda(x_1, x_2) + (1-\lambda)(y_1, y_2) \in (A, B), \forall \lambda \in [0, 1]$.*

Definition 3.7. *A binary linear topology is called locally convex if there exists a local base at $(0, 0)$ whose members are convex.*

Definition 3.8. *A BLTS is locally bounded if $(0, 0)$ has a bounded neighbourhood, i.e. a neighbourhood (E, F) such that $\forall (N, M) \in \mathcal{N}_0$, the set of neighbourhoods of $(0, 0)$, there exists $s \in \mathbb{R}$ such that $\forall t > s$, $(E, F) \subseteq t(N, M)$.*

Proposition 3.9. *Let (V_1, V_2, M) be a BLTS. Then for every $(W_1, W_2) \in \mathcal{N}_0$, $\exists$ balanced and symmetric sets $(X_1, Y_1), (X_2, Y_2) \in \mathcal{N}_0$ such that $(X_1, Y_1) + (X_2, Y_2) \subset (W_1, W_2)$.*

Proof. If $(W_1, W_2) \in \mathcal{N}_0$, then W_1 and W_2 are neighbourhoods of 0 in $(V_1, \tau(M))$ and $(V_2, \tau'(M))$ respectively. By the property of linear topologies there exists symmetric balanced neighbourhoods of $0, X_1, X_2 \in \tau(M)$ and $Y_1, Y_2 \in \tau'(M)$ such that $X_1 + X_2 \subset W_1$ and $Y_1 + Y_2 \subset W_2$. Now X_1, Y_1 are balanced $\Rightarrow \forall \alpha \in \mathbb{R}$ with $|\alpha| \leq 1$, $\alpha X_1 \subset X_1$ and $\alpha Y_1 \subset Y_1$. So $\alpha(X_1, Y_1) = (\alpha X_1, \alpha Y_1) \subset (X_1, Y_1)$. Thus (X_1, Y_1) and (X_2, Y_2) are balanced. By the symmetry of X_1 and Y_1, we get $X_1 = -X_1, Y_1 = -Y_1 \Rightarrow (X_1, Y_1) = (-X_1, -Y_1) = -(X_1, Y_1)$. Thus (X_1, Y_1) is symmetric and similarly (X_2, Y_2) is also symmetric. $(X_1, Y_1) + (X_2, Y_2) = (X_1 + X_2, Y_1 + Y_2) \subset (W_1, W_2)$. □

Proposition 3.10. *Let V_1 and V_2 be real vector spaces and U_1 be a convex set in V_1 and U_2 be a convex set in V_2, then (U_1, U_2) is convex in $\wp(V_1) \times \wp(V_2)$.*

Proof. Let $(x_i, y_i) \in (U_1, U_2)$ for $i = 1, 2$. Then $x_i \in U_1$ and $y_i \in U_2$ for $i = 1, 2 \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in U_1$ for $0 \leq \lambda \leq 1$. And $\lambda y_1 + (1-\lambda)y_2 \in U_2$ for $0 \leq \lambda \leq 1$. So $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in (U_1, U_2)$. Consider

$\lambda(x_1, y_1) + (1-\lambda)(x_2, y_2) = (\lambda x_1, \lambda y_1) + ((1-\lambda)x_2, (1-\lambda)y_2) = (\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \in (U_1, U_2)$ for $0 \leq \lambda \leq 1$. Thus (U_1, U_2) is convex. □

Corollary 3.11. *If (V_1, τ_1) and (V_2, τ_2) are both locally convex topological vector spaces, then their binary product, $(V_1, V_2, \tau_1 \times \tau_2)$ is a locally convex BLTS.*

Proposition 3.12. *Let U_1 and U_2 be bounded sets in two real vector spaces V_1 and V_2 respectively, then (U_1, U_2) is also bounded.*

Proof. Since U_1 is bounded, for every neighbourhood $E_1 \in \mathcal{N}_0(V_1)$, $\exists s_1 \in \mathbb{R}$ such that $\forall t > s_1, U_1 \subset tE_1$. Similarly for every neighbourhood $E_2 \in \mathcal{N}_0(V_2)$, $\exists s_2 \in \mathbb{R}$ such that $\forall t > s_2, U_2 \subset tE_2$. Let $(E, F) \in \mathcal{N}_0$. Then $E \in \mathcal{N}_0(V_1)$ and $F \in \mathcal{N}_0(V_2)$. Let $t_1 \in \mathbb{R}$ correspond to E and $t_2 \in \mathbb{R}$ correspond to F. Then $\forall t > t_1, U_1 \subset tE$ and $\forall t > t_2, U_2 \subset tF$. So $\forall t > s$, where $s = \max\{t_1, t_2\}, U_1 \subset tE$ and $U_2 \subset tF$ i.e. $(U_1, U_2) \subset t(E, F), \forall t > s$. Thus (U_1, U_2) is bounded. □

Corollary 3.13. *If (V_1, τ_1) and (V_2, τ_2) are both locally bounded topological vector spaces, then their binary product, $(V_1, V_2, \tau_1 \times \tau_2)$ is a locally bounded BLTS.*

Proposition 3.14. *Let (V_1, τ_1) be a topological vector space and V_2 be another vector space such that the map $T : V_1 \to V_2$ is an isomorphism. Then $\tau_2 = \{T(A) : A \in \tau_1\}$ is a linear topology in V_2 and hence $\tau_1 \times \tau_2$ is a binary linear topology from V_1 to V_2.*

Proof. Since T is an isomorphism, $T(\phi) = \phi$ and $T(V_1) = V_2$ and so $\phi, V_2 \in \tau_2$. Let $A, B \in \tau_2$. Then $A = T(A')$ and $B = T(B')$ for some A' and $B' \in \tau_1$. So $A' \cap B' \in \tau_1$ and $T(A' \cap B') \in \tau_2 . T(A' \cap B') = T(A') \cap T(B') = A \cap B$. Thus $A \cap B \in \tau_2$. Now let $\{A_\alpha\}_{\alpha \in I} \in \tau_2$ for some index set I. Then there exists $\{B_\alpha\}_{\alpha \in I} \in \tau_1$ such that $A_\alpha = T(B_\alpha)$ for each $\alpha \in I$. Then $\cup_{\alpha \in I} B_\alpha \in \tau_1$ and $\cup_{\alpha \in I} A_\alpha = \cup_{\alpha \in I} T(B_\alpha) = T(\cup_{\alpha \in I} B_\alpha) \in \tau_2$. Thus τ_2 is a topology on V_2. Let $x_2, y_2 \in V_2$ and there exists $B \in \tau_2$ such that $x_2 + y_2 \in B$. Then there exist $x_1, y_1 \in V_1$ such that $T(x_1) = x_2$ and $T(y_1) = y_2$. Let $A = T^{-1}(B) \in \tau_1$. So $x_1 + y_1 \in A$ and there exists $A_1, A_2 \in \tau_1$ such that $A_1 + A_2 \in A$. This implies $T(A_1 + A_2) \in T(A)$. Let $B_1 = T(A_1)$ and $B_2 = T(A_2)$. Then $B_1, B_2 \in \tau_2$ and $x_1 \in A_1 \Rightarrow x_2 = T(x_1) \in T(A_1) = B_1, y_1 \in A_2 \Rightarrow y_2 = T(y_1) \in T(A_2) = B_2$. Also $B_1 + B_2 = T(A_1) + T(A_2) = T(A_1 + A_2) \subseteq T(A) = B$. Let $y \in V_2$ and $\lambda y \in U \in \tau_2$ for some scalar λ. Then $y = T(x)$ for some $x \in V_1$ and $U = T(W)$ for some $W \in \tau_1$. $y = T(x) \Rightarrow \lambda y = \lambda T(x) = T(\lambda x)$. So $\lambda y \in U \Rightarrow T(\lambda x) \in U \Rightarrow \lambda x \in W$. Since τ_1 is a linear topology, there exists W' in τ_1 such that $\lambda W' \subseteq W$. So $U' = T(W') \in \tau_2, y = T(x) \in T(W') = U'$ and $T(\lambda W') = \lambda T(W') = \lambda U' \subseteq T(W) = U$. Thus τ_2 is a linear topology and hence $\tau_1 \times \tau_2$ is a binary linear topology. □

4. Binary Metrizable and Binary Normable BLTS

Definition 4.1. *A binary metric on two sets V_1 and V_2 is a map $d : (V_1 \times V_2) \times (V_1 \times V_2) \to \mathbb{R}$ satisfying the following axioms: If $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ then*

(1). $d[(x_1, x_2), (y_1, y_2)] \geq 0$

(2). $d[(x_1, x_2), (y_1, y_2)] = 0 \Leftrightarrow x_1 = x_2$ and $y_1 = y_2$

(3). $d[(x_1, x_2), (y_1, y_2)] = d[(y_1, y_2), (x_1, x_2)]$ and

(4). $d[(x_1, x_2), (y_1, y_2)] \leq d[(x_1, x_2), (z_1, z_2)] + d[(z_1, z_2), (y_1, y_2)]$ for every $(z_1, z_2) \in V_1 \times V_2$.

Definition 4.2. *Let (V_1, V_2, M) be a BLTS. A binary topology M is metrizable with a binary metric d if for any (x, y) in some binary open set $(A, B) \in M$, $\exists\ r > 0$ such that $B_r(x, y) \subset (A, B)$ i.e. $\pi_1(B_r(x, y)) \subset A$ and $\pi_2(B_r(x, y)) \subset B$, where π_i is the projection map to V_i for $i = 1, 2$.*

Proposition 4.3. *If (V_1, τ_1) and (V_2, τ_2) are two linear topological spaces such that τ_1 and τ_2 are both metrizable with metrics d_1 and d_2 respectively, then $\tau_1 \times \tau_2$ is binary metrizable.*

Proof. Consider the map $d : (V_1 \times V_2) \times (V_1 \times V_2) \to \mathbb{R}$ defined by

$$d((x_1, x_2), (y_1, y_2)) = \frac{d_1(x_1, y_1) + d_2(x_2, y_2)}{2}, \forall (x_1, x_2), (y_1, y_2) \in (V_1 \times V_2)$$

If $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ then

(1). $d[(x_1, x_2), (y_1, y_2)] = \frac{d_1(x_1,y_1)+d_2(x_2,y_2)}{2} \geq 0$, since $d_1(x_1, y_1)$ and $d_2(x_2, y_2)$ are both non-negative.

(2). $d[(x_1, x_2), (y_1, y_2)] = \frac{d_1(x_1,y_1)+d_2(x_2,y_2)}{2} = 0 \Leftrightarrow d_1(x_1, y_1) = 0$ and $d_2(x_2, y_2) = 0$. This happens if and only if $x_1 = x_2$ and $y_1 = y_2$ i.e. when $(x_1, y_1) = (x_2, y_2)$.

(3). $d((x_1, x_2), (y_1, y_2)) = \frac{d_1(x_1,y_1)+d_2(x_2,y_2)}{2} = \frac{d_1(y_1,x_1)+d_2(y_2,x_2)}{2} = d((y_1, y_2), (x_1, x_2))$ and if $(z_1, z_2) \in V_1 \times V_2$

(4). $d[(x_1, x_2), (y_1, y_2)] = \frac{d_1(x_1,y_1)+d_2(x_2,y_2)}{2} \leq \frac{[d_1(x_1,z_1)+d_1(z_1,y_1)]+[d_2(x_2,z_2)+d_2(z_2,y_2)]}{2} = \frac{d_1(x_1,z_1)+d_2(x_2,z_2)}{2} + \frac{d_1(z_1,y_1)+d_2(z_2,y_2)}{2} = d[(x_1, x_2), (z_1, z_2)] + d[(z_1, z_2), (y_1, y_2)]$

Thus d is a binary metric. Let $(A, B) \in \tau_1 \times \tau_2$ and $(x, y) \in (A, B)$. Then $x \in A \in \tau_1$ and $y \in B \in \tau_2$. Since τ_1 and τ_2 are metrizable, $\exists r_1, r_2 > 0$ with respect to d_1 and d_2 respectively such that $B_{r_1}(x) \subset A$ and $B_{r_2}(y) \subset B$. i.e. if $d_1(x, x_1) < r_1$, then $x_1 \in B_{r_1}(x)$ and if $d_2(y, y_1) < r_2$, then $y_1 \in B_{r_2}(y) \Rightarrow (x_1, y_1) \in (A, B)$. Let $r = \min\{r_1, r_2\}$ and $(u, v) \in B_{r/2}(x, y)$. Then $d((x, y), (u, v)) < \frac{r}{2}$. i.e. $\frac{d_1(x,u)+d_2(y,v)}{2} < r/2$. So $d_1(x, u) + d_2(y, v) < r \Rightarrow d_1(x, u) < r < r_1$ and $d_2(y, v) < r < r_2$. Hence $u \in B_{r_1}(x) \subset A$ and $v \in B_{r_2}(y) \subset B$. Thus $(u, v) \in (A, B)$ showing that $B_{r/2}(x, y) \subset (A, B)$. □

Definition 4.4. *A binary seminorm on two vector spaces V_1 and V_2 is a map, $\| \cdot \| : V_1 \times V_2 \to \mathbb{R}$ such that for each $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$*

(1). $\|(x_1, x_2)\| \geq 0$

(2). $\|\alpha(x_1, x_2)\| = \mid \alpha \mid \|(x_1, x_2)\|$

(3). $\|(x_1, x_2) + (y_1, y_2)\| \leq \|(x_1, x_2)\| + \|(y_1, y_2)\|$ *A binary seminorm becomes a binary norm if the following condition holds.*

(4). $\|(x_1, x_2)\| = 0 \Leftrightarrow (x_1, x_2) = (0, 0)$

Proposition 4.5. *If (V_1, τ_1) and (V_2, τ_2) are both normable topological vector spaces, then their binary product is binary normable.*

Proof. Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be the norms corresponding to τ_1 and τ_2 respectively. Then we get two metrics d_1 and d_2, defined by $d_i((x_1, x_2), (y_1, y_2)) = \|(x_1, x_2) - (y_1, y_2)\|_i, i = 1, 2$ and $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$, with which τ_1 and τ_2 are metrizable respectively. So by Proposition 4.3 $\tau_1 \times \tau_2$ is metrizable with the binary metric $d((x_1, x_2), (y_1, y_2)) = \frac{d_1(x_1,y_1)+d_2(x_2,y_2)}{2}, \forall (x_1, x_2), (y_1, y_2) \in (V_1 \times V_2)$. Hence the binary norm $\| \cdot \|$ defined by $\|(x_1, x_2)\| = d((x_1, x_2), (0, 0))$ for $(x_1, x_2) \in V_1 \times V_2$ corresponds to the topology $\tau_1 \times \tau_2$. But this norm is same as $\frac{\|\cdot\|_1 + \|\cdot\|_2}{2}$ since $\|(x_1, x_2)\| = d((x_1, x_2), (0, 0)) = \frac{d_1(x_1,0)+d_2(x_2,0)}{2} = \frac{\|x_1-0\|_1+\|x_2-0\|_2}{2} = \frac{\|x_1\|_1+\|x_2\|_2}{2}$. □

Lemma 4.6. *Let V_1 and V_2 be two vector spaces and p be a binary seminorm on $V_1 \times V_2$ Then there exists two seminorms p_1 and p_2 on V_1 and V_2 respectively.*

Proof. Let $p_1 : V_1 \to \mathbb{R}$ be defined by $p_1(x) = \inf_y\{p(x,y) : y \in V_2\}$. Since $p(x,y) \geq 0, \forall (x,y) \in V_1 \times V_2, p_1(x) \geq 0 \forall x \in V_1$. For $x \in V_1$ and $\alpha \in K$

$$\begin{aligned} p_1(\alpha x) &= \inf_y\{p(\alpha x, y) : y \in V_2\} \\ &= \inf_y\{\mid \alpha \mid p(x, \frac{1}{\alpha}y) : y \in V_2\} \\ &=\mid \alpha \mid \inf_y p(x, \frac{1}{\alpha}y) : y \in V_2\} \\ &=\mid \alpha \mid p_1(x) \end{aligned}$$

For $x, y \in V_1$

$$\begin{aligned} p_1(x+y) &= \inf_z\{p(x+y, z) : z \in V_2\} \\ &= \inf_{z=z_1+z_2}\{p(x+y, z_1+z_2) : z = z_1+z_2 \in V_2\} \\ &= \inf_{z_1,z_2}\{p[(x,z_1)+(y,z_2)] : z_1, z_2 \in V_2\} \\ &\leq \inf_{z_1,z_2}\{p(x,z_1)+p(y,z_2) : z_1, z_2 \in V_2\} \end{aligned}$$

Thus $p_1(x+y) \leq p_1(x) + p_1(y)$

Hence p_1 is a seminorm on V_1 and similarly $p_2 : V_2 \to \mathbb{R}$ defined by $p_2(y) = \inf_x\{p(x,y) : x \in V_1\}$ is a seminorm on V_2. ☐

Proposition 4.7. *Given a family of binary seminorms on two vector spaces V_1 and V_2, then a locally convex binary linear topology is formed between V_1 and V_2.*

Proof. Let $\{p_\alpha\}_{\alpha \in J}$ be a family of binary seminorms on $V_1 \times V_2$. Corresponding to each $p_\alpha, \alpha \in J$, there exists two seminorms $p_{1\alpha}$ and $p_{2\alpha}$ on V_1 and V_2 respectively. Thus we get a family of seminorms $\{p_{i\alpha}\}_{\alpha \in J}$ on $V_i, i = 1, 2$. Hence by theorem 2.7 there exists a locally convex linear topology, τ_i on V_i induced by $\{p_{i\alpha}\}_{\alpha \in J}, i = 1, 2$. Then $\tau_1 \times \tau_2$ is a locally convex binary linear topology between V_1 and V_2. ☐

5. Conclusion

In this paper we have introduced the concept of linear topological spaces to situations in which we have to deal with two vector spaces and a topology between the spaces. This helps to study both the spaces simultaneously. The concept of topological vector space is well used in mathematics, engineering and science and particularly in quantum mechanics. Hence our theory of Binary Linear Topological Spaces helps in the further development of such areas.

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