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A Study on Factoriangular Numbers

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Abstract: In this paper, we prove that there is no factoriangular number that is also factorial. An expression of factoriangular number explicitly in terms of triangular number is given. Bounds of the ratio of consecutive factoriangular number are given. It is shown that for n $\geq$ 5, there exists no Ft_n which divides Ft_{n+1}. Patterns in factoriangular number modulo n is also observed. It is conjectuerd that there exists no factoriangular number that is a perfect square. It has also been conjectured that for n $\geq$ 6, 8n! + 1 is not a perfect square. A conjecture [3] that there is no factoriangular number that is an even perfect number is proved.

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1. Introduction

Definition 1.1 ([9]). *Triangular number is a number obtained by adding all positive integers less than or equal to a given positive integer n, i.e.,* $T_n = \dfrac{n(n+1)}{2}$.

Definition 1.2 ([3]). *Factoriangular number is defined as the sum of the first n natural numbers plus the factorial of n. i.e.* $Ft_n = \dfrac{n(n+1)}{2} + n!$.

Definition 1.3 ([7]). *Generalized factoriangular number is defined is also called (n, k) - factoriangular numbers. The (n, k) - factoriangular number is defined by the formula* $Ft_{n,k} = n! + T_k$ *where n! = 1.2.3...n and* $T_k = 1+2+3+\ldots+k = \dfrac{k(k+1)}{2}$ *for natural numbers* $n, k \geq 1$.

Definition 1.4 ([2]). *Perfect number is a positive integer that is equal to the sum of its proper positive divisors, that is, the sum of its positive divisors excluding the number itself. i.e.* $\sigma(n) = 2n$. *For example, the first few perfect numbers are 6, 28, 496, 8128, …*

It has been found [5] that $Ft_6 = T_{38} = 741$ is one of the numbers that is both Triangular and Factoriangular number. An open question [5] has been raised if there is any other number that is both triangular and Factoriangular. We will prove that except $Ft_1 = 2$, there is no Factoriangular number that is also Factorial.

Theorem 1.5. *Except* $Ft_1 = 2$*, there is no Factoriangular number that is also Factorial.*

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Proof. If $n = 1$, it is easily shown that $Ft_1 = 2$ and $2! = 2$. (Proof by contradiction). For $n \geq 2$, assume there exist n and x where $x > n$, such that

$$\begin{aligned} Ft_n &= x! \\ n! + \frac{n(n+1)}{2} &= x! \\ \frac{n(n+1)}{2} &= x! - n! \\ n(n+1) &= 2n![x(x-1)..(n-1)! - 1] \\ (n+1) &= 2(n-1)!k \quad \text{where } k = [x(x-1)...(n-1)! - 1] \text{ and } k > n. \\ n+1 &= 2(n-1)k! > 2(n-1)!n. \end{aligned}$$

Hence $n + 1 > 2n!$ But $n + 1 < 2n!$ whenever $n \geq 2$. Hence we reach a contradiction. □

A number which is simultaneously square and triangular is called Square Triangular number. Let T_n denote the nth triangular number and S_m the m$^{\text{th}}$ square number, then a number which is both triangular and square satisfies the equation $T_n = S_m$. [15] The first few solutions are (x,y)=(3,2), (17, 12), (99, 70), (577, 408), ... In 1730, Euler showed that there are an infinite number of such solutions. [15] A natural question is to ask whether there is any Factoriangular number that is a perfect square.

n	Ft_n	Prime Factors
1	2	2
2	5	5
3	12	$2^2.3$
4	34	2.17
5	135	$3^3.5$
6	741	3.13.19
7	5068	$2^2.7.181$
8	40356	$2^2.3^2.19.59$
9	362955	$3^3.5^2.1613$
10	3628855	5.557.3303
11	39916866	2.3.11.604801
12	479001678	2.3.79833613
13	6227020891	$7^2.13.9775543$
14	87178291305	3.5.7.14779.56179
15	1307674368120	$2^3.3.5.10897286401$
16	20922789888136	$2^3.29.90184439173$
17	355687428096153	$3^2.17.298373.7791437$
18	6402373705728171	$3^2.317693.2239189583$
19	121645100408832190	2.5.19.2801.228574570001
20	2432902008176640210	2.3.5.7.59.251.383.2042588183

By the observation of prime factors of Ft_n, there is no number that is a perfect square and also the number of factor keeps on increasing as n gets larger. Hence it can be conjectured that there is no factoriangular number that is a perfect square. Now we will give the expression of the Factoriangular number explicitly in terms of Triangular number.

Theorem 1.6. *If m is a positive integer where $m \geq 1$, then $Ft_m = T_{m!} - T_{m!-1} + T_m$.*

Proof. We can always write $m! = \dfrac{m!(m!+1)}{2} - \dfrac{m!(m!-1)}{2} = T_{m!} - T_{m!-1}$. Adding T_m on both sides we get $Ft_m = T_{m!} - T_{m!-1} + T_m$. □

It has been conjectured that For $n, x \geq 1$, and $n \neq x$, Ft_n is the n$^{\text{th}}$ factoriangular number and T_i is the i$^{\text{th}}$ triangular number, the only solution for $Ft_n = T_x + T_n$ is $(n, x) = (5, 15)$. [5] It is consistent with the conjecture given by Christopher

Tomaszewski that 1, 6 and 120 are the only numbers which are both triangular and factorials. [12] ie. $m, n, s \geq 0$, $m! = \dfrac{n(n+1)}{2} = s$ if and only if $s \in \{1, 6, 120\}$. Let's look at how this is possible. $Ft_n = T_x + T_n \Rightarrow T_n + n! = T_x + T_n \Rightarrow$ $n! = \dfrac{x(x+1)}{2} = s$ if and only if $s \in \{1, 6, 20\}$.

Case 1: $n = 1$, then $\dfrac{x(x+1)}{2} = 1 \Rightarrow x^2 + x - 2 = 0 \Rightarrow x = 1$ or -2. Negative value can be avoided. Hence $x = 1$. Therefore $(n, x) = (1, 1)$. But this value can be discarded as $n \neq x$.

Case 2: $n! = 6$, then $n = 3$ and $\dfrac{x(x+1)}{2} = 6 \Rightarrow x^2 + x - 12 = 0 \Rightarrow x = 3$ or -4. We can neglect negative values and hence $x = 3$. ie., $(n, x) = (3, 3)$. But this can be discarded as well since $n \neq x$.

Case 3: $n! = 120 = \dfrac{x(x+1)}{2}$, then $n = 5$ and $x^2 + x - 240 = 0 \Rightarrow x = 15$ or -16. Neglecting negative values $x = 15$. i.e $(n, x) = (5, 15)$.

Hence for $n, x \geq 1$ and $n \neq x$, the only solution for $Ft_n = T_n + T_x$ is $(n, x) = (5, 15)$. Furthermore it has been proved that for $n, x \geq 1$, Ft_n is the n$^{\text{th}}$ triangular number and T_i is the i$^{\text{th}}$ triangular number, $Ft_n = T_x + T_n$ if and only if $8n! + 1$ is a square [5]. $Ft_n = T_x + T_n \Rightarrow n! = T_x \Rightarrow n! = T_x = \{1, 6, 120\}$ i.e. $n = 1, 3, 5$. Then a conjecture can be stated as follows: For $n \geq 6$, $8n! + 1$ is not a square. It was checked and found to be true for $n \leq 100$. If a counterexample is found the same number is a counterexample to the conjecture given by [12] Christopher Tomaszewski. Now we look under what condition the Factoriangular number is twice the triangular number.

Theorem 1.7. *If Ft_n is a factoriangular number and n! is a factorial then, $Ft_n = 2n!$ if and only if $n = 1, 3$.*

Proof. If Ft_n is a factoriangular number and n! is a factorial number $Ft_n = 2n! \Leftrightarrow n! + T_n = 2n! \Leftrightarrow T_n = n! \Leftrightarrow$ $\dfrac{n(n+1)}{2} = n!$. Clearly this is true for $n = 1$. If $\dfrac{n+1}{2} < n - 1$, then $\dfrac{n(n+1)}{2} < n(n-1) < n!$. So there exists no solution such that $\dfrac{n+1}{2} < n - 1 \Rightarrow n + 1 < 2n - 2 \Rightarrow 3 < n$. Hence n = 1, 2, 3 are the only solution but 2 can be discarded since it doesn't satisfy the equation. Hence 1 and 3 are the only solutions. □

2. Bounds of Factoriangular Numbers

We will now look into the upper bound and lower bound for the ratio of Triangular number, Factoriangular number and Factorial number and look at their limiting values.

Observation 2.1. *For all $n > 3$, $\dfrac{Ft_n}{n!} < 2$.*

Proof. We can easily see that $n! > \dfrac{n(n+1)}{2}$ for all $n > 3$. $Ft_n = T_n + n! < n! + n! < 2n!$. Hence $\dfrac{Ft_n}{n!} < 2$. □

Observation 2.2. *For all $n \geq 5$, $\dfrac{Ft_{n+1}}{Ft_n} > n$.*

Proof. Assume to the contrary $\dfrac{Ft_{n+1}}{Ft_n} \leq n \Rightarrow Ft_{n+1} \leq nFt_n$

$$\Rightarrow (n+1)! + \frac{(n+1)(n+2)}{2} \leq n[n! + \frac{n(n+1)}{2}]$$
$$\Rightarrow n! + \frac{n(n+1)}{2} + (n+1) \leq \frac{n^2(n+1)}{2}$$
$$\Rightarrow n! \leq (n+1)[\frac{n^2}{2} - \frac{n}{2} + 1]$$
$$\Rightarrow n! \leq \frac{(n+1)^2(n-2)}{2}.$$

For $n \geq 5$, $n! > \dfrac{(n+1)^2(n-2)}{2}$ which contradicts our assumption. Hence, the proof. □

Observation 2.3. *For all $n \geq 2$, $\dfrac{Ft_{n+1}}{Ft_n} < n + 1$.*

Proof. $\dfrac{Ft_{n+1}}{Ft_n} = \dfrac{(n+1)[n! + \frac{n+2}{2}]}{n! + \frac{n(n+1)}{2}} \Rightarrow \dfrac{Ft_{n+1}}{Ft_n} = (n+1)k$ where $k = \dfrac{n! + \frac{n+2}{2}}{n! + \frac{n(n+1)}{2}}$. If $k < 1$, then the proof follows since $\dfrac{Ft_{n+1}}{Ft_n} = (n+1)k < n+1$. To show $k < 1$, assume to the contrary that $k \geq 1$, then

$$\frac{n! + \frac{n+2}{2}}{n! + \frac{n(n+1)}{2}} \geq 1 \Rightarrow n! + \frac{n+2}{2} \geq n! + \frac{n(n+1)}{2} \Rightarrow \frac{n+2}{2} \geq \frac{n(n+1)}{2} \Rightarrow n+2 \geq n(n+1) \Rightarrow 2 \geq n^2.$$

But for $n \geq 2$, we get a contradiction. Hence $k < 1$. □

From Observation 2.2 and 2.3, we can deduce that for $n \geq 5$, no Ft_n divides Ft_{n+1}.

Theorem 2.4. *If Ft_n is the n^{th} Factoriangular number and $n!$ is the n^{th} Factorial number then,* $\lim_{n\to\infty} \dfrac{Ft_n}{n!} = 1$.

Proof.

$$\begin{aligned}
\frac{Ft_n}{n!} &= \frac{n(n-1)(n-2)...1 + \frac{n(n+1)}{2}}{n(n-1)(n-2)....1} \\
&= 1 + \frac{\frac{n(n+1)}{2}}{n(n-1)(n-2)...1)} \\
&= 1 + \frac{n+1}{2(n-1)(n-2)...1} \\
&= 1 + \frac{n}{2(n-1)(n-2)...1} + \frac{1}{2(n-1)(n-2)...1} \\
&= 1 + \frac{1}{2(1-1/n)(n-2)...1} + \frac{1}{2(n-1)(n-2)...1}
\end{aligned}$$

Hence, $\lim_{n\to\infty} \dfrac{Ft_n}{n!} = 1$. □

Theorem 2.5. *If Ft_{n+1} is the $(n+1)^{th}$ Factoriangular number and Ft_n is the n^{th} Factoriangular number then,* $\lim_{n\to\infty} \dfrac{Ft_{n+1}}{(n+1)Ft_n} = 1$.

Proof.

$$\begin{aligned}
\frac{Ft_{n+1}}{(n+1)Ft_n} &= \frac{(n+1)! + (n+1)(n+2)/2}{(n+1)[n! + \frac{n(n+1)}{2}]} \\
&= \frac{n! + \frac{n+2}{2}}{n! + \frac{n(n+1)}{2}} \\
&= \frac{1 + \frac{n+2}{2n!}}{1 + \frac{n(n+1)}{2n!}} \\
&= \frac{1 + \frac{1}{2(n-1)!} + \frac{1}{n!}}{1 + \frac{n+1}{2(n-1)!}} \\
&= \frac{1 + \frac{1}{2(n-1)!} + \frac{1}{n!}}{1 + \frac{n}{2(n-1)!} + \frac{1}{2(n-1)!}} \\
&= \frac{1 + \frac{1}{2(n-1)!} + \frac{1}{n!}}{1 + \frac{1}{2(1-1/n)(n-2)..1} + \frac{1}{2(n-1)!}}
\end{aligned}$$

Hence, $\lim_{n\to\infty} \dfrac{Ft_{n+1}}{(n+1)Ft_n} = 1$. □

3. Patterns in Factoriangular Number mod n

We look at the triangular number modulo a positive integer k [9]. Reading the triangular numbers modulo 2, we get the following pattern which repeats every 4 steps:

$$1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, ...$$

Similarly, T_n modulo 3 gives the following pattern of numbers which repeats every 3 steps:

$$1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, ...$$

T_n modulo 4, we get a sequence that repeats every 8 steps:

$$1, 3, 2, 2, 3, 1, 0, 0, 1, 3, 2, 2, 3, 1, 0, 0, ...$$

T_n modulo 5, we get a sequence that repeats every 5 steps:

$$1, 3, 1, 0, 0, 1, 3, 1, 0, 0, ...$$

and T_n modulo 6, we get this pattern:

$$1, 3, 0, 4, 3, 3, 4, 0, 3, 1, 0, 0, 1, 3, 0, 4, ...$$

Since Factoriangular number is the sum of triangular and factorial number, we expect some patterns while looking at factoriangular number mod n. Now we look at the Factoriangular number mod n and we have seen a pattern in Ft_n mod 2, 3, 4, 5 and 6. We will give the pattern in the following table.

n	Ft_n	Ft_n mod 2
1	2	0
2	5	1
3	12	0
4	34	0
5	135	1
6	741	1
7	5068	0
8	40356	0
9	362955	1
10	3628855	1
11	39916866	0
12	479001678	0
13	6227020891	1
14	87178291305	1
15	1307674368120	0
16	20922789888136	0
17	355687428096153	1
18	6402373705728171	1
19	121645100408832190	0
20	2432902008176640210	0

Looking at the table, after n $\geq$ 2, Ft_n mod 2 gives the following pattern which repeats after every 4 steps:

$$1, 0, 0, 1, 1, 0, 0, 1, ...$$

Theorem 3.1. *For $n \geq 2$, Ft_n mod 2 is 0 if n is of the form 4k for integer $k \geq 1$, 4k+3 for integer $k \geq 0$ and Ft_n mod 2 is 1 if n is of the form $4k+1$ for integer $k \geq 1$, $4k+2$ for integer $k \geq 0$.*

Proof. Four cases are considered as follows:

Case 1: The natural number n is of the form 4k, for integer $k \geq 1$. (Note that if $k = 0$, then $n = 0$, which is not considered here). If $n = 4k$, then $Ft_{4k} = (4k)! + \frac{4k(4k+1)}{2} = 2[4k(4k-1)...3.1 + k(4k+1)]$ which is in the form $Ft_{4k} = 2a$, where $a = [4k(4k-1)...3.1 + k(4k+1)]$. Hence $Ft_{4k}\ mod\ 2 = 0$.

Case 2: The natural number n is of the form $4k+1$ for integer $k \geq 1$. If $n = 4k+1$, then $Ft_{4k+1} = (4k+1)! + \frac{(4k+1)(4k+2)}{2} = (4k+1)[(4k)! + (4k+1)(2k+1)]$. But $(4k+1)$ is always odd, $(4k)!$ is even and $4k+1$ and $2k+1$ is odd. And the product of $4k+1$ and $2k+1$ is odd. Ft_{4k+1} is odd. Hence Ft_{4k+1} mod 2 is 1.

Case 3: If the natural number n is of the form $4k+2$, for integer $k \geq 0$. If $n = 4k+2$, then $Ft_{4k+2} = 2m + \frac{(4k+1)(4k+3)}{2} = 2m + (2k+1)(4k+3)$. But The product of two odd is odd and hence Ft_{4k+2} is odd and Ft_{4k+2} mod 2 is 1.

Case 4: If the natural number n is of the form $4k+3$, for integer $k \geq 0$, then $Ft_{4k+3} = 2m + \frac{(4k+3)(4k+4)}{2} = 2m + (4k+3)(2k+2)$. But the product of even and odd is even. Hence Ft_{4k+3} mod 2 is 0. □

n	Ft_n	Ft_n mod 3
1	2	2
2	5	2
3	12	0
4	34	1
5	135	0
6	741	0
7	5068	1
8	40356	0
9	362925	0
10	3628855	1
11	39916866	0
12	479001678	0
13	6227020891	1
14	87178291305	0
15	1307674368120	0
16	20922789888136	1
17	355687428096153	0
18	6402373705728171	0
19	121645100408832190	1
20	2432902008176640210	0

For n$\geq$ 3, Ft_n mod 3 gives the following pattern which repeats after every 3 steps:

$$0, 1, 0, 0, 1, 0, 0, 1, 0, ...$$

Theorem 3.2. *For $n \geq 3$, Ft_n mod 3 is 0 if n is of the form $3k$ for integer $k \geq 1$, $3k+2$ for integer $k \geq 1$ and Ft_n mod 3 is 1 if n is of the form $3k+1$ for integer $k \geq 1$.*

Proof. Three cases will be considered where each $n \geq 3$ can be written in the form of either $3k$, $3k+1$ or $3k+2$ for $k \geq 1$.

Case 1: The natural number n is of the form $3k$ for integer $k \geq 1$. If $n = 3k$, then $Ft_{3k} = (3k)! + \frac{3k(3k+1)}{2} = 3[3k(3k-1)..4.2.1 + \frac{k(3k+1)}{2}]$. Now Ft_{3k} can be written in the form $3m$, m is an integer where $m = 3[3k(3k-1)..4.2.1 + \frac{k(3k+1)}{2}]$. Hence $Ft_{3k}\ mod\ 3 = 0$.

Note: The problem arises if $\frac{k(3k+1)}{2}$ is fraction but we are assured that this is not the case. If k is even, $3k+1$ is odd hence the product of even and odd number is even and 2 will be cancelled out. If k is odd, $3k+1$ is always even and it follows that 2 will be cancelled out.

Case 2: The natural number n is of the form $3k+1$, for integer $k \geq 1$. If $n = 3k+1$, then $Ft_{3k+1} = (3k+1)! + \frac{(3k+1)(3k+2)}{2} = (3k+1)! + \frac{9k^2+9k+2}{2} = (3k+1)! + \frac{9k(k+1)}{2} + 1 = 3[(3k+1).3k..4.2 + \frac{3k(k+1)}{2}] + 1$. Now Ft_{3k+1}

can be written in the form $3m+1$ where m is an integer and $m = (3k+1).3k..4.2 + \frac{3k(k+1)}{2} = (3k+1).3k..4.2 + 3T_k$. Hence $Ft_{3k+1}\ mod\ 3 = 1$.

Case 3: The natural number n is of the form $3k+2$, for integer $k \geq 1$. If $n = 3k+2$, then $Ft_{3k+2} = (3k+2)! + \frac{(3k+2)(3k+3)}{2} = 3[(3k+2)(3k+1)..4.2.1 + \frac{(3k+2)(k+1)}{2}]$. Now Ft_{3k+2} can be written in the form of $3m$ where m is an integer and $m = (3k+2)(3k+1)..4.2.1 + \frac{(3k+2)(k+1)}{2}$. To make sure $\frac{(3k+2)(k+1)}{2}$ is an integer, If k is odd then $k+1$ is even cancelling out 2. If k is even, $3k+2$ is always even and hence cancelling out 2. Therefore, $Ft_{3k+2}\ mod\ 3 = 0$. $\square$

n	Ft_n	Ft_n mod 4
1	2	2
2	5	1
3	12	0
4	34	2
5	135	3
6	741	1
7	5068	0
8	40356	0
9	362925	1
10	3628855	3
11	39916866	2
12	479001678	2
13	6227020891	3
14	87178291305	1
15	1307674368120	0
16	20922789888136	0
17	355687428096153	1
18	6402373705728171	3
19	121645100408832190	2
20	2432902008176640210	2

For $n \geq 4$, we see a pattern in Ft_n mod 4 which repeats after every 8 steps:

2, 3, 1, 0, 0, 1, 3, 2, 2, 3, 1, 0, 0, 1, 3, 2, ...

n	Ft_n	Ft_n mod 5
1	2	2
1	5	0
3	12	2
4	34	4
5	135	0
6	741	1
7	5068	3
8	40356	1
9	362925	0
10	3628855	0
11	39916866	1
12	479001678	3
13	6227020891	1
14	87178291305	0
15	1307674368120	0
16	20922789888136	1
17	355687428096153	3
18	6402373705728171	1
19	121645100408832190	0
20	2432902008176640210	0

For $n \geq 5$, we see a pattern in Ft_n mod 5 which repeats after every 5 steps:

$$0, 1, 3, 1, 0, 0, 1, 3, 1, 0, ...$$

Theorem 3.3. *For $n \geq 5$, Ft_n mod 5 = 0 if n is of the form $5k$ or $5k+4$ for the integer $k \geq 1$; Ft_n mod 5 = 1 if n is of the form $5k+1$ or $5k+3$ for the integer $k \geq 1$ and Ft_n mod 5 = 3 if n is of the form $5k+2$ for the integer $k \geq 1$.*

Proof. Five cases will be considered as follows.

Case 1: If $n = 5k$, $Ft_{5k} = (5k)! + \frac{5k(5k+1)}{2} = 5[5k(5k-1)...6.4! + \frac{k(5k+1)}{2}] = 5m$, m is an integer where $m = 5k(5k-1)...6.4! + \frac{k(5k+1)}{2}$. The problem arises when $\frac{k(5k+1)}{2}$ is fraction. But that is not the case because if k is even, 2 will be cancelled out and if k is odd, $5k+1$ is always even cancelling out 2. Hence Ft_n mod $5 = 0$ when n is of the form $5k$ for $k \geq 1$.

Case 2: If $n = 5k+1$, $Ft_{5k+1} = (5k+1)! + \frac{(5k+1)(5k+2)}{2} = (5k+1)! + \frac{25k^2+15k+2}{2} = 5[(5k+1)5k...6.4! + \frac{5k(5k+3)}{2}] + 1 = 5m+1$ where m is an integer and $m = (5k+1)5k...6.4! + \frac{5k(5k+3)}{2}$. We have to check whether $\frac{5k(5k+3)}{2}$ is an integer to make sure m is an integer. If k is even then 2 will be cancelled out and if k is odd then $5k+3$ is always even which will cancel out 2. Hence m is always an integer and Ft_n mod $5 = 1$ if n is of the form $5k+1$.

Case 3: If n is of the form $5k+2$ where $k \geq 1$, then $Ft_{5k+2} = (5k+2)! + \frac{(5k+2)5k+3}{2} = (5k+2)! + \frac{25k^2+25k+6}{2} = 5[(5k+2)...6.4! + \frac{5k(k+1)}{2}] + 3 = 5m+3$ where m is an integer and $m = (5k+2)..6.4! + \frac{5k(k+1)}{2} = (5k+2)...6.4! + 5T_k$. Hence Ft_n mod $5 = 3$ when n is of the form $5k+2$.

Case 4: If n is of the form $5k+3$ where $k \geq 1$, then $Ft_{5k+3} = (5k+3)! + \frac{(5k+3)(5k+4)}{2} = (5k+3)! + \frac{25k^2+35k+12)}{2} = 5[(5k+3)...6.4! + \frac{5k(k+7)}{2} + 1] + 1 = 5m+1$ where m is an integer and $m = (5k+3)...6.4! + \frac{5k(k+7)}{2} + 1$. We can show m is an integer by showing that $\frac{k(k+7)}{2}$ is an integer. If k is even, 2 will be cancelled out and If k is odd, $k+7$ is even and 2 will be cancelled. Hence it is an integer. Therefore Ft_n mod $5 = 1$.

Case 5: If $n = 5k+4$, $Ft_{5k+4} = (5k+4)! + \frac{(5k+4)(5k+5)}{2} = 5[(5k+4)...6.4! + \frac{(5k+4)(k+1)}{2}] = 5m$ where m is an integer and $m = (5k+4)...6.4! + \frac{(5k+4)(k+1)}{2}$. To show m is an integer it suffices to show $\frac{(5k+4)(k+1)}{2}$ is an integer. If k is even, $5k+4$ is even and 2 will be cancelled out. If k is odd, $k+1$ is even and 2 will be cancelled out. Hence Ft_{5k+4} mod $5 = 0$. □

After some experimentation, we are able to guess that the Ft_n mod k repeats every k steps for $n \geq k$, if k is odd, and every $2k$ steps if k is even. We give the condition under which $Ft_{n,k} = 2Ft_n$.

Theorem 3.4. *For $n, k \geq 1$, $Ft_{n,k} = 2Ft_n$ if and only if $8Ft_n + 8T_n + 1$ is a perfect square.*

Proof.

$$Ft_{n,k} = 2Ft_n$$

$$\Leftrightarrow n! + T_k = 2n! + 2T_n$$

$$\Leftrightarrow T_k = n! + 2T_n$$

$$\Leftrightarrow T_k = Ft_n + T_n$$

$$\Leftrightarrow \frac{k(k+1)}{2} = Ft_n + T_n$$

$$\Leftrightarrow k^2 + k = 2Ft_n + 2T_n$$

$$\Leftrightarrow k^2 + k - (2Ft_n + 2T_n) = 0$$

$$\Leftrightarrow k = \frac{-1 \pm \sqrt{1 + 4(2Ft_n + 2T_n)}}{2}.$$

For k to be an integer , $8Ft_n + 8T_n + 1$ must be a perfect square. □

4. Proof of a Conjecture

We will prove the conjecture that there is no factoriangular number that is an even perfect number [3].

Theorem 4.1. *There is no factoriangular number that is an even perfect number.*

Proof. We will prove this by contradiction. We know that every even perfect number n can be represented in the form $n = 2^{p-1}(2^p - 1)$ where 2^{p-1} is a prime number. Assume to the contrary that there exists a factoriangular number Ft_n such that $Ft_n = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is a prime number. Then

$$n! + \frac{n(n+1)}{2} = 2^{p-1}(2^p - 1) \tag{1}$$

Equation (1) $\Rightarrow 2n! + n(n+1) = 2^p(2^p - 1) \Rightarrow n[2(n-1)! + (n+1)] = 2^p(2^p - 1)$. Then $n \mid 2^p(2^p - 1)$ which implies $n \mid 2^p$ or $n \mid (2^p - 1)$. Our claim is that $n \nmid 2^p$ nor $n \nmid (2^p - 1)$ which will lead us to contradiction and hence proving the theorem. Since $2^p - 1$ is prime and for n to divide $2^p - 1$, either $n = 1$ or $n = 2^p - 1$ itself. When $n = 1$, $Ft_n = 2$ which is not equal to an even perfect number so we can discard $n = 1$. When $n = 2^p - 1$, Equation (1) $\Rightarrow n! + \frac{n(n+1)}{2} = \frac{n(n+1)}{2} \Rightarrow n! = 0$ which gives that $n = 0$. When $n = 0$, Ft_n does not exist so we can discard this value of n as well.
Hence the only way is $n \mid 2^p$. Since we are considering even perfect number, Ft_n must be even and We have a Theorem [3] Ft_n is even if and only if $n = 4k$ or $4k+3$. But $n \neq 4k+3$ since $4k+3 \nmid 2^p$. Hence it must be of the form $n = 4k = 2^2k$. Now $n = 4k$, $4k \mid 2^p \Rightarrow k \mid 2^{p-2}$. This can happen if and only if k is of the form 2^a where a is some integer. $k \mid 2^{p-2} \Rightarrow 2^{p-2} = km$ where m is of the form $2^b \Rightarrow 2^p = 4km$ implies

$$2^p - 1 = (4km - 1) \tag{2}$$

Equation (1) $\Rightarrow (4k)! + \frac{4k(4k+1)}{2} = 2^{p-1}(2^p - 1) \Rightarrow 4k[(4k-1)! + \frac{4k+1}{2}] = 2km(4km - 1)$ [From Equation (2)] $\Rightarrow 2(4k-1)! + (4k+1) = m(4km - 1)$ which is a contradiction since $2(4k-1)!$ is even and $(4k+1)$ is odd and the sum of even and odd number is odd But m is even and $4km - 1$ is odd and the product of even and odd number is even. $\square$

5. Conclusion

It was proved that there is no Factoriangular number that is factorial number except $Ft_1 = 2$. Factoriangular number was expressed explicitly in terms of sum of Triangular number. We have also found some bounds of the ratio of factoriangular numbers. It is conjectured that there is no factoriangular number that is a perfect square. It is also conjectured that for $n \geq 6$, $8n! + 1$ is not a perfect square. A conjecture has been proved that there exists no factoriangular number that is an even perfect number [3]. There are many open problems [3, 5] left to the reader. There is lot of scope for the further study on the generalized Factoriangular Number.

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