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Fixed Point Theorem for Occasionally Weakly Compatible Mappings With Common Limit Range Property Satisfying Generalized (ψ, ϕ)-Weak Contractive Conditions

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Abstract: In this paper, we prove some common fixed point theorems for occasionally weakly compatible mappings in metric spaces satisfying generalized (ψ, ϕ)-contractive conditions under the common limit range property. We present fixed point theorem for four finite families of self-mappings which can be utilized to derive common fixed point theorems involving any number of finite mappings. Our results improve and extend the corresponding results of Mohammad Imdad, Sunny Chauhan and ZoranKadelburg(Mathematical Sciences 7(16): 1-8, 2013).

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1. Introduction

The famous Banach Contraction Principle which is also referred as the Banach fixed point theorem continues to be a very popular and powerful tool in solving existence problems in pure and applied sciences which include biology, medicine, physics, and computer science. It evidently plays a crucial role in nonlinear analysis. This theorem states that if (X, d) is a complete metric space and $T : X \to X$ is contraction mapping, i.e.,

$$d(Tx,\ Ty) \leq kd\,(x, y)$$

for all $x, y \in X$, and k is a non-negative real number such that $k < 1$, then T has a unique fixed point in X. Moreover, this fixed point can be explicitly obtained as a limit of repeated iteration of the mapping, initiating at any point of the underlying space. Obviously, every contraction is a continuous function but not conversely. Many mathematicians (e.g., [5, 9, 10, 23–25]) proved several fixed point theorems to explore some new contraction-type mappings in order to generalize the classical Banach Contraction Principle. The concept of weak contraction was introduced by Alber and Guerre-Delabriere [4] in 1997, wherein the authors introduced the following notion for mappings defined on a Hilbert space X. Consider the following set of real functions $\Phi = \{\phi : [0, +\infty) \to [0, +\infty) : \phi$ is lower semi-continuous and $\phi^{-1}(\{0\}) = \{0\}\}$. A mapping $T : X \to X$ is called a ϕ-weak contraction if there exists a function $\phi \in \Phi$ such that $d(Tx,\ Ty) \leq d\,(x, y) - \phi(d(x, y))$, for

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all $x, y \in X$. Alber and Guerre-Delabriere [4] also showed that each ϕ-weak contraction on Hilbert space has a unique fixed point. Thereafter, Rhoades [30] showed that the results contained in [4] are also valid for any Banach Contraction Principle which follows in case one chooses $\phi(t) = (1-k)t$. Zhang and Song proved a common fixed point theorem for two mappings using ϕ-weak contraction . This result was extended by Doric [11] and Dutta and Choudhury [13] to a pair of (ψ, ϕ)-weak contractive mappings. However, the main fixed point theorem for a self-mapping satisfying (ψ, ϕ)-weak contractive condition contained in Dutta and Choudhury [13] runs as follows:

Let us consider the following set of real functions:

$$\Psi = \left\{\psi : [0, +\infty) \to [0, +\infty) : \psi \text{ is continuous non-decreasing and } \psi^{-1}(\{0\}) = \{0\}\right\}.$$

2. Preliminaries

Theorem 2.1. *Let (X, d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying $\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y))$, for some $\psi \in \Psi$ and $\phi \in \Phi$ and all $x, y \in X$. Then, T has a unique fixed point in X.*

In recent years, many researchers utilized (ψ, ϕ)-weak contractive conditions to prove a number of metrical fixed point theorems (e.g., [2, 3, 6–8, 12, 27, 29]). In an important paper, Jachymski [18] showed that some of the results involving two functions $\psi \in \Psi$ and $\phi \in \Phi$ can be reduced to one function $\phi' \in \Phi$. Popescu [26] proved a fixed point theorem in a complete metric space and showed that the conditions on functions ψ and ϕ can be weakened. His result improved the corresponding results of Dutta and Choudhury [13] and Doric [11]. A common fixed point result generally involves conditions on commutativity, continuity, and contraction along with a suitable condition on the containment of range of one mapping into the range of the other. Hence, one is always required to improve one or more of these conditions in order to prove a new common fixed point theorem. It can be observed that in the case of two mappings $A, S : X \to X$, one can consider the following classes of mappings for the existence and uniqueness of common fixed points:

$$d(Ax, Ay) \leq F(m(x, y)), \tag{1}$$

Where F is some function and m(x, y) is the maximum of one of the sets:

$$M^5_{A,S}(x, y) = \{d(Sx, Sy), d(Sx, Ax), d(Sy, Ay), d(Sx, Ay), d(Sy, Ax)\},$$
$$M^4_{A,S}(x, y) = \{d(Sx, Sy), d(Sx, Ax), d(Sy, Ay), \frac{1}{2}(d(Sx, Ay) + d(Sy, Ax))\},$$
$$M^3_{A,S}(x, y) = \{d(Sx, Sy), \frac{1}{2}(d(Sx, Ax) + d(Sy, Ay)), \frac{1}{2}(d(Sx, Ay) + d(Sy, Ax))\}$$

A further possible generalization is to consider four mappings instead of two and ascertain analogous common fixed point theorems. In the case of four mappings $A, B, S, T : X \to X$, the corresponding sets take the form

$$M^5_{A,B,S,T}(x, y) = \{d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)\},$$
$$M^4_{A,B,S,T}(x, y) = \{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}(d(Sx, By) + d(Ty, Ax))\},$$
$$M^3_{A,B,S,T}(x, y) = \{d(Sx, Ty), \frac{1}{2}(d(Sx, Ax) + d(Ty, By)), \frac{1}{2}(d(Sx, By) + d(Ty, Ax))\}.$$

In this case (1) in usually replaced by

$$d(Ax, By) \leq F(m(x, y)), \tag{2}$$

Where $m(x,y)$ is the maximum of one of the M sets. Similarly, we can define the M sets for six mappings $A,B,P,Q,S,T: X\to X$ as

$$M^5_{A,B,P,Q,S,T}(x,y)=\{d(SQx,TPy),d(SQx,Ax),d(TPy,By),d(SQx,By),d(TPy,Ax)\}, \tag{3}$$
$$M^4_{A,B,P,Q,S,T}(x,y)=\{d(SQx,TPy),d(SQx,Ax),d(TPy,By),\frac{1}{2}(d(SQx,By)+d(TPy,Ax))\},$$
$$M^3_{A,B,P,Q,S,T}(x,y)=\{d(SQx,TPy),\frac{1}{2}(d(SQx,Ax)+d(TPy,By)),\frac{1}{2}(d(SQx,By)+d(TPy,Ax))\}.$$

and the contractive condition is again in the form (2). Using different arguments of control functions. Radenovic [28] proved some common fixed point results for two and three mappings using (ψ,ϕ)-weak contractive conditions and improved several known metrical fixed point theorems. Motivated by these results, we prove some common fixed point theorems for two pairs of occasionally weakly compatible mappings with common limit range property satisfying generalized (ψ,ϕ)-weak contractive conditions.Many known fixed point results are improved, especially the ones proved in [28] and also contained in the references cited therein. We also obtain a fixed point theorem for four finite families of self-mappings. Some related results are also derived besides furnishing illustrative examples.

Definition 2.2. *Let A and S be two mappings from a metric space (X,d) into itself. Then, the mappings are said to*

(1). be commuting if $ASx=SAx$ for all $x\in X$,

(2). be compatible [19] if $\lim_{n\to\infty} d(ASx_n,SAx_n)=0$ for each sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n$,

(3). be non-compatible [25] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n$ but $\lim_{n\to\infty} d(ASx_n,SAx_n)$ is either non zero or nonexistent,

(4). be weakly compatible [20] if they commute at their coincidence points, that is, $ASx=SAx$ whenever $Ax=Sx$, for some $x\in X$,

(5). be occasionally weakly compatible(owc) [21] if and if there is a point $z\in X$ which is a coincidence point of A and S at which A and S commute. i.e., there exists a point $z\in X$ such that $Az=Sz$ and $ASz=SAz$.

(6). satisfy the property (E.A) [1] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=t$, for some $t\in X$.

For further details, comparisons, and illustrations on systematic spaces, we refer to Singh and Tomar [31] and Murthy[23]. Any pair of compatible as well as non-compatible self mappings of a metric space (X,d) satisfies the property (E.A), but a pair of mappings satisfying the property (E.A) need not be non-compatible (see Example 1 of [14]). In 2005, Liu [22] defined the notion of common property (E.A) for hybrid pairs of mappings, which contain the property (E.A).

Definition 2.3 ([22])**.** *Two pairs (A,S) and (B,T) of self-mappings of a metric space (X,d) are said to satisfy the common property (E.A) if two sequences $\{x_n\}$ and $\{y_n\}$ in X exist such that $\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} By_n=\lim_{n\to\infty} Ty_n=t$, for some $t\in X$.*

In can be observed that the fixed point results usually require closedness of the underlying subspaces for the existence of common fixed points under the property (E.A) and common property (E.A). In 2011, Sintunavarat and Kumam [32] coined the idea of common limit range property' (see also [33]). Most recently, Imdad [17] extended the notion of common limit range propertyto two pairs of self-mappings which relax the closedness requirements of the underlying subspaces.

Definition 2.4 ([33]). *A pair (A, S) of self-mappings of a metric space (X, d) is said to satisfy the common limit range property with respect to S, denoted by (CLR_S), if the exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in S(X)$. Thus, one can infer that a pair (A, S) satisfying the property (E.A) along with the closedness of the subspaces $S(X)$ always enjoys the (CLR_S) property with respect to the mapping S (see Examples 2.16-2.17 of [17]).*

Definition 2.5 ([17]). *Two pairs (A, S) and (B, T) of self-mappings of a metric space (X, d) are said to satisfy the common limit range property with respect to mappings S and T, denoted by (CLR_{ST}) if two sequences $\{x_n\}$ and $\{y_n\}$ in X exist such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$, for some $t \in S(X) \cap T(X)$.*

Definition 2.6 ([15]). *Two families of self mappings $\{A_i\}_{i=1}^m$ and $\{S_k\}_{k=1}^n$ are said to be pairwise commuting if*

(1). $A_iA_j = A_jA_i$ for all $i, j \in \{1, 2, \dots, m\}$,

(2). $S_kS_l = S_lS_k$ for all $k, l \in \{1, 2, \dots, n\}$,

(3). $A_iS_k = S_kA_i$ for all $i \in \{1, 2, \dots, m\}$ and $k \in \{1, 2, \dots, n\}$.

Definition 2.7 ([21]). *Let X be a set, S and T be occasionally weakly compatible(owc) self maps on X. If S and T have a unique point of coincidence $w = Sx = Tx$ for $x \in X$, then w is the unique common fixed point of S and T.*

3. Main Results

Now, we state and prove our main results for six mappings employing the common limit range property in metric space. Firstly, we prove the following lemma.

Lemma 3.1. *Let A, B, P, Q, S and T be self-mappings of a metric space (X, d) Suppose that.*

(i). The pair (P, AB) satisfies the (CLR_{AB}) property (respectively (Q, ST) satisfies the (CLR_{ST}) property).

(ii). $P(X) \subset ST(X)$ (respectively $Q(X) \subset AB(X)$),

(iii). $ST(X)$ (respectively $AB(X)$) is a closed subset of X,

(iv). $\{Qy_n\}$ converges for every sequence $\{y_n\}$ in X whenever $\{STy_n\}$ converges (respectively $\{Px_n\}$ converges for every sequence $\{x_n\}$ in X whenever $\{ABx_n\}$ converges),

(v). There exist $\phi \in \Phi$ and $\psi \in \Psi$ such that.

(vi). $\psi(d(Px, Qy)) \leq \psi(m(x, y)) - \phi(m(x, y))$, for all $x, y \in X$, where $m(x, y) = \max M^5_{P,Q,AB,ST}(x, y)$

Then, the pairs (P, AB) and (Q, ST) share the (CLR_{AB}) property.

Proof. Since the pair (P, AB) satisfies the (CLR_{AB}) property, a sequence $\{x_n\}$ in X exists such that $\lim_{n\to\infty} Px_n = \lim_{n\to\infty} ABx_n = t$ where $t \in AB(X)$. By Lemma 3.1 item (ii), $P(X) \subset ST(X)$, and for each sequence $\{x_n\}$, there exist a sequence $\{y_n\}$ in X such that $Px_n = STy_n$. Therefore, due to the closedness of $ST(X)$, $\lim_{n\to\infty} STy_n = \lim_{n\to\infty} Px_n = t$, so that $t \in ST(X)$ and in all $t \in AB(X) \cap ST(X)$. Thus, we have $Px_n \to t$, $ABx_n \to t$ and $STy_n \to t$ as $n \to \infty$. By Lemma 3.1 item (iv), the sequence $\{Qy_n\}$ converges, and in all, we need to show that $Qy_n \to t$ as $n \to \infty$. Let, on the contrary that $Qy_n \to z(\neq t)$ as $\to \infty$. On using inequality (vi) with $x = x_n$, $y = y_n$, we have

$$\psi(d(Px_n, Qy_n)) \leq \psi(m(x_n, y_n)) - \phi(m(x_n, y_n)), \tag{4}$$

Where

$$m\left(x_n, y_n\right)=\max \left\{d\left(ABx_n, STy_n\right), d\left(ABx_n, Px_n\right), d\left(STy_n, Qy_n\right), d\left(ABx_n, Qy_n\right), d(STy_n, Px_n)\right\}.$$

Taking the limit as $n \rightarrow \infty$ in (4), we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi(d\left(Px_n, Qy_n\right)) &\leq \lim_{n \rightarrow \infty} \psi\left(m\left(x_n, y_n\right)\right)-\lim_{n \rightarrow \infty} \phi(m\left(x_n, y_n\right)) \\ \lim_{n \rightarrow \infty} \psi(d\left(t, z\right)) &\leq \psi(\lim_{n \rightarrow \infty}\left(m\left(x_n, y_n\right)\right)-\phi(\lim_{n \rightarrow \infty} m\left(x_n, y_n\right)) \end{aligned} \tag{5}$$

Where.

$$\begin{aligned} \lim_{n \rightarrow \infty} m(x_n, y_n) &= \max \left\{d\left(t, t\right), d\left(t, t\right), d\left(t, z\right), d\left(t, z\right), d(t, t)\right\} \\ &= \max\{0, 0, d(t, z), d(t, z), 0\} \\ &= d(t, z) \end{aligned}$$

From (5) we obtain $\psi(d\left(t, z\right)) \leq \psi\left(d\left(t, z\right)\right)-f(d(t, z))$, so that $f\left(d\left(t, z\right)\right)=0$, so that $d\left(t, z\right)=0$ i.e., $t=z$ which is a contradiction. Hence $Qy_n \rightarrow t$ which shows that the pairs (P, AB) and (Q, ST) share the $\left(CLR_{(AB)(ST)}\right)$ property. This concludes the proof. □

Theorem 3.2. *Let P, Q, A, B ,S and T be self-mappings of a metric space (X, d) satisfying the inequality (vi) of Lemma 3.1. If the pairs (P, AB) and (Q, ST) satisfy the $\left(CLR_{(AB)(ST)}\right)$ property, then (P, AB) and (Q, ST) have a coincidence point each. Moreover both the pairs (P, AB) and (Q, ST) are occasionally weakly compatible, then P, Q, AB and ST have a unique common fixed point. Further if (A, B), (S, T), (A, P) and (S, Q) are commuting maps then A, B, S, T, P and Q have a unique common fixed point.*

Proof. If the pair (P, AB) and (Q, ST) enjoy the $\left(CLR_{(AB)(ST)}\right)$ property, then two sequences $\{x_n\}$ and $\{y_n\}$ in X exist such that $\lim_{n \rightarrow \infty} Px_n=\lim_{n \rightarrow \infty} ABx_n=\lim_{n \rightarrow \infty} Qy_n=\lim_{n \rightarrow \infty} STy_n=t$, where $t \in AB(X) \cap ST(X)$. Since $t \in AB(X)$, a point $u \in X$ exists such that $ABu=t$. We assert that $Pu=ABu$ using Lemma 3.1(vi) with $x=u$, $y=y_n$ we get

$$\psi(d(Pu, Qy_n) \leq \psi\left(m\left(u, y_n\right)\right)-\phi(m\left(u, y_n\right)), \tag{6}$$

Where

$$m\left(u, y_n\right)=\max \left\{d\left(ABu, STy_n\right), d\left(ABu, Pu\right), d\left(STy_n, Qy_n\right), d\left(ABu, Qy_n\right), d(STy_n, Pu)\right\}.$$

Taking the limit as $n \rightarrow \infty$ in (6), we get.

$$\begin{aligned} \lim_{n \rightarrow \infty} \psi(d\left(Pu, Qy_n\right)) &\leq \lim_{n \rightarrow \infty} \psi\left(m\left(u, y_n\right)\right)-\lim_{n \rightarrow \infty} \phi(m\left(u, y_n\right)) \\ \lim_{n \rightarrow \infty} \psi(d\left(Pu, t\right)) &\leq \psi(\lim_{n \rightarrow \infty} m\left(u, y_n\right))-\phi(\lim_{n \rightarrow \infty} m\left(u, y_n\right)) \end{aligned}$$

Where

$$\begin{aligned} \lim_{n \rightarrow \infty} m\left(u, y_n\right) &= \max \left\{d\left(t, t\right), d\left(t, Pu\right), d\left(t, t\right), d\left(t, t\right), d(t, Pu)\right\} \\ &= \max\{0, d(t, Pu), 0, 0, d(t, Pu)\} \\ &= d(Pu, t), \end{aligned}$$

Which in turn yields

$$\psi(d(Pu,t)) \leq \psi(d(Pu,t)) - \phi(d(Pu,t)),$$

So that $\phi(d(Pu,t)) = 0$ i.e., $(Pu,t) = 0$. Hence $Pu = ABu = t$. Therefore, u is a coincidence point of the pair (P, AB). As $t \in ST(X)$, there exists a point $v \in X$ such that $STv = t$. We show that $Qv = STv$. Using Lemma 3.1(vi) with $x = u$, $y = v$, we get.

$$\psi(d(t,Qv) = \psi(d(Pu,Qv)) \leq \psi(m(u,v)) - \phi(m(u,v)), \tag{7}$$

Where

$$\begin{aligned} m(u,v) &= \max\{d(ABu,STv), d(ABu,Pu), d(STv,Qv), d(ABu,Qv), d(STv,Pu)\}. \\ &= \max\{d(t,t), d(t,t), d(t,Qv), d(t,Qv), d(t,t)\} \\ &= \max\{0,0,d(t,Qv), d(t,Qv), 0\} \\ &= d(t,Qv). \end{aligned}$$

Which in turn yields.

$$\psi(d(t,Qv)) \leq \psi(d(t,Qv)) - \phi(d(t,Qv)),$$

So that $\phi(d(t,Qv)) = 0$ i.e., $d(t,Qv) = 0$. Hence, $Qv = STv = t$, which shows that v is a coincidence point of the pair (Q, ST). Since the pair (P, AB) are occasionally weakly compatible so by definition there exists a point $u \in X$ such that $Pu = ABu$ and $P(AB)u = (AB)Pu$. Since the pair (Q, ST) are occasionally weakly compatible so by definition there exists a point $v \in X$ such that $Qv = STv$ and $Q(ST)v = (ST)Qv$. Moreover, if there is another point z such that $Pz = ABz$, then, using Lemma 3.1(vi) it follows that $Pz = ABz = Qv = STv$, or $Pu = Pz$ and $w = Pu = ABu$ is unique point of coincidence of P and AB. By Lemma 2.7, w is the unique common fixed point of P and AB. i.e., $w = Pw = ABw$. Similarly there is a unique point $z \in X$ such that $z = Qz = STz$.

Uniqueness: Suppose that $w \neq z$. Using Lemma 3.1(vi) with $x = w$, $y = z$, we get

$$\psi(d(Pw,Qz)) \leq \psi(m(w,z)) - \phi(m(w,z))$$

Where

$$\begin{aligned} m(w,z) &= \max\{d(ABw,STz), d(ABw,Pw), d(STz,Qz), d(ABw,Qz), d(STz,Pw)\} \\ &= \max\{d(w,z), d(w,w), d(z,z), d(w,z), d(z,w)\}. \\ &= \max\{d(w,z), 0, 0, d(w,z), d(z,w)\} \\ &= d(w,z). \end{aligned}$$

which in turn yields

$$\psi(d(w,z)) \leq \psi(d(w,z)) - \phi(d(w,z))$$

So that $f(d(w,z)) = 0$ i.e.,$d(w,z) = 0$. Hence $z = w$. Hence, z is a unique common fixed point of the mappings P, Q, AB and ST. Finally we need to show that z is a common fixed point of A, B, P, Q, S and T. Since (A, B), (A, P) are commutative $Az = A(ABz) = A(BAz) = (AB)Az$; $Az = APz = PAz$; $Bz = B(ABz) = (BA)Bz = (AB)Bz$; $Bz = BPz = PBz$. Which shows that Az, Bz are common fixed point of (AB, P) yielding then by $Az = z = Bz = Pz = ABz$ in the view of uniqueness of common fixed point of the pairs (P, AB). Similarly using the commutativity of (S, T) and (S, Q) it can be shown that $Sz = z = Tz = Qz = Az = Bz = Pz$. Which shows that z is a common fixed point of A, B, P, Q, S and T. We can easily prove the uniqueness of z from Lemma 3.1(vi). □

Remark 3.3. *Theorem 3.2 improves the relevant results of Radenovic [28] as the requirements on the closedness and containment among the ranges of the involved mappings are not needed.*

Now, we furnish an illustrative example which demonstrates the validity of the hypotheses and degree of generality of our main result over comparable ones from the existing literature.

Example 3.4. *Consider X= [3, 14) equipped with the usual metric. Define the self mappings P, Q, A, B, S and T by*

$$Px = \begin{cases} 3, & if\ x \in \{3\} \cup (8,14), \\ 11, & if\ x \in (3,8]; \end{cases} \qquad Qx = \begin{cases} 3, & if\ x \in \{3\} \cup (8,14), \\ 5, & if\ x \in (3,8]; \end{cases}$$

$$Ax = \begin{cases} 3, & if\ x = 3, \\ 13, & if\ x \in (3,8], \\ \frac{x+1}{3}, & if\ x \in (8,14); \end{cases} \qquad Sx = \begin{cases} 3, & if\ x = 3, \\ 10, & if\ x \in (3,8], \\ x-5, & if\ x \in (8,14); \end{cases}$$

$Bx = x \;\; \forall\ x \in [3,14)$ and $Tx = x \;\; \forall\ x \in [3,14)$. Consider two sequences $\{x_n\} = \{8 + \frac{1}{n}\}_{n\in\mathbb{N}}$, $\{y_n\} = \{3\}$ $\left(\{x_n\} = \{3\}, \{y_n\} = \{8 + \frac{1}{n}\}_{n\in\mathbb{N}}\right)$. The pairs (P, AB) and (Q, ST) satisfy the $\left(CLR_{(AB)(ST)}\right)$ property:

$$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} ABx_n = \lim_{n\to\infty} Qy_n = \lim_{n\to\infty} STy_n = 3 \in AB(X) \cap ST(X).$$

Also, $P(X) = \{3, 11\} \nsubseteq [3, 11) = ST(X)$ and $Q(X) = \{3, 5\} \nsubseteq [3, 5) \cup \{13\} = AB(X)$. Take $\psi \in \Psi$ and $\phi \in \Phi$ given by $\psi(t) = 2t$, $\phi(t) = \frac{2}{7}t$. In order to check the contractive condition Lemma 3.1(vi), consider the following nine Cases: (I). $x = y = 3$; (II). $x = 3$, $y \in (3,8]$; (III). $x = 3$, $y \in (8,14)$; (IV). $x \in (3,8]$, $y = 3$; (V). $x, y \in (3,8]$; (VI). $x \in (3,8]$, $y \in (8,14)$; (VII). $x \in (8,14)$, $y = 2$; (VIII). $x \in (8,14)$, $y \in (3,8]$; (IX). $x, y \in (8,14)$. In the cases I, III, VII, and IX, we get that $d(Px, Qy) = 0$; Lemma 3.1(vi) is trivially satisfied. In the cases II and VIII, it is $d(Px, Qy) = 2$ and $m(x, y) = 7$; so, Lemma 3.1(vi) reduces to

$$\psi(2) = 4 \leq 12 = \psi(7) - \phi(7).$$

In the case IV we get that $d(Px, Qy) = 8$ and $m(x, y) = 10$, and again we have

$$\psi(8) = 16 \leq \frac{120}{7} = \psi(10) - \phi(10).$$

In the case VI we get that $d(Px, Qy) = 8$ and $m(x, y) = 15$, and again we have

$$\psi(8) = 16 \leq \frac{180}{7} = \psi(15) - \phi(15).$$

(P,AB) and (Q,ST) are OWC. Hence, all the conditions of Theorem 3.2 are satisfied, and 3 is a unique common fixed point of the pairs (P, AB) and (Q, ST) which also remains a point of coincidence. Here, one may notice that the involved mappings P, Q, A, S are discontinuous at their unique common fixed point 3. However, notice that the subspaces AB(X) and ST(X) are not closed subspaces of X, and required inclusions among the ranges of the involved maps do not hold. Therefore, the results of Radenovic [28] cannot be used in the context of this example which establishes the genuineness of our extension.

In view of Theorem 3.2 and Lemma 3.1, the following corollary is immediate.

Corollary 3.5. *Let P, Q, A, B, S and T be self-mappings of a metric space (X,d) satisfying all the hypotheses of Lemma 3.1. Then P, Q, AB and ST have a unique common fixed point, provided that both the pairs (P, AB) and (Q, ST) are weakly compatible.*

Proof. Owing to Lemma 3.1, it follows that the pairs (P, AB) and (Q,ST) enjoy the $\left(CLR_{(AB)(ST)}\right)$ property. Hence, all the conditions of Theorem 3.1 are satisfied; P, Q, AB and ST have a unique common fixed point provided that both the pairs (P, AB) and (Q,ST) are occasionally weakly compatible. Here, it is worth noting that the conclusions in Example 3.4 cannot be obtained using Corollary 3.5 as conditions (i) and (ii) of Lemma 3.1 are not fulfilled. In what follows, we present another example which creates a situation wherein a conclusion can be reached using Corollary 3.5. □

Example 3.6. *In the setting of Example 3.4, replace the self-mappings A and S by the following, retaining the rest:*

$$Ax = \begin{cases} 3, & if \quad x = 3, \\ 6, & if \ x \ \in (3,8]\,, \\ \frac{x-2}{2}, & if \ x \in (8,14)\,; \end{cases} \qquad Sx = \begin{cases} 3, & if \quad x = 3, \\ 11, & if \ x \ \in (3,8]\,, \\ x-5, & if \ x \in (8,14)\,. \end{cases}$$

Then, as in the earlier example, the pairs (P,AB) and (Q,ST) satisfy the $\left(CLR_{(AB)(ST)}\right)$ property. Moreover, inequality Lemma 3.1(vi) can be verified as earlier. Also, as earlier define,

$$\psi\,(t) = 2t, \quad \phi\,(t) = \frac{2}{7}t.$$

Hence, $P(X) = \{3, 11\} \subset [3, 11] = ST(X)$ and $Q(X)$ =$\{3, 5\} \subset [3, 6] = AB(X)$ holds. Thus, all the conditions of Corollary 3.5 are satisfied and 3 is a unique common fixed point of the involved mappings P, Q, AB and ST.

Remark 3.7. *The conclusions of Lemma 3.1, Theorem 3.2 and Corollary 3.5 remain true if we choose $m\,(x, y) = \max M^4_{P,Q,AB,ST}\,(x, y)$ or $m\,(x, y) = \max M^3_{P,Q,AB,ST}(x, y)$. By setting P, Q, A, B, S and T suitably, we can deduce corollaries involving two as well as three self-mappings. As a sample, we can deduce the following corollary involving two self-mappings:*

Corollary 3.8. *Let P and A be self-mappings of a metric space (X,d). Suppose that*

1. *The pair (P,A) satisfies the $\left(CLR_{(A)}\right)$ property,*
2. *There exist $\phi \in \Phi$ and $\psi \in \Psi$ such that*

$$\psi(d\,(Px, Py)) \leq \psi\,(m\,(x, y)) - \phi\,(m\,(x, y))\,,$$

for all $x, y \in X$, where $m\,(x, y) = \max M^k_{P,A}\,(x, y)$ and $k \in \{3, 4, 5\}$. Then, (P, A) has a coincidence point. Moreover, if the pair (P, A) is occasionally weakly compatible, then the pair has a unique common fixed point in X.

As an application of Theorem 3.2, we have the following result involving four finite families of self-mappings.

Theorem 3.9. *Let $\{A_i\}_{i=1}^m$, $\{B_j\}_{j=1}^n$, $\{S_k\}_{k=1}^p$, and $\{T_l\}_{l=1}^q$ be four finite families of self-mappings of a metric space (X, d) with $A = A_1, A_2, \ldots, A_m$, $B = B_1, B_2, \ldots, B_n$, $S = S_1, S_2, \ldots, S_p$, and $T = T_1, T_2, \ldots, T_q$ satisfying the condition (2). Suppose that the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property, then (A, S) and (B, T) have a point of coincidence each.*

Moreover $\{A_i\}_{i=1}^m$, $\{B_j\}_{j=1}^n$, $\{S_k\}_{k=1}^p$, and $\{T_l\}_{l=1}^q$ have a unique common fixed point if the families $(\{A_i\}\,,\ \{S_k\})$ and $(\{B_j\}\,,\ \{T_l\})$ pairwise OWC where $i \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, p\}$, $j \in \{1, 2, \ldots, n\}$ and $l \in \{1, 2, \ldots, q\}$.

Proof. The proof of this theorem can be completed on the lines of Theorem 3.2 of Imdad [16]. □

Remark 3.10. *A result similar to Theorem 3.8 can be outlined using Corollary 3.5 .*

Remark 3.11. *Theorem 3.8 extends the results of Radenovic [28] and Abbas and Doric [2].*

Now, we indicate that Theorem 3.8 can be utilized to derive common fixed point theorems for any finite number of mappings. As a sample, we can derive a common fixed point theorem for six mappings by setting two families of two members, while the rest by two of single members.

Corollary 3.12. *Let A, B, H, R, S, and T be self-mappings of a metric space (X,d). Suppose that*

1. *The pairs (A, SR) and (B, TH) share the $\left(CLR_{(SR)(TH)}\right)$ property,*
2. *There exist $\phi \in \Phi$ and $\psi \in \Psi$ such that $\psi(d\left(Ax, By\right)) \le \psi\left(m\left(x,y\right)\right) - \phi\left(m\left(x,y\right)\right)$, for all $x, y \in X$, where $m\left(x,y\right) = \max M^{k}_{A,B,H,R,S,T}\left(x,y\right)$, and $k \in \{3,4,5\}$.*

(A, SR) and (B, TH) are OWC. Then (A,SR) and (B,TH) have a coincidence point each. Moreover, A, B, H, R, S, and T have a unique common fixed point provided $AS = SA$, $SR = RS$, $BT = TB$, and $TH = HT$.

By choosing $A_1 = A_2 = \cdots = A_m = A$, $B_1 = B_2 = \cdots = B_n = B$, $S_1 = S_2 = \cdots = S_p = S$, and $T_1 = T_2 = \cdots = T_q = T$ in Theorem 3.8, we get the following corollary:

Corollary 3.13. *Let A, B, S and T be self-mappings of a metric space (X,d). Suppose that*

1. *the pairs (A^m, S^p) and (B^n, T^q) share the $(CLR_{S^pT^q})$ property, where m, n, p, and q are fixed positive integers;*
2. *there exist $\phi \in \Phi$ and $\psi \in \Psi$ such that $\psi(d\left(A^m x, B^n y\right) \le \psi\left(m\left(x,y\right)\right) - \phi\left(m\left(x,y\right)\right)$, for all $x, y \in X$, where $m\left(x,y\right) = \max M^{k}_{A^m,B^n,S^p,T^q}\left(x,y\right)$, and $k \in \{3,4,5\}$.*

Then A, B, S and T have a unique common fixed point provided that (A,S) and (T,B) are OWC.

Remark 3.14. *Notice that Corollary 3.12 is a slight but partial generalization of Theorem 3.2.*

Remark 3.15. *Results similar to Corollary 3.12 can be derived form Corollary 3.5.*

Remark 3.16. *It may be pointed out that the earlier proved results, namely Theorems 3.2 and 3.8 (also Corollaries 3.5, 3.7, 3.11, 3.12) remain valid in symmetric space (X,d) whenever d is continuous.*

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