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Representation of Soft Subsets by Products

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Abstract: In this paper, we show that for any soft set over a universal set, 1. there is a crisp set such that the complete lattice of all soft subsets of the given soft set is complete epimorphic to a complete lattice of certain subsets of the crisp set, where the join in the later complete lattice is the meet induced join and 2. there is a crisp set such that the complete lattice of all regular soft subsets of the given soft set is complete isomorphic to a complete lattice of certain subsets of the crisp set, where the meet in the former complete lattice is the join induced meet and the join in the later complete lattice is the meet induced join.

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1. Introduction

Molodtsov [16] introduced the notion of soft set as a mathematical tool for modelling uncertainties. It has a set E of all possible adjectives or descriptors or attributes, called parameters pertaining to a collection of objects U, called a universal set. Since any one attribute e in E can naturally be associated with a sub collection $\sigma_E e$ of the collection of objects U, intrinsically it has a map σ_E from the set of all adjectives E to the set of all sub collections $P(U)$ of the given collection. Thus, a soft set is a pair consisting of a parameter set E and a map σ_E from this parameter set E into the power set $P(U)$. Ever since the notion of soft set is introduced, several mathematicians imposed various algebraic, topological and topologically algebraic structures and substructures on the soft sets studying some of the basic properties and now there are well over a thousand papers available in print on the net.

For studies in various soft algebraic structures and substructures, one can refer to Aktas-Cagman [3] for soft groups; Sezgin-Atagun [22] for soft groups and normalistic soft groups; Feng-Jun-Zhao [12] for soft semi rings; Acer-Koyuncu-Tanay [1] for soft rings; Sun-Zhang-Liu [24] for soft modules; Sezgin-Atagun-Ayugn [21] for soft near-rings and idealistic soft near-rings; Atagun-Sezgin [6] for soft substructures of rings, fields and modules, Murthy-Maheswari [17, 18] for $f(p)$-soft τ-algebras and their ω-subalgebras and Changphas-Thongkam [8] for soft algebras in a general viewpoint; for studies in soft topological structures and substructures, one can refer to Shabir-Naz [23], Peyghan-Samadi-Tayebi [20] and Cagman-Karatas-Enginoglu [7] for soft topological space and soft topology; Ahmad-Hussain [2] for some structures of soft topology and Kannan [14] for soft generalized closed sets in soft topological spaces; for studies in soft topological algebraic structures and substructures, one can refer to Das-Majumdar-Samanta [9] for soft linear spaces and soft normed linear spaces; Das-Samanta [10, 11] for

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soft metric spaces and soft inner product spaces etcetera.

Please notice that the above list is far from being complete and our aim is to suggest a few papers for a beginner in each direction. However, our aim in this paper is to construct, for any soft set, a crisp set in such a way that 1. the complete poset of all soft subsets of the former is complete epimorphic to a complete poset of certain subsets of the later, where the join in the later is the meet induced join and 2. the complete poset of all regular soft subsets of the given soft set is complete isomorphic to a complete poset of certain subsets of the crisp set, where the meet in the former complete poset is the join induced meet and the join in the later complete poset is the meet induced join.

2. Preliminaries

We assume the following notions from Lattice Theory: poset, least upper bound, greatest lower bound, (meet/join) complete poset, (meet/join) complete subposet, (meet/join) complete homomorphism (isomorphism) of (meet/join) complete posets, complete ideal (filter), one can refer to any standard text books on Lattice Theory for them. Observe that by a meet (join) complete poset we mean a poset in which every *non-empty* subset S has infimum (supremum), denoted by $\wedge S$ ($\vee S$); by a meet complimented meet complete poset we mean a meet complete poset in which for every element α there exists β such that $\alpha \wedge \beta = 0$, where 0 is the least element; by a complete poset or a complete lattice we mean a poset which is both a meet complete poset and a join complete poset; a subset of a meet (join) complete poset is a meet (join) complete subposet iff it is closed under infimum (supremum) for all its *non-empty* subsets; a subset of a complete lattice is a complete sublattice iff it is both a meet complete subposet and a join complete subposet; by a meet (join) complete homomorphism we mean any map between meet (join) complete posets which preserves infimums (supremums) for all *non-empty* subsets; by a complete homomorphism we mean any map between complete lattices which preserves infimums and supremums for all *non-empty* subsets; by a complete isomorphism we mean any complete homomorphism between complete lattices which is both one-one and onto; by a complete ideal I of a complete lattice L we mean a subset I of L which is closed under the supremum for every *non-empty* subset of I and also closed under all the elements of L that are smaller than elements of I and by a complete filter F of a complete lattice L we mean a subset F of L which is closed under the infimum for every *non-empty* subset of F and also closed under all the elements of L that are larger than elements of F.

Proofs are omitted for two reasons: 1. to minimize the size of the document and 2. in most cases, they are either easy or straight forward and a little involving.

2.1. Lattices

In what follows we recall some of the set theoretic and lattice theoretic results which are used in the main results:

(a). For any index set I and for any family of sets $(U_i)_{i\in I}$ such that $\Pi_{i\in I}U_i = U$, $P(U)$, the power set of U, is a complete boolean algebra with the partial ordering given by the usual set inclusion, the meet given by the usual set intersection, the join given by the usual set union and the complement given by the usual set complement.

(b). For any index set I and for any family of complete lattices $(L_i)_{i\in I}$, the product set $\Pi_{i\in I}L_i = L$ is a complete lattice with the partial ordering, the meet and the join defined as follows: for $f, g \in L$, $f \leq g$ iff $f(i) \leq g(i)$ for all $i \in I$, for $(f_j)_{j\in J}$ in L, $\wedge_{j\in J} f_j$ and $\vee_{j\in J} f_j$ in L are given by $(\wedge_{j\in J} f_j)(i) = \wedge_{j\in J}\, f_j(i)$ and $(\vee_{j\in J} f_j)(i) = \vee_{j\in J}\, f_j(i)$.

(c). For any index set I and for any family of complete Boolean algebras $(B_i)_{i\in I}$, the product set $\Pi_{i\in I}B_i = B$ is a complete Boolean algebra with the partial ordering, the meet, the join and the complement defined as follows: for $f, g \in B$, $f \leq g$ iff $f(i) \leq g(i)$ for all $i \in I$, for $(f_j)_{j\in J}$ in B, $\wedge_{j\in J} f_j$ and $\vee_{j\in J} f_j$ in B are given by $(\wedge_{j\in J} f_j)(i) = \wedge_{j\in J}\, f_j(i)$ and

$(\vee_{j\in J} f_j)(i) = \vee_{j\in J} f_j(i)$ and $(f^c)(i) = (f(i))^c$.

(d). Let L be a meet complete poset with the largest element 1_L. For any non-empty subset S of L define $\overline{\vee} S = \wedge_{\alpha\leq\beta,\, \alpha\in S,\, \beta\in L}\beta$. Then L is a complete lattice, where the join is given by $\overline{\vee}$.

(e). For any meet complete poset L with 1_L, the join defined as above is called the *meet induced join* in L and the complete lattice L defined as above is called the *associated complete lattice for the meet complete poset L.*

(f). Let L be a meet complete subposet of the complete lattice M with $1_L = 1_M$. Then the associated complete lattice for the meet complete poset L still remains as a meet complete subposet of M without necessarily being a complete sublattice of M.

In other words, in the above, when L is a meet complete subposet of a complete lattice M, it may so happen that L may *not* even be closed under the $\vee$ of M but its meet induced join in L exists so that the associated complete lattice L is *not* necessarily a complete sublattice of M.

(g). However, if L is a complete sublattice of the complete lattice M then for any non-empty subset S of L, $\overline{\vee}_L S = \vee_L S = \vee_M S = \overline{\vee}_M S$.

(h). Let L be a join complete poset with the least element 0_L. For any non-empty subset S of L define $\overline{\wedge} S = \vee_{\beta\leq\alpha,\, \alpha\in S,\, \beta\in L}\beta$. Then L is a complete lattice, where the meet is given by $\overline{\wedge}$.

(i). For any join complete poset L with 0_L, the meet defined as above is called the *join induced meet* in L and the complete lattice L defined as above is called the *associated complete lattice for the join complete poset L.*

(j). Let L be a join complete subposet of the complete lattice M with $0_L = 0_M$. Then the associated complete lattice for the join complete poset L still remains as a join complete subposet of M without necessarily being a complete sublattice of M.

In other words, in the above, when L is a join complete subposet of a complete lattice M, it may so happen that L may *not* even be closed under the $\wedge$ of M but its join induced meet in L exists so that the associated complete lattice L is *not* necessarily a complete sublattice of M.

(k). However, if L is a complete sublattice of the complete lattice M then for any non-empty subset S of L, $\overline{\wedge}_L S = \wedge_L S = \wedge_M S = \overline{\wedge}_M S$.

(l). For any subset $(\alpha_i)_{i\in I}$ of a meet complete subposet L with 1_L of a complete lattice M, $\vee_{i\in I}\alpha_i$ in $M \leq_M \overline{\vee}_{i\in I}\alpha_i$ in L. However, $\overline{\vee}_{i\in I}\alpha_i =_L \vee_{i\in I}\alpha_i$ whenever $\vee_{i\in I}\alpha_i \in L$.

(m). For any subset $(\alpha_i)_{i\in I}$ of a join complete subposet L with 0_L of the complete lattice M, $\overline{\wedge}_{i\in I}\alpha_i$ in $L \leq_M \wedge_{i\in I}\alpha_i$ in M. However, $\overline{\wedge}_{i\in I}\alpha_i =_L \wedge_{i\in I}\alpha_i$ whenever $\wedge_{i\in I}\alpha_i \in L$.

(n). For any meet complete isomorphism $\phi : L \to M$ of a meet complete poset L with 1_L to a complete lattice M, ϕ is a join complete isomorphism (and hence a complete isomorphism), where the join in L is the join induced by the meet in L.

(o). For any join complete isomorphism $\phi : L \to M$ of a join complete poset L with 0_L to a complete lattice M, ϕ is a meet complete isomorphism (and hence a complete isomorphism), where the meet in L is the meet induced by the join in L.

(p). For any meet complete isomorphism $\phi : L \to M$ of a complete lattice L to a meet complete poset M with 1_M, $\phi^{-1} : M \to L$ is a meet complete isomorphism.

(q). For any join complete isomorphism $\phi : L \to M$ of a complete lattice L to a join complete poset M with 0_M, $\phi^{-1} : M \to L$ is a join complete isomorphism.

(r). For any meet complete isomorphism $\phi : L \to M$ of a complete lattice L to a meet complete poset M with 1_M, ϕ is a join complete isomorphism (and hence a complete isomorphism), where the join in M is the join induced by the meet in M.

(s). For any join complete isomorphism $\phi : L \to M$ of a complete lattice L to a join complete poset M with 0_M, ϕ is a meet complete isomorphism (and hence a complete isomorphism), where the meet in M is the meet induced by the join in M.

2.2. Soft Sets

In what follows we recall some of the basic notions in Soft Set Theory which are used in the main results:

(a). [16] Let U be a universal set, $P(U)$ be the power set of U and E be a set of parameters. A pair (F, E) is called a *soft set* over U iff $F : E \to P(U)$ is a mapping defined by for each $e \in E$, $F(e)$ is a subset of U. In other words, a soft set over U is a parametrized family of subsets of U. Notice that a collective presentation of all the notions algebras, soft sets, fuzzy soft sets, f-soft algebras, f-fuzzy soft algebras in the single paper, Murthy-Maheswari[17] raised some serious notational conflicts and to fix the same we deviated from the above notation for a soft set and adapted the following notation for convenience as follows:
Let U be a universal set. A typical *soft set* over U is an ordered pair $\mathsf{E} = (\sigma_E, E)$, where E is a set of *parameters*, called the (underlying) parameter set for E, $P(U)$ is the power set of U and $\sigma_E : E \to P(U)$ is a map, called the *underlying set valued map* for E. Some times σ_E is also called the *soft structure* on E.

(b). [5] The *empty soft set* over U is a soft set with the empty parameter set, denoted by $\Phi = (\sigma_\phi, \phi)$. Clearly, it is unique.

(c). [5] A soft set E over U is said to be a *null soft set*, denoted by Φ_E, iff $\sigma_E e = \phi$, the empty set, for all $e \in E$.

(d). [4] A soft set E over U is said to be a *whole soft set*, denoted by U_E, iff $\sigma_E e = U$ for all $e \in E$.

(e). [19] For any pair of soft sets A and B over U, A is a *soft subset* of B, denoted by $\mathsf{A} \subseteq \mathsf{B}$, iff (i) $A \subseteq B$ (ii) $\sigma_A a \subseteq \sigma_B a$ for all $a \in A$. The set of *all soft subsets* of E is denoted by $\mathcal{S}_U(\mathsf{E})$.

(f). [4] The *complement* of a soft set $\mathsf{E} = (\sigma_E, E)$, denoted by $\mathsf{E}^c = (\sigma_E, E)^c$, is defined by $(\sigma_E, E)^c = (\sigma_E^c, E)$, where $\sigma_E^c : E \to P(U)$ is a mapping given by $\sigma_E^c(e) = U - \sigma_E(e)$ for all $e \in E$. For any pair of soft sets A, B.

(g). A is a *d-total* soft subset of B iff A is a soft subset of B and $A = B$. The set of *all d-total soft subsets* of E is denoted by $\mathcal{S}_U^d(\mathsf{E})$.

(h). A is a *strong* soft subset of B iff A is a soft subset of B and $\sigma_A a = \sigma_B a$ for all $a \in A$. Notice that the empty soft set Φ is trivially strong soft subset of E because there is *no* $e \in \phi$ such that $\sigma_\phi e \neq \sigma_E e$. The set of *all strong soft subsets* of E is denoted by $\mathcal{S}_U^s(\mathsf{E})$.

(i). A soft subset A of E is *regular* soft subset iff $\sigma_A a \neq \phi$ for all $a \in A$. Notice that the empty soft set Φ is trivially regular because there is *no* $e \in \phi$ such that $\sigma_\phi e = \phi$. The set of *all regular soft subsets* of E is denoted by $\mathcal{S}_U^r(\mathsf{E})$.

(j). The following are easy to see:

(1). Always the empty soft set Φ is a soft subset of every soft set A

(2). $\mathsf{A} = \mathsf{B}$ iff $\mathsf{A} \subseteq \mathsf{B}$ and $\mathsf{B} \subseteq \mathsf{A}$ iff $A = B$ and $\sigma_A = \sigma_B$.

For any family of soft subsets $(\mathsf{A}_i)_{i\in I}$ of E,

(k). [12] the *soft union* of $(\mathsf{A}_i)_{i\in I}$, denoted by $\cup_{i\in I}\mathsf{A}_i$, is defined by the soft set A, where (i) $A = \cup_{i\in I} A_i$ (ii) $\sigma_A a = \cup_{i\in I_a}\sigma_{A_i} a$, where $I_a = \{i \in I / a \in A_i\}$, for all $a \in A$

(l). the *soft intersection* of $(\mathsf{A}_i)_{i\in I}$, denoted by $\cap_{i\in I}\mathsf{A}_i$, is defined by the soft set A, where (i) $A = \cap_{i\in I} A_i$ (ii) $\sigma_A a = \cap_{i\in I}\sigma_{A_i} a$ for all $a \in A$. Notice that $\cap_{i\in I}\mathsf{A}_i$ can become empty resulting the soft intersection is the empty soft set.

(m). [17] For any soft set E over U, the following are true:

(1). The complete lattice of all (crisp) subsets of the underlying set E is complete isomorphic to the complete sublattice of all strong soft subsets of E

(2). Whenever E is a whole soft set, the complete sublattice of all d-total soft subsets of E is complete isomorphic to the complete lattice of all $P(U)$-fuzzy subsets of E (in the sense of Goguen [13]).

Thus, soft sets are a proper generalization of crisp sets and in a sense a special case of Goguen fuzzy sets.

In what follows essentially we show that for any soft set E over U, 1. the complete lattice of all soft subsets of E is complete epimorphic to the complete filter of all the factorizable subsets of the associated product set for the extended soft subset of E over $\overline{U}$, where the join in the later is the meet induced join and 2. the complete lattice of all regular soft subsets of E is complete isomorphic to the complete filter of all the factorizable subsets of the associated product set for the extended soft subset of E over $\overline{U}$, where the meet in the former is the join induced meet and the join in the later is the meet induced join.

3. Factorizable Sets

In this section we study the lattice theoretic properties of the collection of all factorizable subsets of a product set. Notice that through out this paper, we assume that all the index sets are non-empty.

Lemma 3.1. *For a pair of index sets I, J, for a pair of families of sets $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ and for any family of sets $(A_{j,i})_{(j,i)\in(J\times I)}$, the following are true:*

(a). $A_i \subseteq B_i$ for all $i \in I$ implies $\Pi_{i\in I} A_i \subseteq \Pi_{i\in I} B_i$ but not conversely. However, converse is also true whenever $\Pi_{i\in I} A_i \neq \phi$.

(b). $\cap_{j\in J}(\Pi_{i\in I} A_{j,i}) = \Pi_{i\in I}(\cap_{j\in J} A_{j,i})$.

Proof. (a) is straight forward and (b) follows from (a). □

The following Example shows that converse of (a) in the above Lemma need *not* be true:

Example 3.2. *Let $A_1 = \phi = B_2$ and $A_2 = U = B_1$. Then $A_1 \times A_2 = B_1 \times B_2$ but $A_2 \not\subseteq B_2$.*

Definitions 3.3. *Let I be an index set, $(U_i)_{i\in I}$ be a family of sets such that $\Pi_{i\in I} U_i = U$ and $P(U)$ be the power set of U. Then* *(a). a set $A \in P(U)$ is factorizable iff $A = \Pi_{i\in I}A_i$, $A_i \subseteq U_i$ for all $i \in I$. Notice that ϕ is factorizable and whenever $A = \Pi_{i\in I}A_i$, $A \neq \phi$ iff $A_i \neq \phi$ for all $i \in I$. Thus, the set of all factorizable subsets of U, denoted by $F(U)$, is given by $F(U) = \{A \in P(U)/\phi \neq A = \Pi_{i\in I}A_i, A_i \subseteq U_i \text{ for all } i \in I\} \cup \{\phi\}$.*

(b). the binary relation $\preceq$ on $F(U)$, defined by $\preceq = \{(\phi, A)/A \in F(U)\} \cup \{(A, B)/A \neq \phi, A_i \subseteq B_i \text{ for all } i \in I\}$, is clearly a partial order on $F(U)$ so that $(F(U), \preceq)$ is a poset.

Lemma 3.4. *For any family of sets $(U_i)_{i\in I}$, $|F(\Pi_{i=1}^{m}U_i)| = 1 + \Pi_{i=1}^{m}(2^{|U_i|} - 1)$. In particular, $|F(U^E)| = 1 + (2^{|U|} - 1)^{|E|}$.*

Proof. It is straight forward. □

Theorem 3.5. *For any index set I and for any family of sets $(U_i)_{i\in I}$ such that $\Pi_{i\in I}U_i = U$, the following are true:*

(a). $F(U)$ is a meet complete subposet of $P(U)$ with $1_{F(U)} = U = 1_{P(U)}$ and $0_{F(U)} = \phi = 0_{P(U)}$

(b). The meet complete poset $F(U)$ with the largest element U is a complete lattice with the meet induced join $\bar{\vee}$, given by, for any family $(A_j)_{j\in J}$ in $F(U)$, where $A_j = \Pi_{i\in I}A_{j,i}$, $A_{j,i} \subseteq U_i$ for all $i \in I$, $\bar{\vee}_{j\in J}A_j = \Pi_{i\in I}(\cup_{j\in J}A_{j,i})$. Further, $\vee_{j\in J}A_j$ in $P(U) \leq_{P(U)} \bar{\vee}_{j\in J}A_j$ in $F(U)$.

Proof. (a). follows from 3.3(b), 3.1(b) and

(b). follows from (a), 2.1(d) and 2.1(l). □

Corollary 3.6. *For any index set E and for any set U, the following are true:*

(a). $F(U^E)$ is a meet complete subposet of $P(U^E)$ with $1_{F(U^E)} = U^E = 1_{P(U^E)}$ and $0_{F(U^E)} = \phi = 0_{P(U^E)}$

(b). The meet complete poset $F(U^E)$ with the largest element U^E is a complete lattice with the meet induced join $\bar{\vee}$. Further, for any family $(A_j)_{j\in J}$ in $F(U^E)$, $\vee_{j\in J}A_j$ in $P(U^E) \leq_{P(U^E)} \bar{\vee}_{j\in J}A_j$ in $F(U^E)$.

Proof. It follows from 3.5 above. □

Theorem 3.7. *For any index set I, for any family of sets $(U_i)_{i\in I}$ such that $\Pi_{i\in I}U_i = U$ and for any $\overline{u}$ in U, the following are true:*

(a). $\mathcal{F} = \{A \in F(U)/\overline{u} \in A\}$ is a complete lattice with

(1). the partial ordering defined by: for any $A, B \in \mathcal{F}$, $A \leq B$ iff $A_i \subseteq B_i$ for all $i \in I$

(2). the largest element $1_{\mathcal{F}} = U$ and the least element $0_{\mathcal{F}} = \overline{u}$ and with

(3). for any family $(A_j)_{j\in J}$ in $\mathcal{F}$, $\wedge_{j\in J}A_j = \cap_{j\in J}A_j$ and $\vee_{j\in J}A_j = \bar{\vee}_{j\in J}A_j$ (cf.3.5(b)), where $\bar{\vee}$ is the meet induced join in $\mathcal{F}$. Further, $\mathcal{F}$ is also a complete filter in $F(U)$

(b). $\mathcal{F} = \{A \in P(U)/\overline{u} \in A\}$ is a complete lattice with

(1). the partial ordering defined by: for any $A, B \in \mathcal{F}$, $A \leq B$ iff $A \subseteq B$

(2). the largest element $1_{\mathcal{F}} = U$ and the least element $0_{\mathcal{F}} = \overline{u}$ and with

(3). for any family $(A_j)_{j\in J}$ in $\mathcal{F}$, $\wedge_{j\in J}A_j = \cap_{j\in J}A_j$ and $\vee_{j\in J}A_j = \cup_{j\in J}A_j$. Further, $\mathcal{F}$ is also a complete filter in $P(U)$.

Proof. (a) follows from 3.1(b) and 2.1(d) and (b) is straight forward. □

Definitions 3.8.

(a). The complete filter in 3.7(a) above is denoted by $(\overline{u})_{F(U)}$ and is called the complete filter of factorizable subsets of U generated by $\overline{u}$.

(b). The complete filter in 3.7(b) above is denoted by $(\overline{u})_{P(U)}$ and is called the complete filter of subsets of U generated by $\overline{u}$.

Notice that (1) $(\overline{u})_{F(U)} \subseteq F(U) \subseteq P(U)$ (2) $(\overline{u})_{F(U)} \subseteq (\overline{u})_{P(U)}$.

4. Various Soft Subsets

In this section we introduce the notions of extended soft subset and reparametrization for a soft subset and study the lattice homomorphic properties of the operators induced by the same and lattice theoretic properties of various types of sub collections of soft subsets.

Definitions and Statements 4.1.

(a). For any set U, the extended set, denoted by $\overline{U}$, is defined by $\overline{U} = U \cup \{\infty\}$, where ∞ is not in U. Notice that such ∞ exists because for example, one can take $\infty = U$.

(b). For any soft subset A of a soft set E over U, the extended soft set for A over $\overline{U}$, denoted by A', is defined by the d-total soft set, where $A' = E$ and $\sigma_{A'} : E \to P(\overline{U})$ is defined by

$$\sigma_{A'}e = \begin{cases} \sigma_A e \cup \{\infty\} & \text{if } \sigma_A e \neq \phi \\ \{\infty\} & \text{if } \sigma_A e = \phi \text{ or } e \in E - A \end{cases}$$

Notice that

(1). the extended soft set for E over $\overline{U}$ given by E', where $E' = E$ and $\sigma_{E'} : E \to P(\overline{U})$ is given by

$$\sigma_{E'}e = \begin{cases} \sigma_E e \cup \{\infty\} & \text{if } \sigma_E e \neq \phi \\ \{\infty\} & \text{if } \sigma_E e = \phi \end{cases}$$

(2). whenever A is a soft subset of E over U, observe that A' is a soft subset of E' over $\overline{U}$.

The set of all extended soft sets over $\overline{U}$ for all soft subsets of E over U, is denoted by $\mathcal{S}_U(\mathsf{E})'$. In other words, $\mathcal{S}_U(\mathsf{E})' = \{\mathsf{A}'/\mathsf{A} \in \mathcal{S}_U(\mathsf{E})\}$. Clearly, for any soft subset A of a soft set E, both of the extended soft subsets A' and E' over $\overline{U}$ are always d-total and regular.

(c). Clearly, for any soft set E over U, E' is a soft set over $\overline{U}$ and for any soft subset A of the soft set E, A' is a soft subset of the soft set E'. Thus, $\mathcal{S}_U(\mathsf{E})' \subseteq \mathcal{S}_{\overline{U}}(\mathsf{E}')$, the set of all soft subsets of E' over $\overline{U}$. In fact, $\mathcal{S}_U(\mathsf{E})' \subseteq \mathcal{S}^{d,r}_{\overline{U}}(\mathsf{E}')$, the set of all d-total and regular soft subsets of E', and the inclusion in the above can be proper as any soft subset with parameter set $A = E$ and $\sigma_A a \neq \phi$ for all $a \in A$ with $\infty \notin \sigma_A a_0 \in P(\overline{U})$ for some $a_0 \in A$, is an element of $\mathcal{S}^{d,r}_{\overline{U}}(\mathsf{E}')$ but not in $\mathcal{S}_U(\mathsf{E})'$.

(d). Observe that the assignment $\mathsf{A} \to \mathsf{A}'$ defines an operator $\varepsilon : \mathcal{S}_U(\mathsf{E}) \to \mathcal{S}_U(\mathsf{E})'$.

(e). Clearly, $\mathcal{S}_U^r(\mathsf{E})$ is a subset of $\mathcal{S}_U(\mathsf{E})$ and the restriction of the operator ε to $\mathcal{S}_U^r(\mathsf{E})$ is denoted by $\varepsilon|\mathcal{S}_U^r(\mathsf{E})$.

Observe that for any regular soft subset A of a soft set E over U, the extended soft subset A' of E' over $\overline{U}$ is given by the d-total soft subset, where $A' = E$ and $\sigma_{A'} : E \to P(\overline{U})$ is given by

$$\sigma_{A'}e = \begin{cases} \sigma_A e \cup \{\infty\} & \text{if } e \in A \\ \{\infty\} & \text{if } e \in E - A \end{cases}$$

(f). For any soft subset A of a soft set E, the support of A, denoted by *Supp*(A), is defined by *Supp*$(\mathsf{A}) = \{a \in A/\sigma_A a \neq \phi\}$.

Theorem 4.2. *For any soft set E over U, the following are true:*

(a). The set $\mathcal{S}_U(\mathsf{E}) = \{\mathsf{A}/\mathsf{A} \subseteq \mathsf{E}\}$ of all soft subsets of E is a complete lattice with

(1). the partial ordering defined by: for any $\mathsf{A}, \mathsf{B} \in \mathcal{S}_U(\mathsf{E})$, $\mathsf{A} \leq \mathsf{B}$ iff $\mathsf{A} \subseteq \mathsf{B}$

(2). the largest element $1_{\mathcal{S}_U(\mathsf{E})} = \mathsf{E}$, the soft set, and the least element $0_{\mathcal{S}_U(\mathsf{E})} = \Phi$, the empty soft set, and with

(3). for any family $(\mathsf{A}_i)_{i\in I}$ in $\mathcal{S}_U(\mathsf{E})$, $\wedge_{i\in I}\mathsf{A}_i = \cap_{i\in I}\mathsf{A}_i$ and $\vee_{i\in I}\mathsf{A}_i = \cup_{i\in I}\mathsf{A}_i$

(b). The set $\mathcal{S}_U^r(\mathsf{E}) = \{\mathsf{A}/\mathsf{A} \subseteq E, \phi \neq \sigma_A e \subseteq \sigma_E e \text{ for all } e \in A\}$ of all regular soft subsets of E is a join complete subposet of $\mathcal{S}_U(\mathsf{E})$ with

(1). the induced partial ordering from the super set $\mathcal{S}_U(\mathsf{E})$

(2). the largest element $1_{\mathcal{S}_U^r(\mathsf{E})} = \mathsf{L}$, where $L =$ *Supp*(E) and $\sigma_L e = \sigma_E e$ for all $e \in L$, and the least element $0_{\mathcal{S}_U^r(\mathsf{E})} = \Phi$ and with

(3). for any family $(\mathsf{A}_i)_{i\in I}$ in $\mathcal{S}_U^r(\mathsf{E})$, $\vee_{i\in I}\mathsf{A}_i = \cup_{i\in I}\mathsf{A}_i$. Notice that $1_{\mathcal{S}_U^r(\mathsf{E})} \leq 1_{\mathcal{S}_U(\mathsf{E})}$ and equality holds whenever E itself is regular. Further, $\mathcal{S}_U^r(\mathsf{E})$ is a complete lattice with the join induced meet $\overline{\wedge}$. In fact,

(4). For any family $(\mathsf{A}_i)_{i\in I}$ in $\mathcal{S}_U^r(\mathsf{E})$, $\overline{\wedge}_{i\in I}\mathsf{A}_i = \mathsf{A}$, where $A = \{e \in \cap_{i\in I}A_i/\cap_{i\in I}\sigma_{A_i}e \neq \phi\}$ and $\sigma_A e = \cap_{i\in I}\sigma_{A_i}e$ for all $e \in A$

(5). However, $\mathcal{S}_U^r(\mathsf{E})$ is $\cap$-closed iff $|U| = 1$ or $|E| = 0$.

(c). The set $\mathcal{S}_U^d(\mathsf{E}) = \{\mathsf{A}/A = E, \sigma_A e \subseteq \sigma_E e \text{ for all } e \in A\}$ of all d-total soft subsets of E is a complete sublattice of $\mathcal{S}_U(\mathsf{E})$ with

(1). the induced partial ordering from the super set $\mathcal{S}_U(\mathsf{E})$

(2). the largest element $1_{\mathcal{S}_U^d(\mathsf{E})} = \mathsf{E}$ and the least element $0_{\mathcal{S}_U^d(\mathsf{E})} = \Phi_E$, null soft set (cf.2.2(c)), and with

(3). for any family $(\mathsf{A}_i)_{i\in I}$ in $\mathcal{S}_U^d(\mathsf{E})$, $\wedge_{i\in I}\mathsf{A}_i = \cap_{i\in I}\mathsf{A}_i$ and $\vee_{i\in I}\mathsf{A}_i = \cup_{i\in I}\mathsf{A}_i$

(d). The set $\mathcal{S}_U^s(\mathsf{E}) = \{\mathsf{A}/\mathsf{A} \subseteq E, \sigma_A e = \sigma_E e \text{ for all } e \in A\}$ of all strong soft subsets of E is a complete sublattice of $\mathcal{S}_U(\mathsf{E})$ with

(1). the induced partial ordering from the super set $\mathcal{S}_U(\mathsf{E})$

(2). the largest element $1_{\mathcal{S}_U^s(\mathsf{E})} = \mathsf{E}$ and the least element $0_{\mathcal{S}_U^s(\mathsf{E})} = \Phi$ and with

(3). for any family $(\mathsf{A}_i)_{i\in I}$ in $\mathcal{S}_U^s(\mathsf{E})$, $\wedge_{i\in I}\mathsf{A}_i = \cap_{i\in I}\mathsf{A}_i$ and $\vee_{i\in I}\mathsf{A}_i = \cup_{i\in I}\mathsf{A}_i$

(e). The set $\mathcal{S}_U^{d,s}(\mathsf{E}) = \{\mathsf{A}/A = E, \sigma_A e = \sigma_E e \text{ for all } e \in A\}$ of all d-total and strong soft subsets of E is $\{\mathsf{E}\}$

(f). The set $\mathcal{S}_U^{r,s}(\mathsf{E}) = \{\mathsf{A}/A \subseteq E, \phi \neq \sigma_A e = \sigma_E e \, for \, all \, e \in A\}$ *of all regular and strong soft subsets of* E *is a complete sublattice of* $\mathcal{S}_U(\mathsf{E})$ *with*

(1). the induced partial ordering from the super set $\mathcal{S}_U(\mathsf{E})$

(2). the largest element $1_{\mathcal{S}_U^{r,s}(\mathsf{E})} = \mathsf{L}$*, where* $L = Supp(\mathsf{E})$ *and* $\sigma_L e = \sigma_E e$ *for all* $e \in L$*, and the least element* $0_{\mathcal{S}_U^{r,s}(\mathsf{E})} = \Phi$ *and with*

(3). for any family $(\mathsf{A}_i)_{i\in I}$ *in* $\mathcal{S}_U^{r,s}(\mathsf{E})$*,* $\wedge_{i\in I}\mathsf{A}_i = \cap_{i\in I}\mathsf{A}_i$ *and* $\vee_{i\in I}\mathsf{A}_i = \cup_{i\in I}\mathsf{A}_i$.

Proof. Straight forward. □

The following Example shows that $\mathcal{S}_U^{r}(\mathsf{E})$ is *not* closed under intersections:

Example 4.3. *Let* $U = \{u_1, u_2\}$*,* $\mathsf{E} = (\{(e, U)\}, \{e\})$*,* $\mathsf{A}_1 = (\{(e, \{u_1\})\}, \{e\})$ *and* $\mathsf{A}_2 = (\{(e, \{u_2\})\}, \{e\})$*. Then* A_1 *and* A_2 *are regular soft subsets of* E *but* $\mathsf{A}_1 \cap \mathsf{A}_2 = \mathsf{A}$*, where* $A = \{e\}$ *and* $\sigma_A e = \{u_1\} \cap \{u_2\} = \phi$*, is not regular.*

In what follows, we study the lattice homomorphic properties of the operator definable by the extended soft subset over $\overline{U}$ for a soft subset over U.

Theorem 4.4. *For any soft set* E *over* U*, the map* $\varepsilon : \mathcal{S}_U(\mathsf{E}) \to \mathcal{S}_U(\mathsf{E})'$ *defined by for any* $\mathsf{A} \in \mathcal{S}_U(\mathsf{E})$*,* $\varepsilon\mathsf{A} = \mathsf{A}'$ *being the extended soft subset of* E' *over* $\overline{U}$*, satisfies the following properties:*

(a). The map ε *is onto*

(b). For any $\mathsf{A}, \mathsf{B} \in \mathcal{S}_U(\mathsf{E})$*,* $\mathsf{A} \subseteq \mathsf{B}$ *implies* $\varepsilon\mathsf{A} \subseteq \varepsilon\mathsf{B}$

For any family $(\mathsf{A}_i)_{i\in I}$ *in* $\mathcal{S}_U(\mathsf{E})$*,*

(c). $\varepsilon(\cup_{i\in I}\mathsf{A}_i) = \cup_{i\in I}\,\varepsilon\mathsf{A}_i$

(d). $\varepsilon(\cap_{i\in I}\mathsf{A}_i) = \cap_{i\in I}\,\varepsilon\mathsf{A}_i$

(e). However, the restricted map $\varepsilon|\mathcal{S}_U^{r}(\mathsf{E}) : \mathcal{S}_U^{r}(\mathsf{E}) \to \mathcal{S}_U(\mathsf{E})'$ *satisfies the following properties:*

(1). $\varepsilon|\mathcal{S}_U^{r}(\mathsf{E})$ *is both one-one and onto*

(2). For any $\mathsf{A}, \mathsf{B} \in \mathcal{S}_U^{r}(\mathsf{E})$*,* $\mathsf{A} \subseteq \mathsf{B}$ *implies* $\varepsilon\mathsf{A} \subseteq \varepsilon\mathsf{B}$

For any family $(\mathsf{A}_i)_{i\in I}$ *in* $\mathcal{S}_U^{r}(\mathsf{E})$*,*

(3). $\varepsilon(\cup_{i\in I}\mathsf{A}_i) = \cup_{i\in I}\,\varepsilon\mathsf{A}_i$

(4). $\varepsilon(\overline{\wedge}_{i\in I}\mathsf{A}_i) = \cap_{i\in I}\,\varepsilon\mathsf{A}_i$.

Proof. (a). is clear by the definition of map;

(b). follows from the definition of map and 2.2(e);

(c). follows from the definition of map, 2.2(k) and 2.2(j)(2);

(d). follows from the definition of map, 2.2(l) and 2.2(j)(2); and

(e). (1) follows from the definition of map, 2.2(j)(2); (2) and (3) are similar as (b) and (c) above and (4) follows from 4.2(b)(4), the definition of map and 2.2(j)(2). □

Definition 4.5. *The complete epimorphism* ε *defined as in 4.4 above is called the extention operator.*

The following Example shows that (1) the map ε is *not* one-one (2) converse of (b) in the above Theorem is *not* true:

Example 4.6. *Let U be a universal set, $\mathsf{E} = (\{(e_1, U), (e_2, U), (e_3, U)\}, \{e_1, e_2, e_3\})$, $\mathsf{A} = (\{(e_1, \phi), (e_3, U)\}, \{e_1, e_3\})$ and $\mathsf{B} = (\{(e_3, U)\}, \{e_3\})$. Then $\mathsf{A}' = (\{(e_1, \infty), (e_2, \infty), (e_3, \overline{U})\}, \{e_1, e_2, e_3\})$ and $\mathsf{B}' = (\{(e_1, \infty), (e_2, \infty), (e_3, \overline{U})\}, \{e_1, e_2, e_3\})$. Clearly, (1) $\mathsf{A}' = \mathsf{B}'$ but $\mathsf{A} \neq \mathsf{B}$ and (2) $\mathsf{A}' \subseteq \mathsf{B}'$ but $\mathsf{A} \not\subseteq \mathsf{B}$.*

Notice that Molodtsov [16] introduced the notion of soft set as a natural mathematical tool for modelling uncertainties, consisting a set E of all possible adjectives or descriptors, called parameters pertaining to a collection of objects U, called a universal set and a map σ_E from the set of all attributes E to the set of all, preferably non-empty sub collections $P(U)$ of the given collection U. Consequently, for a parameter e in E if the associated sub collection $\sigma_E e$ is empty, the parameter may as well be omitted from the parameter set E. In what follows we study some mathematical properties of this process of omission of such parameters from the parameter set E:

Theorem 4.7. *For any soft set E over U, the map $\rho : \mathcal{S}_U(\mathsf{E}) \to \mathcal{S}^r_U(\mathsf{E})$ defined by for any $\mathsf{A} \in \mathcal{S}_U(\mathsf{E})$, $\rho\mathsf{A} = \mathsf{B}$, where $B = Supp(\mathsf{A})$ and $\sigma_B e = \sigma_A e$ for all $e \in B$, satisfies the following properties:*

(a). For any $\mathsf{A} \in \mathcal{S}_U(\mathsf{E})$, $\rho\mathsf{A} \subseteq \mathsf{A}$. However, equality holds whenever A is regular

(b). The map ρ is onto

(c). For any $\mathsf{A}, \mathsf{B} \in \mathcal{S}_U(\mathsf{E})$, $\mathsf{A} \subseteq \mathsf{B}$ implies $\rho\mathsf{A} \subseteq \rho\mathsf{B}$

For any family $(\mathsf{A}_i)_{i\in I}$ in $\mathcal{S}_U(\mathsf{E})$,

(d). $\rho(\cup_{i\in I}\mathsf{A}_i) = \cup_{i\in I}\rho\mathsf{A}_i$

(e). $\rho(\cap_{i\in I}\mathsf{A}_i) = \overline{\wedge}_{i\in I}\rho\mathsf{A}_i$, where $\overline{\wedge}$ is the join induced meet in $\mathcal{S}^r_U(\mathsf{E})$.

Proof. (a) and (c) follows from the definition of map and 2.2(e); (b) is straight forward; (d) follows from 2.2(k), the definition of map and 2.2(j)(2); and (e) follows from 2.2(l), the definition of map, 4.2(b)(4) and 2.2(j)(2). ☐

Definition 4.8. *The complete epimorphism ρ defined as in 4.7 above is called the reparametrization or regularization map.*

The following Example shows that ρ is *not* one-one:

Example 4.9. *Let U be a universal set, $\mathsf{E} = (\{(e_1, U), (e_2, U)\}, \{e_1, e_2\})$, $\mathsf{A} = (\{(e_1, U)\}, \{e_1\})$ and $\mathsf{B} = (\{(e_1, U), (e_2, \phi)\}, \{e_1, e_2\})$. Then $\rho\mathsf{A} = \mathsf{A} = \rho\mathsf{B}$ but $\mathsf{A} \neq \mathsf{B}$.*

Theorem 4.10. *For any soft set E over U, the following are true:*

(a). For any $\mathsf{B} \in \mathcal{S}^d_{\overline{U}}(\mathsf{E}')$ such that $\infty \in \sigma_B b$ for all $b \in B$ there exists unique A in $\mathcal{S}^r_U(\mathsf{E})$ such that $\mathsf{B} = \mathsf{A}'$. In particular, for any $\mathsf{B} \in \mathcal{S}_U(\mathsf{E})'$ there exists unique $\mathsf{A} \in \mathcal{S}^r_U(\mathsf{E})$ such that $\mathsf{A}' = \mathsf{B}$. Consequently, $\mathcal{S}^d_U(\mathsf{E})' = \mathcal{S}^r_U(\mathsf{E})'$

(b). The operator $\nu : \mathcal{S}^d_{\overline{U}}(\mathsf{E}') \to \mathcal{S}^r_U(\mathsf{E})$ defined by $\nu(\mathsf{B}) = \mathsf{A}$, where A in $\mathcal{S}^r_U(\mathsf{E})$ is unique such that $\mathsf{A}' = \mathsf{B}$ as in (a) above, satisfies the following properties:

(1). ν is both one-one and onto

For any family $(\mathsf{B}_i)_{i\in I}$ in $\mathcal{S}_{\overline{U}}(\mathsf{E}')$,

(2). $\nu(\cup_{i\in I}\mathsf{B}_i) = \cup_{i\in I}\nu\mathsf{B}_i$

(3). $\nu(\cap_{i\in I}\mathsf{B}_i) = \overline{\wedge}_{i\in I}\nu\mathsf{B}_i$

(c). For any $B, D \in \mathcal{S}_U(E)'$, define $B \leq D$ in $\mathcal{S}_U(E)'$ iff $A \leq C$ in $\mathcal{S}_U^r(E)$, where $B = A'$ and $D = C'$. Then $\leq$ defines a partial order on $\mathcal{S}_U(E)'$.

Proof. (a) is straight forward; (b) (1) follows from 2.2(j)(2) and the definition of map, (2) follows from 2.2(k), the definition of map and 2.2(j)(2) and (3) follows from 2.2(l), the definition of map, 4.2(b)(4) and 2.2(j)(2) and (c) is straight forward. □

Theorem 4.11. *For any soft set E over U, the following are true:*

(a). The set $\mathcal{S}_U(E)' = \{A'/A \in \mathcal{S}_U(E)\}$ of all extended soft sets over $\overline{U}$ for all soft subsets of E over U is a complete lattice with

(1). the induced partial ordering from the super set $\mathcal{S}_{\overline{U}}(E')$

(2). the largest element $1_{\mathcal{S}_U(E)'} = E'$, the extended soft set for E, and the least element $0_{\mathcal{S}_U(E)} = \Phi'$, where $\phi' = E$ and $\sigma_{\phi'} = \{\infty\}$ for all $e \in \phi'$, and with

(3). for any family $(A_i')_{i\in I}$ in $\mathcal{S}_U(E)'$, $\wedge_{i\in I} A_i' = \cap_{i\in I} A_i'$ and $\vee_{i\in I} A_i' = \cup_{i\in I} A_i'$

(b). In fact, $\mathcal{S}_U(E)'$ is a complete filter of $\mathcal{S}_{\overline{U}}(E')$

(c). Further, $\mathcal{S}_U(E)' = \{B \in \mathcal{S}_{\overline{U}}^d(E')/\infty \in \sigma_B b \, for \, all b \in B\}$.

Proof. Straight forward. □

5. Representation of Soft Subsets by Products

In this section we introduce the notion of an associated product set for a (extended) soft set and study the relations between the complete lattice of all (regular) soft subsets of a soft set, the complete lattice of all factorizable subsets of the associated product set for the given soft set.

Definitions and Statements 5.1.

(a). For a set E and $\infty \in \overline{U}$, $\overline{\infty} : E \to \overline{U}$ is a map defined by $\overline{\infty}(e) = \infty$ for all $e \in E$.

(b). For any soft subset A of a soft set E, the associated product set for A, denoted by ΠA, is defined by $\Pi A = \Pi_{a\in A}\sigma_A a$.
Notice that, the associated product set ΠA for a soft set A can be empty. However, $\Pi A \neq \phi$ iff A is regular.
In particular, for any extended soft subset A' of an extended soft set E', the associated product set for A' is given by $\Pi A' = \Pi_{e\in E}\sigma_{A'} e$.
Notice that, $\Pi A'$ is always non-empty because A' is regular.

(c). For any soft subset A of a soft set E, the set of all factorizable subsets (cf.3.3(a)) of ΠA, denoted by $F(\Pi A)$, is given by $F(\Pi A) = \{X \in P(\Pi A)/\phi \neq X = \Pi_{a\in A} X_a, X_a \subseteq \sigma_A a \, for \, all a \in A\} \cup \{\phi\}$.
In particular, for any extended soft subset A' of E', the set of all factorizable subsets of $\Pi A'$ is given by $F(\Pi A') = \{X \in P(\Pi A')/\phi \neq X = \Pi_{e\in E} X_e, X_e \subseteq \sigma_{A'} e \, for \, all e \in E\} \cup \{\phi\}$.

(d). The binary relation $\preceq$ on $F(\Pi A)$ (cf.3.3(b)), given by $\preceq = \{(\phi, X)/X \in F(\Pi A)\} \cup \{(X, Y)/X \neq \phi, X_a \subseteq Y_a \, for \, all a \in A\}$, is clearly a partial order on $F(\Pi A)$ so that $(F(\Pi A), \preceq)$ is a poset.
In particular, the binary relation $\preceq$ on $F(\Pi A')$, given by $\preceq = \{(\phi, X)/X \in F(\Pi A')\} \cup \{(X, Y)/X \neq \phi, X_e \subseteq Y_e \, for \, all e \in E\}$, is clearly a partial order on $F(\Pi A')$ so that $(F(\Pi A'), \preceq)$ is a poset.

Corollary 5.2. *For any soft set* E *over* U*, the following are true:*

(a). $F(\Pi\mathsf{E}')$ *is a meet complete subposet of* $P(\Pi\mathsf{E}')$ *with* $1_{F(\Pi\mathsf{E}')} = \Pi\mathsf{E}' = 1_{P(\Pi\mathsf{E}')}$ *and* $0_{F(\Pi\mathsf{E}')} = \emptyset = 0_{P(\Pi\mathsf{E}')}$

(b). The meet complete poset $F(\Pi\mathsf{E}')$ *with the largest element* $\Pi\mathsf{E}'$ *is a complete lattice with the meet induced join* $\bar{\vee}$*, given by, for any family* $(X_j)_{j\in J}$ *in* $F(\Pi\mathsf{E}')$*, where* $X_j = \Pi_{e\in E}X_{j,e}$*,* $X_{j,e} \subseteq \sigma_{E'}e$ *for all* $e \in E$*,* $\bar{\vee}_{j\in J}X_j = \Pi_{e\in E}(\cup_{j\in J}X_{j,e})$*.*

Further, $\vee_{j\in J}X_j$ *in* $P(\Pi\mathsf{E}') \leq_{P(\Pi\mathsf{E}')} \bar{\vee}_{j\in J}X_j$ *in* $F(\Pi\mathsf{E}')$*.*

Proof. (a) follows from 3.5(a) and (b) follows from 3.5(b). □

Corollary 5.3. *For any soft set* E *over* U*, the following are true:*

(a). $\mathcal{F} = \{X \in F(\Pi\mathsf{E}')/\overline{\infty} \in X\}$ *is a complete lattice with*

(1). the partial ordering given by: for any $X, Y \in \mathcal{F}$*,* $X \leq Y$ *iff* $X_e \subseteq Y_e$ *for all* $e \in E$

(2). the largest element $1_{\mathcal{F}} = \Pi\mathsf{E}'$ *and the least element* $0_{\mathcal{F}} = \overline{\infty}$ *and with*

(3). for any family $(X_j)_{j\in J}$ *in* $\mathcal{F}$*,* $\wedge_{j\in J}X_j = \cap_{j\in J}X_j$ *and* $\vee_{j\in J}X_j = \bar{\vee}_{j\in J}X_j$*, where* $\bar{\vee}$ *is the meet induced join in* $\mathcal{F}$*. Further,* $\mathcal{F}$ *is a complete filter in* $F(\Pi\mathsf{E}')$

(b). $\mathcal{F} = \{X \in P(\Pi\mathsf{E}')/\overline{\infty} \in X\}$ *is a complete lattice with*

(1). the partial ordering given by: for any $X, Y \in \mathcal{F}$*,* $X \leq Y$ *iff* $X \subseteq Y$

(2). the largest element $1_{\mathcal{F}} = \Pi\mathsf{E}'$ *and the least element* $0_{\mathcal{F}} = \overline{\infty}$ *and with*

(3). for any family $(X_j)_{j\in J}$ *in* $\mathcal{F}$*,* $\wedge_{j\in J}X_j = \cap_{j\in J}X_j$ *and* $\vee_{j\in J}X_j = \cup_{j\in J}X_j$*. Further,* $\mathcal{F}$ *is a complete filter in* $P(\Pi\mathsf{E}')$*.*

Proof. (a) follows from 3.7(a) and (b) follows from 3.7(b). □

Definitions 5.4.

(a). The complete filter in 5.3(a) above is denoted by $(\overline{\infty})_{F(\Pi\mathsf{E}')}$ *and is called the complete filter of factorizable subsets of* $\Pi\mathsf{E}'$ *generated by* $\overline{\infty}$*.*

(b). The complete filter in 5.3(b) above is denoted by $(\overline{\infty})_{P(\Pi\mathsf{E}')}$ *and is called the complete filter of subsets of* $\Pi\mathsf{E}'$ *generated by* $\overline{\infty}$*.*

Notice that (1) $(\overline{\infty})_{F(\Pi\mathsf{E}')} \subseteq F(\Pi\mathsf{E}') \subseteq P(\Pi\mathsf{E}')$ *(2)* $(\overline{\infty})_{F(\Pi\mathsf{E}')} \subseteq (\overline{\infty})_{P(\Pi\mathsf{E}')}$*.*

Theorem 5.5. *For any soft set* E *over* U*, the maps*
$\lambda : \mathcal{S}_U(\mathsf{E})' \to (\overline{\infty})_{F(\Pi\mathsf{E}')}$ *defined by for any* $\mathsf{B} \in \mathcal{S}_U(\mathsf{E})'$*,* $\lambda\mathsf{B}=\Pi_{e\in E}\sigma_B e$ *and*
$\mu : (\overline{\infty})_{F(\Pi\mathsf{E}')} \to \mathcal{S}_U(\mathsf{E})'$ *defined by for any* $X \in (\overline{\infty})_{F(\Pi\mathsf{E}')}$*,* $\mu X = \mathsf{B}$*, where* $B = E$ *and* $\sigma_B e = X_e$ *for all* $e \in B$*,*
satisfy the following properties:

(a). $\lambda \circ \mu = 1_{(\overline{\infty})_{F(\Pi\mathsf{E}')}}$

(b). $\mu \circ \lambda = 1_{\mathcal{S}_U(\mathsf{E})'}$

(c). λ*,* μ *both are one-one and onto*

(d). For any $\mathsf{A}', \mathsf{B}' \in \mathcal{S}_U(\mathsf{E})'$*,* $\mathsf{A}' \subseteq \mathsf{B}'$ *implies* $\lambda\mathsf{A}' \leq \lambda\mathsf{B}'$

(e). For any $X, Y \in (\overline{\infty})_{F(\Pi E')}$, $X \leq Y$ implies $\mu X \subseteq \mu Y$

For any $(A'_i)_{i\in I}$ in $\mathcal{S}_U(E)'$,

(f). $\lambda(\cap_{i\in I} A'_i) = \cap_{i\in I} \lambda A'_i$

(g). $\lambda(\cup_{i\in I} A'_i) = \overline{\nabla}_{i\in I} \lambda A'_i$

For any $(X_i)_{i\in I}$ in $(\overline{\infty})_{F(\Pi E')}$,

(h). $\mu(\cap_{i\in I} X_i) = \cap_{i\in I} \mu X_i$

(i). $\mu(\overline{\nabla}_{i\in I} X_i) = \cup_{i\in I} \mu X_i$.

Proof. (a) and (b) are straight forward; (c) follows from 3.1(a), the definitions of λ, μ and (a), (b) above; (d) and (e) are straight forward; (f) follows from 2.2(l), the definition of λ and 3.1(a); (g) follows from 2.2(k), the definition of λ, 3.5(b) and 3.1(a); (h) follows from 3.1(b), the definition of μ and 2.2(j)(2) and (i) follows from 3.5(b), the definition of μ and 2.2(j)(2). $\square$

The following Example shows that $\cup_{i\in I}\lambda A'_i$ need *not* be in $(\overline{\infty})_{F(\Pi E')}$ and a strict containment can hold in $\cup_{i\in I}\lambda A' \subsetneq \lambda(\cup_{i\in I} A')$:

Example 5.6. *Let $\overline{U} = \{u_1, u_2, \infty\}$, $E' = (\{(e_1, \overline{U}), (e_2, \overline{U})\}, \{e_1, e_2\})$, $A'_1 = (\{(e_1, \{u_1, \infty\}), (e_2, \{u_2, \infty\})\}, \{e_1, e_2\})$, $A'_2 = (\{(e_1, \{u_2, \infty\}), (e_2, \{u_1, \infty\})\}, \{e_1, e_2\})$. Then $A'_1 \cup A'_2 = E'$, $\lambda(A'_1 \cup A'_2) = \overline{U} \times \overline{U}$, $\lambda A'_1 = \{u_1, \infty\} \times \{u_2, \infty\}$, $\lambda A'_2 = \{u_2, \infty\} \times \{u_1, \infty\}$ and $\lambda A'_1 \cup \lambda A'_2 = \{(u_1, u_2), (u_1, \infty), (\infty, u_2), (\infty, \infty), (u_2, u_1), (u_2, \infty), (\infty, u_1)\}$. Clearly, $\lambda A'_1 \cup \lambda A'_2 \notin (\overline{\infty})_{F(\Pi E')}$ because it not factorizable and $\lambda A'_1 \cup \lambda A'_2 \subset \lambda(A'_1 \cup A'_2)$. Further, $\lambda A'_1 \overline{\nabla} \lambda A'_2 = (\{u_1, \infty\} \cup \{u_2, \infty\}) \times (\{u_2, \infty\} \cup \{u_1, \infty\}) = \{u_1, u_2, \infty\} \times \{u_1, u_2, \infty\} = \overline{U} \times \overline{U} = \lambda(A'_1 \cup A'_2)$.*

The following Example shows that $\cup_{i\in I} X_i$ need *not* be in $(\overline{\infty})_{F(\Pi E')}$ so that $\mu(\cup_{i\in I} X_i)$ is *not* defined.

Example 5.7. *Let $\overline{U} = \{u_1, u_2, \infty\}$, $E' = (\{(e_1, \overline{U}), (e_2, \overline{U})\}, \{e_1, e_2\})$, $X_1 = \{u_1, \infty\} \times \{u_2, \infty\}$, $X_2 = \{u_2, \infty\} \times \{u_1, \infty\}$. Then $\Pi E' = \overline{U} \times \overline{U}$, $\mu X_1 = A'_1 = (\{(e_1, \{u_1, \infty\}), (e_2, \{u_2, \infty\})\}, \{e_1, e_2\})$, $\mu X_2 = A'_2 = (\{(e_1, \{u_2, \infty\}), (e_2, \{u_1, \infty\})\}, \{e_1, e_2\})$, $\mu X_1 \cup \mu X_2 = A'_1 \cup A'_2 = E'$, $X_1 \cup X_2 = \{(u_1, u_2), (u_1, \infty), (\infty, u_2), (\infty, \infty)\} \cup \{(u_2, u_1), (u_2, \infty), (\infty, u_1), (\infty, \infty)\} = \{(u_1, u_2), (u_1, \infty), (\infty, u_2), (\infty, \infty), (u_2, u_1), (u_2, \infty), (\infty, u_1)\}$. Clearly, $X_1 \cup X_2 \notin (\overline{\infty})_{F(\Pi E')}$ because it is not factorizable so $\mu(X_1 \cup X_2)$ is not defined. Further, $X_1 \overline{\nabla} X_2 = (\{u_1, \infty\} \cup \{u_2, \infty\}) \times (\{u_2, \infty\} \cup \{u_1, \infty\}) = \{u_1, u_2, \infty\} \times \{u_1, u_2, \infty\} = \overline{U} \times \overline{U}$. Now $\mu(X_1 \overline{\nabla} X_2) = C' = (\{(e_1, \overline{U}), (e_2, \overline{U})\}, \{e_1, e_2\}) = E' = \mu X_1 \cup \mu X_2$.*

Theorem 5.8. *For any soft set E over U, the map*
$\lambda \circ \varepsilon : \mathcal{S}_U(E) \to (\overline{\infty})_{F(\Pi E')}$ defined by for any $A \in \mathcal{S}_U(E)$, $(\lambda \circ \varepsilon)A = \lambda(\varepsilon A) = \lambda(A') = \Pi_{e\in E}\sigma_{A'}e$, satisfies the following properties:

(a). $\lambda \circ \varepsilon$ is onto

For any $(A_i)_{i\in I}$ in $\mathcal{S}_U(E)$,

(b). $(\lambda \circ \varepsilon)(\cap_{i\in I} A_i) = \cap_{i\in I} (\lambda \circ \varepsilon)A_i$

(c). $(\lambda \circ \varepsilon)(\cup_{i\in I} A_i) = \overline{\nabla}_{i\in I} (\lambda \circ \varepsilon)A_i$.

Proof. (a) follows from 4.4(a) and 5.5(c); (b) follows from 4.4(d) and 5.5(f) and (c) follows from 4.4(c) and 5.5(g). $\square$

Theorem 5.9. *For any soft set E over U, the map*
$\lambda \circ \varepsilon : \mathcal{S}^r_U(E) \to (\overline{\infty})_{F(\Pi E')}$ defined by for any $A \in \mathcal{S}^r_U(E)$, $(\lambda \circ \varepsilon)A = \lambda(\varepsilon A) = \lambda(A') = \Pi_{e\in E}\sigma_{A'}e$, satisfies the following properties:

(a). $\lambda \circ \varepsilon$ *is both one-one and onto*

For any $(A_i)_{i\in I}$ *in* $\mathcal{S}_U^r(E)$,

(b). $(\lambda \circ \varepsilon)(\overline{\wedge}_{i\in I} A_i) = \cap_{i\in I} (\lambda \circ \varepsilon)A_i$

(c). $(\lambda \circ \varepsilon)(\cup_{i\in I} A_i) = \overline{\vee}_{i\in I} (\lambda \circ \varepsilon)A_i$.

Proof. (a) follows from 4.4(e)(1) and 5.5(c); (b) follows from 4.4(e)(4) and 5.5(f) and (c) follows from 4.4(e)(3) and 5.5(g). □

Corollary 5.10. *For any soft set* E *over* U,

(a). there is a crisp set P *such that the complete lattice of all soft subsets of* E *is complete epimorphic to a complete lattice of certain subsets of* P*, where the join in the later is the meet induced join*

(b). there is a crisp set P *such that the complete lattice of all regular soft subsets of* E *is complete isomorphic to a complete lattice of certain subsets of* P*, where the meet in the former is the join induced meet and the join in the later is the meet induced join.*

Proof. (a) follows from 5.8 and (b) follows from 5.9. □

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