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Fixed Point Theorem for Mappings Satisfying Certain Rational Inequality

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Abstract: The authors of [1, 2] and [3] proved some fixed point theorems in complete metric space. In this paper we prove some fixed point theorems in complete metric space for mappings satisfying certain rational inequality.

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1. Introduction

The well known Banach contraction principle states that "If X is a complete metric space and T is a contraction mapping on X into itself then T has unique fixed point in X". Many researchers worked on this principle. This paper is another attempt in this direction

Definition 1.1. *A sequence $\{x_n\}$ in metric space (X, d) is called Cauchy sequence if for each $\varepsilon > 0$ there exist a positive integer $n_0$such that $m, n \geq n_0 \Rightarrow d(x_m, x_n) < \varepsilon$.*

Definition 1.2. *A complete metric space is a metric space in which every Cauchy sequence is convergent.*

Definition 1.3. *A sequence $\{x_n\}$ converges to z if $\lim\limits_{n\to\infty} d(x_n, z) = 0$. In this case z is called the limit of $\{x_n\}$ and we write $x_n \to z$.*

2. Main Results

Theorem 2.1. *Let S and T be self mappings defined on a complete metric space (X, d) satisfying*

$$d(Sx, Ty) \leq \alpha \frac{[1 + d(x, Sx)]\, d(y, Ty)}{[1 + d(x, y)]} + \beta \frac{[1 + d(x, Ty)]\, d(x, Sx)}{[1 + d(x, Sx) + d(y, Ty)]} + \gamma d(x, y) \tag{1}$$

for all x, y in X, where $\alpha + \beta + \gamma < 1$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$. Then S and T have a unique common fixed point.

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Proof. Let x_0 be any arbitrary point in X. We define a sequence $\{x_n\}$ in X such that $x_n = Sx_{n-1}$ and $x_{n+1} = Tx_n$ for $n = 1, 2, 3, \dots$. Then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Sx_{n-1}, Tx_n) \\ &\leq \alpha \frac{[1 + d(x_{n-1}, Sx_{n-1})]\, d(x_n, Tx_n)}{[1 + d(x_{n-1}, x_n)]} + \beta \frac{[1 + d(x_{n-1}, Tx_n)]\, d(x_{n-1}, Sx_{n-1})}{[1 + d(x_{n-1}, Sx_{n-1}) + d(x_n, Tx_n)]} + \gamma d(x_{n-1}, x_n) \ \text{[by (1)]} \\ &\leq \alpha \frac{[1 + d(x_{n-1}, x_n)]\, d(x_n, x_{n+1})}{[1 + d(x_{n-1}, x_n)]} + \beta \frac{[1 + d(x_{n-1}, x_{n+1})]\, d(x_{n-1}, x_n)}{[1 + d(x_{n-1}, x_n) + d(x_n, x_{n+1})]} + \gamma d(x_{n-1}, x_n) \\ &\leq \alpha d(x_n, x_{n+1}) + \beta \frac{[1 + d(x_{n-1}, x_{n+1})]\, d(x_{n-1}, x_n)}{[1 + d(x_{n-1}, x_{n+1})]} + \gamma d(x_{n-1}, x_n) \\ &\leq \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) + \gamma d(x_{n-1}, x_n) \\ \Rightarrow d(x_n, x_{n+1}) &\leq \left(\frac{\beta + \gamma}{1 - \alpha}\right) d(x_{n-1}, x_n) \quad \text{or } d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n) \end{aligned}$$

Where $\lambda = \left(\frac{\beta+\gamma}{1-\alpha}\right)$ and $0 < \lambda < 1$ in view of $\alpha + \beta + \gamma < 1$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$. Similarly $d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1})$ and so on. Hence $d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1) \to 0$ as $n \to \infty$ since $0 < \lambda < 1$. This proves that $\{x_n\}$ is a Cauchy's sequence in X which is Complete, so it converges to a point say z in X. Now,

$$\begin{aligned} d(z, Tz) &\leq d(z, x_n) + d(x_n, Tz) \\ &\leq d(z, x_n) + d(Sx_{n-1}, Tz) \\ &\leq d(z, x_n) + \alpha \frac{[1 + d(x_{n-1}, Sx_{n-1})]\, d(z, Tz)}{[1 + d(x_{n-1}, z)]} + \beta \frac{[1 + d(x_{n-1}, Tz)]\, d(x_{n-1}, Sx_{n-1})}{[1 + d(x_{n-1}, Sx_{n-1}) + d(z, Tz)]} + \gamma d(x_{n-1}, z) \\ &\leq d(z, x_n) + \alpha \frac{[1 + d(x_{n-1}, x_n)]\, d(z, Tz)}{[1 + d(x_{n-1}, z)]} + \beta \frac{[1 + d(x_{n-1}, Tz)]\, d(x_{n-1}, x_n)}{[1 + d(x_{n-1}, x_n) + d(z, Tz)]} + \gamma d(x_{n-1}, z) \\ \Rightarrow (1 - \alpha) d(z, Tz) &\leq 0 \ \ as \ \ n \to \infty \end{aligned}$$

Hence $Tz = z$ since $0 < \alpha < 1$. Similarly we can show $Sz = z$. Thus z is the common fixed point of S and T. Suppose y is another fixed point of S and T. Then we have

$$\begin{aligned} d(z, y) &\leq d(Sz, Ty) \\ &\leq \alpha \frac{[1 + d(z, Sz)]\, d(y, Ty)}{[1 + d(z, y)]} + \beta \frac{[1 + d(z, Ty)]\, d(z, Sz)}{[1 + d(z, Sz) + d(y, Ty)]} + \gamma d(z, y) \\ &\leq \alpha \frac{[1 + d(z, z)]\, d(y, y)}{[1 + d(z, y)]} + \beta \frac{[1 + d(z, y)]\, d(z, z)}{[1 + d(z, z) + d(y, y)]} + \gamma d(z, y) \\ &\leq \gamma d(z, y) \end{aligned}$$

$\Rightarrow d(z, y) \leq \gamma d(z, y)$ or $(1 - \gamma) d(z, y) \leq 0 \Rightarrow z = y$ since $0 < \gamma < 1$. This completes the proof. □

Theorem 2.2. *Let $\{T_k\}$ be a sequence of self mappings defined on a complete metric space (X, d) satisfying*

$$d(T_i x, T_j y) \leq \alpha \frac{[1 + d(x, T_i x)]\, d(y, T_j y)}{[1 + d(x, y)]} + \beta \frac{[1 + d(x, T_j y)]\, d(x, T_i x)}{[1 + d(x, T_i x) + d(y, T_j y)]} + \gamma d(x, y) \tag{2}$$

for all x, y in X, where $\alpha + \beta + \gamma < 1$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$. Then $\{T_k\}$ have a unique common fixed point.

Proof. Let x_0 be any arbitrary point in X. We define a sequence $\{x_n\}$ in X such that $x_n = T_i x_{n-1}$ and $x_{n+1} = T_j x_n$ for $n = 1, 2, 3, 4, \dots$. Then

$$d(x_n, x_{n+1}) = d(T_i x_{n-1}, T_j x_n)$$

$$
\begin{aligned}
&\leq \alpha \frac{[1+d(x_{n-1},T_ix_{n-1})]\,d(x_n,T_jx_n)}{[1+d(x_{n-1},x_n)]} + \beta \frac{[1+d(x_{n-1},T_jx_n)]\,d(x_{n-1},T_ix_{n-1})}{[1+d(x_{n-1},T_ix_{n-1})+d(x_n,T_jx_n)]} + \gamma d(x_{n-1},x_n) \ \ \text{[by (2)]}\\
&\leq \alpha \frac{[1+d(x_{n-1},x_n)]\,d(x_n,x_{n+1})}{[1+d(x_{n-1},x_n)]} + \beta \frac{[1+d(x_{n-1},x_{n+1})]\,d(x_{n-1},x_n)}{[1+d(x_{n-1},x_n)+d(x_n,x_{n+1})]} + \gamma d(x_{n-1},x_n)\\
&\leq \alpha d(x_n,x_{n+1}) + \beta \frac{[1+d(x_{n-1},x_{n+1})]\,d(x_{n-1},x_n)}{[1+d(x_{n-1},x_{n+1})]} + \gamma d(x_{n-1},x_n)\\
&\leq \alpha d(x_n,x_{n+1}) + \beta d(x_{n-1},x_n) + \gamma d(x_{n-1},x_n)
\end{aligned}
$$

$$
\Rightarrow d(x_n,x_{n+1}) \leq \left(\frac{\beta+\gamma}{1-\alpha}\right) d(x_{n-1},x_n) \ \ or \ \ d(x_n,x_{n+1}) \leq \lambda d(x_{n-1},x_n)
$$

Where $0<\lambda<1$. Similarly $d(x_{n-1},x_n) \leq \lambda d(x_{n-2},x_{n-1})$ and so on. Hence $d(x_n,x_{n+1}) \leq \lambda^n d(x_0,x_1) \to 0$ as $n \to \infty$ since $0<\lambda<1$. This proves that $\{x_n\}$ is a Cauchy's sequence in X which is Complete, so it converges to a point say z in X. Now,

$$
\begin{aligned}
d(z,T_jz) &\leq d(z,x_n) + d(x_n,T_jz)\\
&\leq d(z,x_n) + d(T_ix_{n-1},T_jz)\\
&\leq d(z,x_n) + \alpha \frac{[1+d(x_{n-1},T_ix_{n-1})]\,d(z,T_jz)}{[1+d(x_{n-1},z)]} + \beta \frac{[1+d(x_{n-1},T_jz)]\,d(x_{n-1},T_ix_{n-1})}{[1+d(x_{n-1},T_ix_{n-1})+d(z,T_jz)]} + \gamma d(x_{n-1},z)\\
&\leq d(z,x_n) + \alpha \frac{[1+d(x_{n-1},x_n)]\,d(z,T_jz)}{[1+d(x_{n-1},z)]} + \beta \frac{[1+d(x_{n-1},T_jz)]\,d(x_{n-1},x_n)}{[1+d(x_{n-1},x_n)+d(z,T_jz)]} + \gamma d(x_{n-1},z)
\end{aligned}
$$

$\Rightarrow (1-\alpha)d(z,T_jz) \leq 0$ as $n \to \infty$. Hence $T_jz = z$ since $0<\alpha<1$. Similarly we can show $T_iz = z$. Thus z is the common fixed point of T_i and T_j. Consequently z is the common fixed point of $\{T_k\}$. Suppose y is another fixed point of T_i and T_j. Then we have

$$
\begin{aligned}
d(z,y) &\leq d(T_iz,T_jy)\\
&\leq \alpha \frac{[1+d(z,T_iz)]\,d(y,T_jy)}{[1+d(z,y)]} + \beta \frac{[1+d(z,T_jy)]\,d(z,T_iz)}{[1+d(z,T_iz)+d(y,T_jy)]} + \gamma d(z,y)\\
&\leq \alpha \frac{[1+d(z,z)]\,d(y,y)}{[1+d(z,y)]} + \beta \frac{[1+d(z,y)]\,d(z,z)}{[1+d(z,z)+d(y,y)]} + \gamma d(z,y)\\
&\leq \gamma d(z,y)
\end{aligned}
$$

$\Rightarrow d(z,y) \leq \gamma d(z,y)$ or $(1-\gamma)d(z,y) \leq 0 \Rightarrow z = y$ since $0<\gamma<1$. This completes the proof. □

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