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Revisit of Ostrowski's Method and Two New Higher Order Methods for Solving Nonlinear Equation

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Abstract: In this paper, we have obtained the Ostrowski's method in a different way and proposed two new methods of order seven and thirteen. The efficiency index of Ostrowsky's method is 1.587 and that of the seventh order method and thirteenth order method are respectively 1.626 and 1.670, which are better than Newton's method (1.414) and Ostrowsky's method. Also it is observed from the numerical illustrations, the proposed methods take less number of iterations than Newton's method. Few other methods are compared with the proposed two methods, where the number of iterations for those methods are either same or more than the presented methods. Some examples are given to illustrate the performance of the new methods.

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1. Introduction

One of the most important root-finding methods for solving nonlinear scalar equation $f(x) = 0$ is Newton's method. In general, to compute a root in a finite number of arithmetic operations is a difficult task by direct methods, hence this requires an iterative method. There are many iterative methods available in the literature and among them some are higher order iterative methods. They are specifically developed and analyzed for solving nonlinear equations that improve classical methods, such as Newton's method (NM) and Halley's iteration method, which are respectively given below:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'(x_n)^2 - f(x_n)f''(x_n)}. \tag{2}$$

Newton's method has second order convergence and it is optimal with two function evaluations. Halley's iteration method has third order convergence with three function evaluations. Obviously, f'' is difficult to calculate and computationally more costly. Therefore, f'' in Equation (2) is approximated using the finite difference; still, the convergence order and total number of function evaluation are maintained. Such a third order method similar to Equation (2) after approximating f'' in Halley's iteration method is given below [5]:

$$y_n = x_n - \beta\frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = x_n - \frac{2\beta f(x_n)}{(2\beta-1)f'(x_n) + f'(y_n)}, \quad \beta \neq 0. \tag{3}$$

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In the past decade, some authors have proposed third order methods with three function evaluations free from f''; for example, [1, 3] and the references therein. The efficiency index (EI) of an iterative method is measured using the formula $p^{\frac{1}{d}}$, where p is the local order of convergence and d is the number of function evaluations per full iteration cycle. Kung-Traub [9] conjectured that the order of convergence of any multi-point without memory method with d function evaluations cannot exceed the bound 2^{d-1}, the "optimal order". Thus, the optimal order for three evaluations per iteration would be four. Jarratt's method [8] is an example of an optimal fourth order method. Recently, some optimal and non-optimal multi-point iterative methods have been developed in [2, 3, 10–12, 16] and the references therein.

In this paper, we have obtained the Ostrowski's method [14] by using a different approach and our aim is to improve the order of this method to seven and thirteen by using divided difference approximation. Section 2 gives the preliminaries and Section 3 presents the development of the new methods. Section 4 discusses the convergence analysis and Section 5 carries out the test on the numerical examples and compare the present methods with few other methods. Finally, Section 6 gives conclusion on our work.

2. Preliminaries

Definition 2.1 ([17]). *If the sequence $\{x_n\}$ tends to a limit x^* in such a way that*

$$\lim_{n\to\infty} \frac{x_{n+1}-x^*}{(x_n-x^*)^p} = C$$

for $p \geq 1$, then the order of convergence of the sequence is said to be p, and C is known as the asymptotic error constant. If $p = 1$, $p = 2$ or $p = 3$, the convergence is said to be linear, quadratic or cubic, respectively. Let $e_n = x_n - x^*$, then the relation

$$e_{n+1} = C\ e_n^p + O\left(e_n^{p+1}\right) = O\left(e_n^p\right). \tag{4}$$

is called the error equation. The value of p is called the order of convergence of the method.

Definition 2.2 ([14]). *The Efficiency Index (EI) is given by*

$$EI = p^{\frac{1}{d}}, \tag{5}$$

where d is the total number of new function evaluations (the values of f and its derivatives) per iteration.

Definition 2.3 (Kung-Traub Conjecture [9]). *Let ψ be an iterative function without memory with d function evaluations. Then*

$$p(\psi) \leq p_{opt} = 2^{d-1}, \tag{6}$$

where p_{opt} is the maximum order.

3. Development of the methods

Let us consider the following third order method, taking $\beta = 1$ in Equation (3):

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(y_n)}. \tag{7}$$

This is known as Arithmetic mean Newton's method(AM) with cubic convergence (see [18]). This method (7) is of order three with three function evaluation per full iteration having EI = 1.442. The following method is known as Harmonic mean Newton's method(HM) with cubic convergence [6]:

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(y_n)}\right). \tag{8}$$

The Newton-Steffensen method(SM) with cubic convergence [15] is given by

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f'(x_n)[f(x_n) - f(y_n)]}. \tag{9}$$

From the literature survey, we observe that the methods (7), (8) and (9) are approximately equal [4] with respect to the convergence order.

By equating (7) and (8), we have

$$f'(y_n) \approx \frac{f'(x_n)^2}{2f'(x_n) - f'(y_n)}. \tag{10}$$

Also, equating (8) and (9), we have

$$f'(y_n) \approx \frac{f'(x_n)[f(x_n) - f(y_n)]}{f(x_n) + f(y_n)} \tag{11}$$

Again, equating (10) and (11), we have

$$f'(y_n) \approx \frac{f'(x_n)[f(x_n) - 3f(y_n)]}{f(x_n) - f(y_n)} \tag{12}$$

Reconstructed fourth order method(M1): Equation (12) is substituted in (7) to obtain the well known Ostrowski's method [14]:

$$z_n = x_n - \frac{f(x_n)}{f'(x_n)}\Big[\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}\Big] \tag{13}$$

The efficiency of the this two-step method (13) is found to be $EI = 1.587$.

Seventh order method(M2): We extend this method (13) to seventh order with one more function evaluation as follows: First we introduce one more Newton's step in the method (13), then we get

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \tag{14}$$

Thus, we obtain a new three-step iterative method (14) with convergence order eight which has five function evaluations. The efficiency of the method (14) $EI = 1.516$ shows that it is higher than Newton's method but lower than Ostrowski's method (13). In order to improve the efficiency index, we modify the method (14) by approximating $f'(z_n)$ with already computed function values using the divided difference,

$$f'(z_n) \approx \frac{f(z_n) - f(y_n)}{z_n - y_n} = p'(z_n),$$

and obtain the following three-step method whose convergence order is six with four function evaluations:

$$x_{n+1} = z_n - \frac{f(z_n)}{p'(z_n)}. \tag{15}$$

However, the method (15) has efficiency index is $EI = 1.565$, but still this method has less efficiency than Ostrowski's method (13). Now, let us consider the modified version of (15) as follows:

$$w_n = z_n - \frac{f(z_n)}{p'(z_n)}\Big[1 - \Big(\frac{f(y_n)}{f(x_n)}\Big)^2\Big], \tag{16}$$

where $\left[1-\left(\frac{f(y_n)}{f(x_n)}\right)^2\right]$ is a weight function included to improve the order and hence efficiency. Thus, we have obtained a new three-step iterative method (16) denoted by **M2** which has convergence order seven with four function evaluations whose efficiency index is $EI = 1.626$.

Thirteenth order method(M3): Babajee *et al.* [2] improved a sixth order Jarratt method to a twelfth order method. Using their technique, we improve (16) by another Newton's step and obtain

$$x_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)}, \tag{17}$$

The above method has fourteenth order convergence with six function evaluations. To reduce the number of function evaluation from six to five, we approximate $f'(w_n)$ as follows [2]:

$$\begin{aligned} f'(w_n) &\approx \frac{1}{z_n - w_n}\left(f'(x_n)(z_n - w_n) + 2f[w_n, x_n, x_n](z_n - x_n)(w_n - x_n) + (f[z_n, x_n, x_n] - 3f[w_n, x_n, x_n])(w_n - x_n)^2\right) \\ &= q'(w_n), \\ f[z_n, x_n, x_n] &= \frac{f[z_n, x_n] - f'(x_n)}{z_n - x_n},\ f[z_n, x_n] = \frac{f(z_n) - f(x_n)}{z_n - x_n}, \\ f[w_n, x_n, x_n] &= \frac{f[w_n, x_n] - f'(x_n)}{w_n - x_n},\ f[w_n, x_n] = \frac{f(w_n) - f(x_n)}{w_n - x_n}. \end{aligned}$$

Thus, we have obtained a new four-step iterative method

$$x_{n+1} = w_n - \frac{f(w_n)}{q'(w_n)}, \tag{18}$$

denoted by **M3** which has convergence order thirteen with five function evaluations whose efficiency index is $EI = 1.670$. This method has the highest efficiency among all the three methods.

4. Convergence Analysis

Theorem 4.1. *Let a sufficiently smooth function $f : D \subset \mathbb{R} \to \mathbb{R}$ has a simple root x^* in the open interval D. If x_0 is chosen in a sufficiently small neighborhood of x^*, then the method (16) is of local seventh order convergence.*

Proof. Let $e_n = x_n - \alpha$. Using the Taylor series, we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + \ldots] \tag{19}$$

$$\text{and} \ \ f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + \ldots], \tag{20}$$

where $c_k = \frac{f^{(k)}(x^*)}{k!f'(x^*)}$, $k \geq 2$. Now substituting (19) and (20) in (7), we get

$$\begin{aligned} y_n = \alpha &+ c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + (-8c_2^4 + 20c_2^2c_3 - 6c_3^2 - 10c_2c_4 + 4c_5)e_n^5 + (16c_2^5 - 52c_2^3c_3 + 33c_2c_3^2 \\ &+ 28c_2^2c_4 - 17c_3c_4 - 13c_2c_5 + 5c_6)e_n^6 - 2(16c_2^6 - 64c_2^4c_3 - 9c_3^3 + 36c_2^3c_4 + 6c_4^2 + 9c_2^2(7c_3^2 - 2c_5) \\ &+ 11c_3c_5 + c_2(-46c_3c_4 + 8c_6) - 3c_7)e_n^7 + \ldots \end{aligned} \tag{21}$$

Expanding $f(y_n)$ about α and taking into account (21), we have

$$\begin{aligned} f(y_n) = f'(\alpha)[&c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 - 2(6c_2^4 - 12c_2^2c_3 + 3c_3^2 + 5c_2c_4 - 2c_5)e_n^5 \\ &+ (28c_2^5 - 73c_2^3c_3 + 34c_2^2c_4 - 17c_3c_4 + c_2(37c_3^2 - 13c_5) + 5c_6)e_n^6 - 2(32c_2^6 - 103c_2^4c_3 - 9c_3^3 + 52c_2^3c_4 \\ &+ 6c_4^2 + c_2^2(80c_3^2 - 22c_5) + 11c_3c_5 + c_2(-52c_3c_4 + 8c_6) - 3c_7)e_n^7 + \ldots]. \end{aligned} \tag{22}$$

Now, using (19),(20) and (22) in (13) then we have

$$\begin{aligned} z_n &= \alpha + (c_2^3 - c_2c_3)e_n^4 - 2(2c_2^4 - 4c_2^2c_3 + c_3^2 + c_2c_4)e_n^5 + (10c_2^5 - 30c_2^3c_3 + 12c_2^2c_4 - 7c_3c_4 + 3c_2(6c_3^2 - c_5))e_n^6 \\ &- 2(10c_2^6 - 40c_2^4c_3 - 6c_3^3 + 20c_2^3c_4 + 3c_4^2 + 8c_2^2(5c_3^2 - c_5) + 5c_3c_5 + c_2(-26c_3c_4 + 2c_6))e_n^7 + \dots \end{aligned} \tag{23}$$

Expanding $f(z_n)$ about α and taking into account (23), we have

$$\begin{aligned} f(z_n) &= f'(\alpha)\Big[(c_2^3 - c_2c_3)e_n^4 - 2(2c_2^4 - 4c_2^2c_3 + c_3^2 + c_2c_4)e_n^5 + (10c_2^5 - 30c_2^3c_3 + 18c_2c_3^2 + 12c_2^2c_4 - 7c_3c_4 - 3c_2c_5)e_n^6 \\ &- 2(10c_2^6 - 40c_2^4c_3 - 6c_3^3 + 20c_2^3c_4 + 3c_4^2 + 8c_2^2(5c_3^2 - c_5) + 5c_3c_5 + c_2(-26c_3c_4 + 2c_6))e_n^7 + \dots\Big]. \end{aligned} \tag{24}$$

Using equations (19) and (22), we have

$$\begin{aligned} \frac{f(y_n)}{f(x_n)} &= c_2e + (-3c_2^2 + 2c_3)e_n^2 + (8c_2^3 - 10c_2c_3 + 3c_4)e_n^3 + (-20c_2^4 + 37c_2^2c_3 - 8c_3^2 - 14c_2c_4 + 4c_5)e_n^4 \\ &+ (48c_2^5 - 118c_2^3c_3 + 55c_2c_3^2 + 51c_2^2c_4 - 22c_3c_4 - 18c_2c_5 + 5c_6)e_n^5 + (-112c_2^6 + 344c_2^4c_3 - 252c_2^2c_3^2 \\ &+ 26c_3^3 - 163c_2^3c_4 + 150c_2c_3c_4 - 15c_4^2 + 65c_2^2c_5 - 28c_3c_5 - 22c_2c_6 + 6c_7)e_n^6 + ... \end{aligned} \tag{25}$$

Using equations (21), (22), (23) and (24), we have

$$p'(z_n) = f'(\alpha)\Big[1 + c_2^2e_n^2 + (-2c_2^3 + 2c_2c_3)e_n^3 + c_2(5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 - 4(c_2(3c_2^4 - 6c_2^2c_3 + c_3^2 + 3c_2c_4 - c_5))e_n^5 + ...\Big]. \tag{26}$$

Finally, using (23), (24), (26) and (25) in (16), we get

$$e_{n+1} = \left(4c_2^6 - 6c_2^4c_3 + 2c_2^2c_3^2\right)e_n^7 + O(e_n^8), \tag{27}$$

which shows seventh order convergence. □

The following theorem is given without proof, which can be worked out with the help of Mathematica.

Theorem 4.2. *Let a sufficiently smooth function $f : D \subset \mathbb{R} \to \mathbb{R}$ has a simple root x^* in the open interval D. If x_0 is chosen in a sufficiently small neighborhood of x^*, then the method (18) is of local thirteenth order convergence.*

$$e_{n+1} = 2c_2^3\left(c_2^2 - c_3\right)\left(2c_2^4 - 3c_2^2c_3 + c_3^2\right)c_4e_n^{13} + O(e_n^{14}).$$

5. Numerical examples

In this section, we denote the different methods as follows: Newton iterative method **(NM)**, Arithmetic mean Newton iterative method **(AM)**, Harmonic mean Newton iterative method **(HM)**, Newton-Steffensen iterative method **(SM)**, method proposed by Hu et al. **(HF)** [7] , method proposed by Noor et al. **(NKNA)** [13] . The methods presented in this paper are denoted as **M1**, **M2** and **M3**. Numerical results on some test functions are given for the proposed methods with some existing methods. Numerical computations have been carried out in the MATLAB software with 500 significant digits. Depending on the precision of the computer, we have used the stopping criteria for the iterative process as $error = |x_N - x_{N-1}| < \epsilon$, where $\epsilon = 10^{-50}$ and N is the number of iterations required for convergence. The computational order of convergence is given by

$$\rho = \frac{\ln|(x_N - x_{N-1})/(x_{N-1} - x_{N-2})|}{\ln|(x_{N-1} - x_{N-2})/(x_{N-2} - x_{N-3})|}.$$

Table 1 shows the efficiency indices of the new methods with some known methods.

Methods	p	d	EI	Optimal/Non-optimal
NM	2	2	1.414	Optimal
AM	3	3	1.442	Non-optimal
HM	3	3	1.442	Non-optimal
SM	3	3	1.442	Non-optimal
HF	7	5	1.475	Non-optimal
NKNA	7	5	1.475	Non-optimal
M1	4	3	1.587	Optimal
M2	7	4	1.626	Non-optimal
M3	13	5	1.670	Non-optimal

Table 1. Comparison of Efficiency Indices (EI) and Optimality

The following test functions and their simple zeros for our study are given below:

$$f_1(x) = \sin(2\cos x) - 1 - x^2 + e^{\sin(x^3)}, \quad x^* = -0.7848959876612125352...$$

$$f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \quad x^* = -1.2076478271309189270...$$

$$f_3(x) = sin(x) + cos(x) + x, \quad x^* = -0.4566247045676308244...$$

$$f_4(x) = (x+2)e^x - 1, \quad x^* = -0.4428544010023885831...$$

$$f_5(x) = x^2 + \sin\left(\frac{x}{5}\right) - \frac{1}{4}, \quad x^* = 0.4099920179891371316...$$

From Table 1, it is observed that the present methods **M1**, **M2** and **M3** have better efficiency when compared with other methods. Table 2 shows the results for $f_1(x)$ to $f_5(x)$. We observe that if the initial points are close to the root, then all the methods take less number of iterations and produce least error. For the test function $f_2(x)$, **HF** method diverges, whereas it converges for the proposed new methods. We also note from the numerical results for the test function $f_5(x)$, the computational order of convergence is found to be higher than the theoretical order.

$f(x)$	Methods	x_0	N	ρ	$error$	$cpu(s)$	x_0	N	ρ	$error$	$cpu(s)$
$f_1(x)$	**NM**	-1.2	7	1.99	1.5646e-060	0.6532	-0.5	8	1.99	6.4194e-071	0.5181
	AM		5	3.00	6.5582e-052	0.6806		6	2.99	5.5304e-147	0.7804
	HM		5	2.99	1.0380e-051	0.6470		5	3.00	4.4040e-057	0.7997
	SM		5	2.99	3.1215e-072	0.5858		6	3.00	7.2647e-130	0.7091
	HF		4	6.99	1.0930e-270	0.6908		4	7.00	8.9500e-064	0.7922
	NKNA		4	7.00	2.2508e-297	0.7258		4	7.00	8.3597e-201	0.7320
	M1		5	3.99	2.1844e-171	0.5631		5	3.99	5.1071e-124	0.6430
	M2		4	6.99	1.1126e-221	0.5776		4	7.00	1.2796e-115	0.5169
	M3		3	12.61	8.9515e-108	0.6438		3	13.60	3.5614e-063	0.6158
$f_2(x)$	**NM**	-1.7	9	2.00	4.3765e-054	0.5677	-0.8	9	2.00	2.1249e-055	0.5854
	AM		7	3.00	4.3181e-124	0.7034		7	3.00	1.0285e-086	0.6239
	HM		6	3.00	1.2897e-072	0.5774		6	3.00	9.8953e-140	0.5470
	SM		6	2.99	2.7850e-051	0.5842		7	3.00	3.5528e-149	0.6173
	HF		4	7.05	1.5088e-062	0.5193		Div	-	-	-
	NKNA		4	6.99	1.6192e-101	0.5843		4	7.00	2.2669e-074	0.5494
	M1		5	4.00	4.4698e-093	0.4831		5	4.00	5.0201e-150	0.5219
	M2		4	7.00	2.1198e-086	0.5063		4	6.99	3.8806e-096	0.5137
	M3		4	13.14	0.0000e-000	0.5037		3	13.19	6.9382e-052	0.5737

$f(x)$	Methods	x_0	N	ρ	$error$	$cpu(s)$	x_0	N	ρ	$error$	$cpu(s)$
$f_3(x)$	**NM**	0.5	7	1.99	1.0799e-055	0.4327	-1.5	7	1.99	1.1601e-058	0.4051
	AM		5	2.99	2.7460e-066	0.4944		6	2.99	9.2298e-149	0.5173
	HM		6	2.99	1.6162e-137	0.5121		6	2.99	5.0536e-143	0.5313
	SM		5	3.00	1.2953e-059	0.5586		5	2.99	1.3916e-107	0.4584
	HF		4	7.00	3.0611e-114	0.5875		4	7.00	1.9419e-253	0.4898
	NKNA		4	7.00	1.5307e-208	0.6733		4	6.99	9.4246e-287	0.5575
	M1		5	3.99	2.0535e-128	0.6138		5	3.99	2.5068e-161	0.4504
	M2		4	7.00	1.0154e-128	0.5417		4	6.99	1.6262e-195	0.4808
	M3		3	13.67	1.6309e-070	0.5009		3	12.88	1.2146e-118	0.4986
$f_4(x)$	**NM**	-0.2	7	1.99	7.2347e-052	0.7061	-0.9	8	1.99	3.1021e-058	0.4480
	AM		5	2.99	2.6714e-061	0.6253		6	2.99	1.4745e-093	0.4895
	HM		5	3.00	2.3754e-082	0.7212		5	2.99	1.8265e-076	0.4350
	SM		5	2.99	1.7485e-065	0.6500		6	2.99	1.8007e-109	0.4834
	HF		4	6.99	1.8551e-274	0.6048		5	7.00	4.0979e-275	0.5416
	NKNA		4	6.99	3.9568e-305	0.7280		4	7.00	3.8579e-171	0.5040
	M1		4	3.99	7.5199e-060	0.5937		5	4.00	1.6339e-154	0.4383
	M2		4	6.99	2.5159e-276	0.5042		4	7.00	7.2869e-142	0.4398
	M3		3	13.00	3.9609e-144	0.4926		3	13.29	2.6606e-077	0.4481
$f_5(x)$	**NM**	0.8	8	1.99	3.2094e-072	0.4551	0.2	8	1.99	8.2490e-076	0.3976
	AM		6	3.00	1.6989e-136	0.5046		6	2.99	2.5980e-143	0.5031
	HM		5	2.99	2.3450e-094	0.4639		5	2.99	1.8391e-098	0.4406
	SM		5	3.00	1.8442e-136	0.5015		6	3.00	2.8208e-143	0.5197
	HF		4	6.99	3.5540e-159	0.4762		4	7.00	2.6957e-099	0.4654
	NKNA		4	6.99	2.5015e-209	0.5584		4	7.00	1.2607e-209	0.5111
	M1		5	3.99	1.1319e-143	0.4608		5	3.99	7.4805e-151	0.4351
	M2		4	6.99	2.9356e-169	0.4266		4	7.00	2.0343e-152	0.4289
	M3		3	13.44	2.5412e-097	0.4389		3	14.24	2.6427e-088	0.4359

Table 2. **Numerical results for test functions**

6. Conclusions

In this work, we have developed a different procedure to obtain Ostrowski's fourth order method which is found to be optimal as per the Kung-Traub conjuncture. This method requires only three function evaluations. Also, we have extended the fourth order method to seventh order along with weight function and extended seventh order method to thirteenth order. It is clear that our extended methods require only four function evaluations per iterative step to obtain seventh order convergence and five function evaluations per iterative step to get thirteenth order convergence. Table 1 compares the efficiency of different methods where we find that the proposed new methods have better efficiency. Moreover, unlike all other methods, the proposed new methods require less cpu time for convergence and hence considered as a competitor to Newton's method and few other higher order methods.

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