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A Fixed Point Result in Dislocated Quasi b-Metric Space

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Abstract: The purpose of this paper is to consider the notion of dislocated quasi b-metric space and establish a fixed point theorem using symmetric rational type contractive condition in the context of dislocated quasi b-metric space as a generalization of Banach contraction principle.

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1. Introduction

As a generalization of metric spaces, Bakhtin [1] introduce the concept of b-metric space. More over Czerwik [2] made the results of Bakhtin [1] known more. Hitzler and Seda [3, 4] introduced the notion of dislocated metric spaces and generalized the celebrated Banach Contraction Principle in such spaces. These metrics play a very important role not only in topology but also in other branches of science involving mathematics especially in logic programming and electronic engineering. Zeyada [5] initiated the concept of dislocated quasi-metric space and generalized the result of Hitzler and Seda in dislocated quasi-metric spaces. Recently Rahman and Sarwar [6] presented the idea of dislocated quasi b-metric space and proved Banach's contraction principle, Kannan and Chatterjea type fixed point results for self mapping in such space. On the other hand, it is well known that the first elegant result on fixed points for various type of contractive type condition is Banach Contraction principle [7]. This principle has been generalized by various authors by putting different type of contractive conditions either on mappings or on the spaces. A comprehensive literature and generalization of the same can be found in Rhoades [8] and Meszaros [9]. In the sequel Khan [10] proved a fixed point theorem using symmetric rational type of contractive condition which extended the Banach contraction principle in complete metric space. In view of above these facts the aim of this paper is to present a version of Khan Fixed point theorem in the context of dislocated quasi b- metric space.

2. General Framework

Consistent with [1, 5, 6], we recall basic definitions and other results that will be needed in the sequel.

Definition 2.1 ([5])**.** *Let X be a non-empty and let $d : X \times X \to [0,\infty)$ be a function, called a distance function, satisfies:*

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d1: $d(x,x) = 0$,

d2: $d(x,y) = d(y,x) = 0$ *then* $x = y$,

d3: $d(x,y) = d(y,x)$,

d4: $d(x,y) \leq d(x,z) + d(z,y)$

for all $x, y, z \in X$*. If* d *satisfies the condition d1-d4, then* d *is called a metric on* X*. If it satisfies the conditions d1, d2 and d4 it is called a quasi metric space. If* d *satisfies conditions d2, d3, d4 it is called a dislocated metric (or simply d-metric) and if* d *satisfies only d2 and d4 then* d *is called a dislocated quasi-metric (or simply dq-metric) on* X*. Non empty set* X *together with dq-metric* d*, i.e.* (X, d) *is called a dislocated quasi-metric space.*

Definition 2.2 ([1])**.** *Let* X *be a non-empty and let* $k \geq 1$ *be a real number then a mapping* $d : X \times X \to [0, \infty)$ *is called b-metric if;*

d1: $d(x,x) = 0$,

d2: $d(x,y) = d(y,x) = 0$ *then* $x = y$,

d3: $d(x,y) = d(y,x)$,

d4: $d(x,y) \leq kd(x,z) + d(z,y)$

for all $x, y, z \in X$*. The pair* (X, d) *is called b-metric space.*

It is clear that b-metric is more generalization of usual metric.

Definition 2.3 ([6])**.** *Let* X *be a non-empty and let* $k \geq 1$ *be a real number then a mapping* $d : X \times X \to [0, \infty)$ *is called dislocated quasi b-metric if;*

d1: $d(x,y) = d(y,x) = 0$ *then* $x = y$,

d2: $d(x,y) \leq kd(x,z) + d(z,y)$.

For all $x, y, z \in X$*. The pair* (X, d) *is called dislocates quasi b-metric or shortly (dq b-metric) space.*

Proposition 2.4 ([6])**.** *Let* X *be a non-empty set such that* d^* *is dq-metric and* d^{**} *is a b-metric with* $k \geq 1$ *on* X*. Then the function* $d : X \times X \to [0, \infty)$ *defined by* $d(x,y) = d^*(x,y) + d^{**}(x,y)$ *for all* $x, y \in X$ *is a dq b-metric on* X*.*

Definition 2.5 ([6])**.** *A sequence* $\{x_n\}$ *is called dq b-convergent in* X *if for* $n \geq N$ *we have* $d(x_n, x) < \epsilon$ *where* $\epsilon > 0$*, then* X *is called the dq b-limit of the sequence* $\{x_n\}$*.*

Definition 2.6 ([6])**.** *A sequence* $\{x_n\}$ *in dq b-metric space* X *is called Cauchy sequence if for* $\epsilon > 0$*,* $\exists \ n_0 \in N$ *such that for all* $m, n \geq n_0$*,* $d(x_m, x_n) < \epsilon$*.*

Definition 2.7 ([6])**.** *A dq b-metric space* (X, d) *is complete if every Cauchy sequence in it is dq b-convergent.*

Lemma 2.8 ([6])**.** *Limit of convergent sequence in a dq b-metric space is unique.*

Lemma 2.9 ([6])**.** *Let* (X, d) *be a dq b-metric space and* $\{x_n\}$ *be a sequence in* X *such that* $d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n)$ *for* $n = 1, 2, 3, \ldots$ *and* $0 \leq \alpha k < 1$*,* $\alpha \in [0, 1)$*, and* k *is defined in dq b-metric space. Then* $\{x_n\}$ *is a Cauchy sequence.*

In [6] authors proved the following fixed point theorems.

Theorem 2.10 ([6]). *Let (X, d) be a complete dq b-metric space and let $T : X \to X$ be a continuous contraction with $\alpha \in [0, 1)$ and $0 \leq \alpha k < 1$, where $k \geq 1$ then T has a unique fixed point in X.*

Theorem 2.11 ([6]). *Let (X, d) be a complete dq b-metric space and let $T : X \to X$ be a continuous self mapping with $\alpha \in \left[0, \frac{1}{2}\right)$ and $k \geq 1$ satisfying the condition $d(Tx, Ty) \leq \alpha \left[d(x, Tx) + d(y, Ty)\right]$ for all $x, y \in X$. Then T has a unique fixed point in X.*

Theorem 2.12 ([6]). *Let (X, d) be a complete dq b-metric space and let $T : X \to X$ be a continuous self mapping with $k\alpha \in \left[0, \frac{1}{4}\right)$ and $k \geq 1$ satisfying the condition $d(Tx, Ty) \leq \alpha \left[d(x, Ty) + d(y, Tx)\right]$ for all $x, y \in X$. Then T has a unique fixed point in X.*

Khan [10] proved the following fixed point theorem using rational type contraction in complete metric spaces.

Theorem 2.13. *Let T be a self- map defined on a complete metric space (X, d). Further, let T satisfies the following contractive condition:*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx).d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)}$$

for all $x, y \in X$ and for $\alpha \in [0, 1)$ then T has a unique fixed point. Later on Fisher [11] improved this result by implying the condition$d(x, Ty) + d(y, Tx) \neq 0$.

3. Main Results

Now we establish the modify form of Khan's theorem [10] in the setting of dislocated quasi b-metric space as follows.

Theorem 3.1. *Let (X, d) be a complete dislocated quasi b-metric space with $k \geq 1$ and let $T : X \to X$ be a continuous self mapping satisfying the condition*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx).d(x, Ty) + d(y, Ty)d(y, Tx)}{d(x, Ty) + d(y, Tx)} + \beta d(x, y) \tag{1}$$

for all $x, y \in X$ and $\alpha, \beta \geq 0$, $d(x, Ty) + d(y, Tx) \neq 0$ with $k(2\alpha + \beta) < 1$. Then T has a unique fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X such that $x_1 = T(x_0), x_2 = T(x_1), \ldots, x_{n+1} = T(x_n), \ldots$. Replace X by x_{n-1} and y by x_n in (1), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha \frac{d(x_{n-1}, Tx_{n-1}).d(x_{n-1}, Tx_n) + d(x_n, Tx_n).d(x_n, Tx_{n-1})}{d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})} + \beta d(x_{n-1}, x_n) \\ &\leq \alpha \frac{d(x_{n-1}, x_n).d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1}).d(x_n, x_n)}{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)} + \beta d(x_{n-1}, x_n) \end{aligned}$$

using triangular inequality

$$\begin{aligned} &\leq \alpha \frac{d(x_{n-1}, x_n).k(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) + d(x_n, x_{n+1}).k(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))}{k(d(x_{n-1}, x_n) + d(x_n, x_{n+1}))} + \beta d(x_{n-1}, x_n) \\ &\leq \alpha \left(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right) + \beta d(x_{n-1}, x_n) \end{aligned}$$

therefore

$$d(x_n, x_{n+1}) \leq \frac{\alpha + \beta}{1 - \alpha} d(x_{n-1}, x_n) = hd(x_{n-1}, x_n) \tag{2}$$

where $h = \frac{\alpha+\beta}{1-\alpha} < 1$ as $k(2\alpha+\beta) < 1$. In the same way, we have

$$d(x_{n-1}, x_n) \leq hd(x_{n-2}, x_{n-1})$$

by (2) we get,

$$d(x_n, x_{n+1}) \leq h^2 d(x_{n-2}, x_{n-1})$$

Continue this process, we get in general

$$d(x_n, x_{n+1}) \leq h^n d(x_1, x_0)$$

Since $0 \leq h < 1$ as $n \to \infty$, $h^n \to 0$. Hence by lemma 2.9, $\{x_n\}$ is a Cauchy sequence in complete dq b-metric space X. Thus $\{x_n\}$ dislocated quasi b-converges to some u in X such that $T(u) = \lim T(x_n) = \lim x_{n+1} = u$. Thus u is a fixed point of T.

Uniqueness: Let $x \in X$ is a fixed point of T. Then by (1),

$$\begin{aligned} d(x,x) &= d(Tx,Tx) \\ &\leq \alpha \frac{d(x,x)d(x,x) + d(x,x)d(x,x)}{d(x,x) + d(x,x)} + \beta d(x,x) \\ &\leq (\alpha+\beta)d(x,x) \end{aligned}$$

which is true only if $d(x,x) = 0$, since $0 \leq k(2\alpha+\beta) < 1$ and $d(x,x) \geq 0$. Thus$d(x,x) \geq 0$, if X is a fixed point of T. Suppose that X and y in X are two fixed point of T i.e. $Tx = x$ and $Ty = y$. Then by (1) we have,

$$d(x,y) = d(Tx,Ty) \leq \beta d(x,y)$$

Which gives $d(x,y) = 0$, since $0 \leq \beta < 1$ and $d(x,y) \geq 0$. Similarly $d(y,x) = 0$ and hence $x = y$. Thus fixed point of T is unique. □

Remark 3.2. *If we put $\alpha = 0$ in (1), we get the result in [6]. Thus Theorem 3.1 is a generalization of Banach contaction principle.*

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