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An Alternative Failure Model of Three Parameter Exponential Distribution: Study of its Structural Properties and Parameters Estimation

Shradha Dwivedi[1,*]

1 IT unit, Salalah college of technology, Salalah, Sultanate of Oman.

Abstract: Here we studied a three parameter exponential distribution which can work as an alternative model for various other available exponential distribution in Reliability and Life testing theory.It has its own structural properties which makes it a unique and flexible model for various real life data. It is a three parameter model with different shape parameter but having the same location and scale parameter.

Keywords: Three parameter exponential distribution, Mode, Moment Estimator, Maximum Likelihood Estimator.
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1. Introduction

Classical Exponential distribution has *pdf* of the form :

$$f(x/\sigma) = \frac{1}{\theta} * \exp\left(\frac{-x}{\theta}\right), \quad where\, x \geq 0, \sigma > 0 \tag{1}$$

here θ is a scale parameter.

2. New Exponential Distribution

Cumulative distribution function of exponential distribution is :

$$F(x,\mu,p,\theta) = 1 - \exp\left(-\frac{(x-\mu)^p}{\theta^p}\right), \; where\;\; 0 < x < \infty, \mu \geq 0, \theta > 0 \tag{2}$$

here μ is a location parameter,θ *and* p are scale and shape parameters respectively. The *pdf* of new three parameter Exponential distribution is

$$f(x,\mu,p,\theta) = p\; \frac{(x-\mu)^{p-1}}{\theta^p} \exp\left(-\frac{(x-\mu)^p}{\theta^p}\right), \; where\;\; 0 < x < \infty,\; \mu \geq 0,\; \theta > 0 \tag{3}$$

If we consider Location Parameter $\mu = 0$,then the above *pdf* reduces to

$$f(x,\mu,p,\theta) = p\; \frac{x^{p-1}}{\theta^p} \exp\left(-\frac{x^p}{\theta^p}\right), \; where\;\; 0 < x < \infty,\; \mu \geq 0,\; \theta > 0 \tag{4}$$

Special Case: If Location parameter $\mu = 0$ and the shape parameter $p = 1$ then the above *pdf* reduces to the *pdf* of classical model one parameter Exponential Distribution, Equation (1).

* *E-mail: k.getshradha@gmail.com*

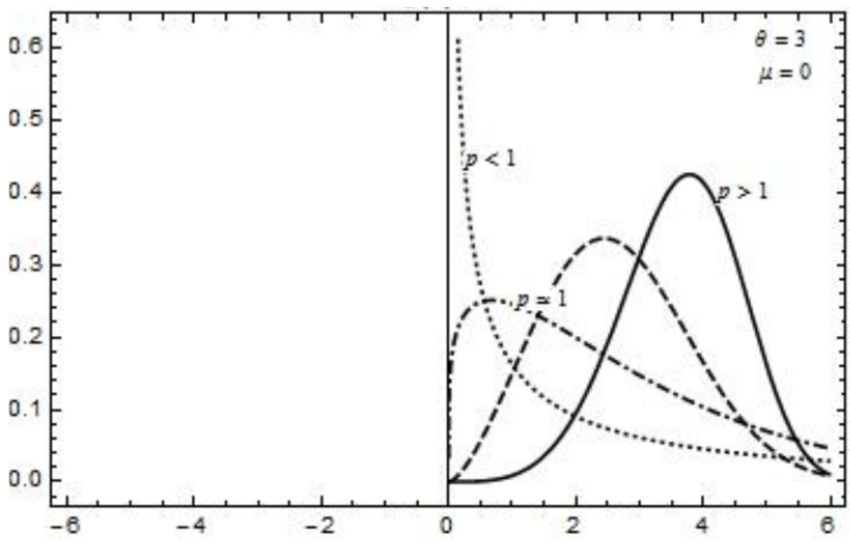

Figure 1.

Hazard Rate: function and Reliability functions are as follows

$$h(x) = exp(-\frac{x^p}{\theta^p}), \;\; where \;\; 0 < x < \infty, \;\; \theta > 0 \tag{5}$$

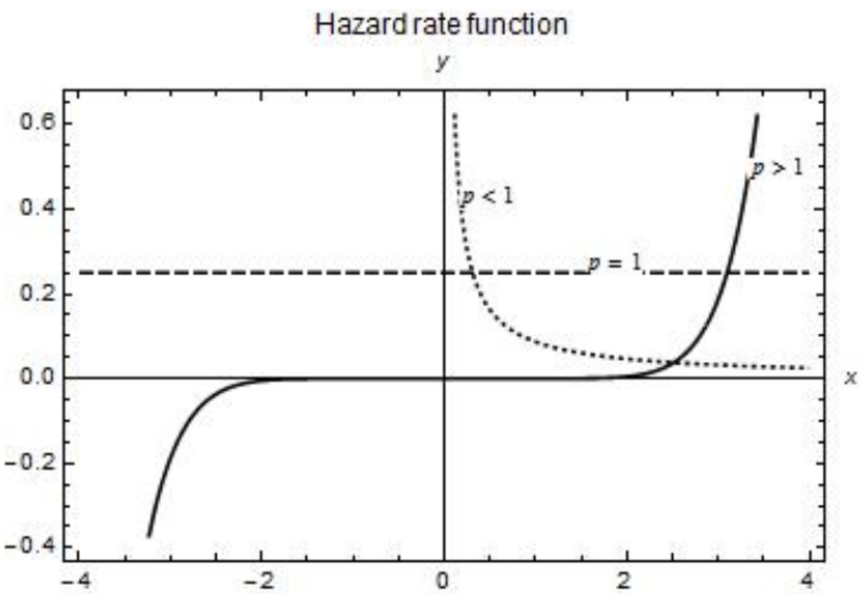

Figure 2.

Corollary 2.1.

(1). If $p > 1$ for all $x > 0$, then the hazard function is monotonically increasing.

(2). If $p < 1$ for all $x > 0$, then the hazard function is monotonically decreasing.

(3). If $p = 1$ for all $x > 0$, then the hazard function is constant.

(4). In addition, $\lim_{x \to +\infty} f(x) = \infty$, then hazard rate function is upside down bathtub shape.

$$R(x) = p\,\frac{x^{p-1}}{\theta^p}, \;\; where \;\; 0 < x < \infty, \;\; \theta > 0 \tag{6}$$

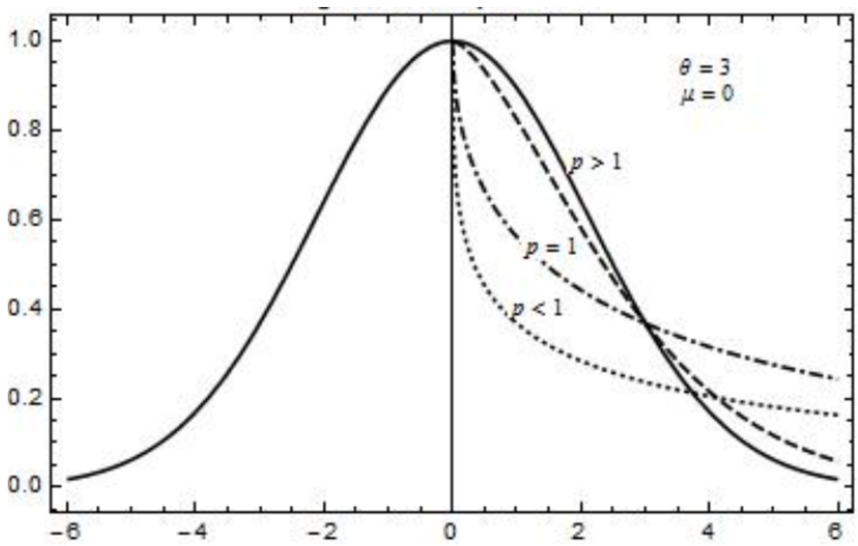

Figure 3.

Note: Like a classical exponential distribution Equation (1), above distribution has also the property of 'forgetfulness' or 'no memory'. It means if a unit has survived for t hours,then the probability of its surviving for an addition h hours is exactly same as the probability of surviving h hours of a new item.

We consider the exponential density eqn(3) with $\mu = 0$, as follows

$$f(x, \theta, p) = p\ \frac{x^{p-1}}{\theta^p} exp\left(-\frac{x^p}{\theta^p}\right), x, \theta, p > 0$$

$$P(x \geq t + h \mid X \geq t) = \frac{\int\limits_{t+h}^{\infty} p\ \frac{x^{p-1}}{\theta^p} exp\left(-\frac{x^p}{\theta^p}\right) dx}{\int\limits_{t}^{\infty} p\ \frac{x^{p-1}}{\theta^p} exp\left(-\frac{x^p}{\theta^p}\right) dx}$$

$$= exp\left(-\frac{h^p}{\theta^p}\right) = P(X \geq h)$$

2.1. Structural Properties

Here we discuss the general properties of exponential distribution like Mean,Mode,Moments,Variance ,skewness and kurtosis.

Mode: The probability distribution function of new model is

$$f(x, \mu, p, \theta) = p\ \frac{(x-\mu)^{p-1}}{\theta^p}\ exp\left(-\frac{(x-\mu)^p}{\theta^p}\right),\quad where\ \ 0 < x < \infty,\ \mu \geq 0,\ \theta > 0 \tag{7}$$

here, μ is a location parameter, θ and p are scale and shape parameters respectively. In order to discuss monotonicity of proposed model we take logarithm of its pdf:

$$\log f(x, \mu, p, \theta) = \log\left[p\ \frac{(x-\mu)^{p-1}}{\theta^p}\ exp\left(-\frac{(x-\mu)^p}{\theta^p}\right)\right]$$

$$= log\ \left[\frac{p}{\theta^p}\right] + log\left[(x-\mu)^{p-1}\right] - \frac{(x-\mu)^p}{\theta^p}$$

$$\frac{\partial f}{\partial x} = 0 \Leftrightarrow x = \left[\frac{\theta^p(p-1)}{p}\right]^{\frac{1}{p}} + \mu$$

The Mode of proposed model is given by

$$x = \theta\left[\frac{(p-1)}{p}\right]^{\frac{1}{p}} + \mu \tag{8}$$

Mean, Moments and Variance, Skewness and Kurtosis: If we consider location parameter $\mu = 0$, then the first Moment of new Exponential distribution is

$$\boldsymbol{Mean} = E(X) = m_1^{'} = \int_0^{\infty} p \ \frac{x^{p-1}}{\theta^p} \ exp\left(-\frac{x^p}{\theta^p}\right) \ dx$$

$$= \theta \, \Gamma\left(\frac{1}{p} + 1\right), \quad where \ \frac{1}{p} > 1 \tag{9}$$

$$E(X^2) = m_2^{'} = \int_0^{\infty} p \ \frac{x^{p-1}}{\theta^p} \ exp\left(-\frac{x^p}{\theta^p}\right) \ dx$$

$$= \theta^2 \, \Gamma\left(\frac{2}{p} + 1\right), \quad where \ \frac{2}{p} > 1 \tag{10}$$

and

$$E(X^3) = m_3^{'} = \theta^3 \, \Gamma\left(\frac{3}{p} + 1\right) \tag{11}$$

In this way the r^{th} ***moment*** of the model will be,

$$E(X^r) = m_r^{'} = \theta^r \, \Gamma\left(\frac{r}{p} + 1\right) \tag{12}$$

where, $\Gamma\left(\frac{r}{p} + 1\right)$ is an incomplete Gamma function. Now, the ***second and third central Moments*** are

$$m_2 = m_2^{'} - (m_1^{'})^2 = \theta^2 \left[\Gamma\left(\frac{2}{p} + 1\right) - \Gamma\left(\frac{1}{p} + 1\right)^2\right] \tag{13}$$

Or,

$$\boldsymbol{variance} = \theta^2 \left[\Gamma\left(\frac{2}{p} + 1\right) - \Gamma\left(\frac{1}{p} + 1\right)^2\right] \tag{14}$$

$$m_3 = m_3^{'} - 3m_1^{'}m_2^{'} + 2m_2^{'3} = \theta^3 \left[\Gamma\left(\frac{3}{p} + 1\right) - 3\Gamma\left(\frac{1}{p} + 1\right)\Gamma\left(\frac{2}{p} + 1\right) + 2\Gamma\left(\frac{1}{p} + 1\right)^3\right] \tag{15}$$

and

$$m_4 = m_4^{'} - 4m_1^{'}m_3^{'} + 6m_1^{'2}m_2^{'} - 3m_1^{'4}$$

$$= \theta^4 \left[\Gamma\left(\frac{4}{p} + 1\right) - 4\Gamma\left(\frac{1}{p} + 1\right)\Gamma\left(\frac{3}{p} + 1\right) + 6\Gamma\left(\frac{1}{p} + 1\right)^2\Gamma\left(\frac{2}{p} + 1\right) + 2\Gamma\left(\frac{1}{p} + 1\right)^3\right] \tag{16}$$

We know that $m = m_1^{'}$ and $m_0^{'} = 1$. Based on the first four central moments of the model, the measures of ***skewness*** $A(\Phi)$ and ***kurtosis*** $k(\Phi)$ can obtained as,

$$A(\Phi) = \frac{m_3(\theta) - 3m_1(\theta)m_2(\theta) + 2m_1^3(\theta)}{[m_2(\theta) - m_1^2(\theta)]^{\frac{3}{2}}} \tag{17}$$

and

$$k(\Phi) = \frac{m_4(\theta) - 4m_1(\theta)m_3(\theta) + 6m_1^2(\theta)m_2(\theta) - 3m_1^4(\theta)}{[m_2(\theta) - m_1^2(\theta)]^2} \tag{18}$$

3. Statistical Properties

Here, we will discuss the Moment generating function,Characteristic function, Fisher Information Matrix and different Entropies.

3.1. Moment generating function

The Moment generating function of proposed exponential distribution, equation (4) is defined as:

$$\begin{aligned} E(e^{tx}) &= \int_0^\infty e^{tx} f(x)dx \\ &= \int_0^\infty \left(1 + t\frac{x}{1!} + t^2\frac{x^2}{2!} + t^3\frac{x^3}{3!} + \dots\right) f(x)dx \\ &= 1 + tm_1 + t^2 m_2 + t^3 m_3 + \dots \\ &= \sum_{j=0}^{\infty} \frac{t^j\, m_j}{j!}, \quad where\ j = 1, 2, 3, \dots \end{aligned}$$

where m_n is the nth moment of the distribution.

3.2. Characteristic Function

The Characteristic function of proposed exponential distribution, eqn(4) is defined as:

$$\begin{aligned} E(e^{itx}) &= \int_0^\infty e^{itx} f(x)dx \\ &= \int_0^\infty \left(1 + (it)\frac{x}{1!} + (it)^2\frac{x^2}{2!} + (it)^3\frac{x^3}{3!} + \dots\right) f(x)dx \\ &= 1 + (it)m_1 + (it)^2 m_2 + (it)^3 m_3 + \dots \\ &= \sum_{j=0}^{\infty} \frac{(it)^j\, m_j}{j!}, \quad where\ j = 1, 2, 3, \dots \end{aligned}$$

where m_n is the nth moment of the distribution.

3.3. Fisher Information matrix

Fisher Information matrix of a random variable X is given by

$$I(\theta) = E\left[\frac{\partial}{\partial\theta}\log\left(f(x;\theta)\right)\right]^2$$

Now, if $logf(x,\theta)$ is twice differentiable w.r.t. θ under certain conditions, Fisher Information is given by;

$$I(\theta) = E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log\left(f(x;\theta)\right)\right] \tag{19}$$

Probability density function of proposed exponential distribution is given by equation (3),as follows

$$f(x, \mu, p, \theta) = p\ \frac{(x-\mu)^{p-1}}{\theta^p}\ exp\left(-\frac{(x-\mu)^p}{\theta^p}\right), \quad where\ \ 0 < x < \infty,\ \mu \geq 0,\ \theta > 0$$

Applying logarithm both sides we get,

$$\begin{aligned} \log f(x, \mu, p, \theta) &= \log\left[p\ \frac{(x-\mu)^{p-1}}{\theta^p}\ exp\left(-\frac{(x-\mu)^p}{\theta^p}\right)\right] \\ &= \log\left[\frac{p}{\theta^p}\right] + \log\left[(x-\mu)^{p-1}\right] - \frac{(x-\mu)^p}{\theta^p} \\ \log f(x, \mu, p, \theta) &= \log p - p\log\theta + (p-1)\log(x-\mu) - \frac{(x-\mu)^p}{\theta^p} \end{aligned} \tag{20}$$

differentiating both sides w.r.t. μ, p and θ, we get

$$\frac{\partial \log f}{\partial \mu} = -\frac{(p-1)}{(x-\mu)} + \frac{p(x-\mu)^{p-1}}{\theta^p} \tag{21}$$

$$\frac{\partial \log f}{\partial p} = \frac{1}{p} - \log\theta + \log(x-\mu) - (x-\mu)^p\theta^{-p}[\log(x-\mu) - \log\theta] \tag{22}$$

$$\frac{\partial \log f}{\partial \theta} = -\frac{p}{\theta} + \frac{p(x-\mu)^p}{\theta^{p+1}} \tag{23}$$

Differentiating again the above equations w.r.t. μ, p and θ, we have

$$\frac{\partial^2 \log f}{\partial \mu^2} = -\frac{(p-1)}{(x-\mu)^2} - \frac{p(p-1)x^{p-2}}{\theta^p} \tag{24}$$

$$\frac{\partial^2 \log f}{\partial p \partial \mu} = -\frac{1}{(x-\mu)} + \frac{p(x-\mu)^{p-1}}{\theta^p} - p(x-\mu)^{p-1}\theta^{-p}\log(x-\mu) - \log\theta) \tag{25}$$

$$\frac{\partial^2 \log f}{\partial \theta \partial \mu} = -\frac{p(x-\mu)^{p-1}}{\theta^{p+1}} \tag{26}$$

$$\frac{\partial^2 \log f}{\partial p^2} = -\frac{1}{p^2} - \frac{x^{p-1}(p\theta - x)(\log x - \log\theta)}{\theta^{p+1}} \tag{27}$$

$$\frac{\partial^2 \log f}{\partial \mu \partial p} = -\frac{1}{(x-\mu)} - p(x-\mu)^{p-1}\theta^{-p}[\log(x-\mu) - \log\theta)] + \frac{p(x-\mu)^{p-1}}{\theta^p} \tag{28}$$

$$\frac{\partial^2 \log f}{\partial \theta \partial p} = -\frac{1}{\theta} + \frac{x^p}{\theta^{p+1}}[p(\log x - \log\theta) + 1] \tag{29}$$

$$\frac{\partial^2 \log f}{\partial \theta^2} = \frac{p}{\theta^2} - \frac{p(p+1)x^p}{\theta^{p+2}} \tag{30}$$

$$\frac{\partial^2 \log f}{\partial \mu \partial \theta} = -\frac{p(x-\mu)^{p-1}}{\theta^{p+1}} \tag{31}$$

$$\frac{\partial^2 \log f}{\partial p \partial \theta} = -\frac{1}{\theta} + \frac{x^p}{\theta^{p+1}}[p(\log x - \log\theta) + 1] \tag{32}$$

Taking expectations on both sides of the above equations, we get

$$-E\left[\frac{\partial^2 \log f}{\partial \mu^2}\right] = -E\left[-\frac{(p-1)}{(x-\mu)^2} - \frac{p(p-1)(x-\mu)^{p-2}}{\theta^p}\right] \tag{33}$$

$$-E\left[\frac{\partial^2 \log f}{\partial p \partial \mu}\right] = -E\left[-\frac{1}{(x-\mu)} + \frac{p(x-\mu)^{p-1}}{\theta^p} - p(x-\mu)^{p-1}\theta^{-p}(\log(x-\mu) - \log\theta)\right] \tag{34}$$

$$-E\left[\frac{\partial^2 \log f}{\partial \theta \partial \mu}\right] = -E\left[-\frac{p(x-\mu)^{p-1}}{\theta^{p+1}}\right] \tag{35}$$

$$-E\left[\frac{\partial^2 \log f}{\partial p^2}\right] = -E\left[-\frac{1}{p^2} - \frac{(x-\mu)^{p-1}(p\theta - (x-\mu))(\log(x-\mu) - \log\theta)}{\theta^{p+1}}\right] \tag{36}$$

$$-E\left[\frac{\partial^2 \log f}{\partial \mu \partial p}\right] = -E\left[-\frac{1}{(x-\mu)} - p(x-\mu)^{p-1}\theta^{-p}\left[\log(x-\mu) - \log\theta)\right] + \frac{p(x-\mu)^{p-1}}{\theta^p}\right] \quad I(2,2) \tag{37}$$

$$-E\left[\frac{\partial^2 \log f}{\partial \theta \partial p}\right] = -E\left[-\frac{1}{\theta} + \frac{(x-\mu)^p}{\theta^{p+1}}\left[p(\log(x-\mu) - \log\theta) + 1\right]\right] \tag{38}$$

$$-E\left[\frac{\partial^2 \log f}{\partial \theta^2}\right] = -E\left[\frac{p}{\theta^2} - \frac{p(p+1)(x-\mu)^p}{\theta^{p+2}}\right] \tag{39}$$

$$-E\left[\frac{\partial^2 \log f}{\partial \mu \partial \theta}\right] = -E\left[-\frac{p(x-\mu)^{p-1}}{\theta^{p+1}}\right] \tag{40}$$

$$-E\left[\frac{\partial^2 \log f}{\partial p \partial \theta}\right] = -E\left[-\frac{1}{\theta} + \frac{(x-\mu)^p}{\theta^{p+1}}\left[p(\log(x-\mu) - \log\theta) + 1\right]\right] \tag{41}$$

For equation (33) to equation (41), we need to put the following expectation values

$$E(\log(x-\mu)) = 1 + \frac{1}{\gamma}, \quad where \;\; \gamma \approx -0.577216 \;\; is\, the\, Euler - Maheroni\;\; constant$$

$$E\left[(x-\mu)^p\right] = \theta^p$$

$$E\left[(x-\mu)^{p-1}\right]=\theta^{p-1}\Gamma\left(2-\frac{1}{p}\right)$$
$$E\left[(x-\mu)^{p-2}\right]=\theta^{p-2}\Gamma\left(2-\frac{2}{p}\right)$$
$$E\left[(x-\mu)^{2}\right]=\theta^{2}\Gamma\left(\frac{2}{p}+1\right)$$
$$E\left[(x-\mu)\right]=\theta\Gamma\left(\frac{1}{p}+1\right)$$

by entering above values in equation (33) to equation (41) we get the Fisher Information matrix as

$$I(1,1)=-E\left[\frac{\partial^2}{\partial\mu^2}\right]=\frac{(p-1)}{\theta^2\Gamma\left(\frac{2}{p}+1\right)}+\frac{p(p-1)\Gamma\left(2-\frac{2}{p}\right)}{\theta^2}$$
$$I(1,2)=-E\left[\frac{\partial^2}{\partial p\partial\mu}\right]=\frac{1}{\theta\Gamma\left(\frac{1}{p}+1\right)}-\frac{p\Gamma\left(2-\frac{1}{p}\right)}{\theta}+p\theta^{p-1}\Gamma\left(2-\frac{1}{p}\right)\theta^{-p}\left(1+\frac{1}{\gamma}\right)-\log\theta$$
$$I(1,3)=-E\left[\frac{\partial^2}{\partial\theta\partial\mu}\right]=\frac{p\Gamma\left(2-\frac{1}{p}\right)}{\theta^2}$$
$$I(2,1)=-E\left[\frac{\partial^2\log f}{\partial p^2}\right]=\frac{1}{p^2}+\frac{\Gamma\left(2-\frac{1}{p}\right)(p\theta-\theta)\Gamma\left(\frac{1}{p}+1\right)\left[\left(1+\frac{1}{\gamma}\right)-\log\theta\right]}{\theta^2}$$
$$I(2,2)=-E\left[\frac{\partial^2}{\partial\mu\partial p}\right]=\frac{1}{\theta\Gamma\left(\frac{1}{p}+1\right)}+p\theta^{-1}\Gamma\left(2-\frac{1}{p}\right)\left[\left(1+\frac{1}{\gamma}\right)-\log\theta\right]-\frac{p\Gamma\left(2-\frac{1}{p}\right)}{\theta}$$
$$I(2,3)=-E\left[\frac{\partial^2}{\partial\theta\partial p}\right]=\frac{1}{\theta}-\frac{1}{\theta}\left[p\left(\left(1+\frac{1}{\gamma}\right)-\log\theta\right)+1\right]$$
$$I(3,1)=-E\left[\frac{\partial^2}{\partial\theta^2}\right]=\frac{p}{\theta^2}+\frac{p(p+1)}{\theta^2}$$
$$I(3,2)=-E\left[\frac{\partial^2}{\partial\mu\partial\theta}\right]=\frac{p\Gamma\left(2-\frac{1}{p}\right)}{\theta^2}$$
$$I(3,3)=-E\left[\frac{\partial^2}{\partial p\partial\theta}\right]=\frac{1}{\theta}-\frac{1}{\theta}\left[p\left(\left(1+\frac{1}{\gamma}\right)-\log\theta\right)+1\right]$$

3.4. Entropy estimation

Here we will discuss the Shannon's entropy ,Renyi' entropy and q-entropy. **Shannon's Entropy:** By equation (4) consider the pdf of three parameter Exponential distribution:

$$f(x,\theta,p)=p\;\frac{x^{p-1}}{\theta^p}\exp\left(-\frac{x^p}{\theta^p}\right),\quad x,\theta,p>0$$

taking log likelihood of both sides

$$Lf(x,\theta,p)=n\left[\log p-np\log\theta+n(p-1)\sum_{i=1}^{n}\log x_i\right]-\frac{1}{\theta^p}\sum_{i=1}^{n}\log x_i^p$$
$$Lf(x,\theta,p)=n\left[\log p-p\log\theta+(p-1)\sum_{i=1}^{n}\log x_i\right]-\frac{1}{\theta^p}\sum_{i=1}^{n}\log x_i^p$$
$$\frac{Lf(x,\theta,p)}{n}=\left[\log p-p\log\theta+(p-1)\sum_{i=1}^{n}\log x_i\right]-\frac{1}{n\theta^p}\sum_{i=1}^{n}\log x_i^p$$

Shannon's entropy of the new model is

$$\widehat{H}=-\left[\frac{Lf(x,\theta,p)}{n}\right]$$

Or,

$$\widehat{H} = -\left[\log \widehat{p} - \widehat{p} \log \widehat{\theta} + (\widehat{p} - 1)\sum_{i=1}^{n} \log x_i - \frac{1}{\widehat{\theta}^{\widehat{p}}} \sum_{i=1}^{n} \log \overline{x_i^p}\right] \tag{42}$$

Renyi' and q-Entropy: The Renyi' entropy for a random sample $X = x_1, x_2, x_3, \ldots$ is defined as

$$I_R(\delta) = \frac{1}{1-\delta} \log \mid I(\delta) \mid \tag{43}$$

where

$$I(\delta) = \int_R f^{\delta}(x)\, dx, \quad \delta > 0 \; and \; \delta \neq 1 \tag{44}$$

by considering *pdf f(x)* from equation (4) to equation (21) we get,

$$\begin{aligned}\int_0^{\infty} f^{\delta}(x) dx &= \frac{p^{\delta}}{\theta^{\delta p}} \int_0^{\infty} x^{\delta(p-1)} \exp\left(-\delta \frac{x^p}{\theta^p}\right) dx \\ &= \frac{p^{\delta-1}}{\theta^{\delta+1}}\ \delta^{\frac{\delta(p-1)+1}{p}}\ \Gamma\left(\frac{\delta(p-1)+1}{p}\right)\end{aligned}$$

Hence the ***Renyi' Entropy*** reduces to

$$I_R(\delta) = \frac{1}{1-\delta} \log \left[\frac{p^{\delta-1}}{\theta^{\delta+1}}\ \delta^{\frac{\delta(p-1)+1}{p}}\ \Gamma\left(\frac{\delta(p-1)+1}{p}\right)\right] \tag{45}$$

The ***q-Entropy*** say $H_q(f)$, is defined by

$$H_q(f) = \frac{1}{q-1} \log\left[1 - I_q(f)\right] \tag{46}$$

Where,

$$I_q(f) = \int_R f^q(x)\, dx, \quad q > 0 \; and \; q \neq 1$$

We get the q-Entropy for the new model is as follows,

$$H_q(f) = \frac{1}{q-1} \log\left[1 - \frac{p^{q-1}}{\theta^{q+1}}\, q^{\frac{q(p-1)+1}{p}}\ \Gamma\left(\frac{q(p-1)+1}{p}\right)\right] \tag{47}$$

4. Parameters estimation

In this section we will discuss the the Moment estimates and Maximum likelihood Functions.

4.1. Moment estimates

From the pdf of equation (3) we obtain the Moment estimates of the shape, scale and location parameters by using the corresponding values as follows

$$\widehat{\widehat{p}} = \frac{4}{b_1}, \quad where \;\; b_1 = \frac{m_3^2}{m_2^3} \tag{48}$$

and $m_3 = m_3' - 3m_1'm_2' + 2m_2'^3$; $m_2 = m_2' - (m_1')^2$ by equation (14) and (15) we can estimate shape parameter. For the scale parameter θ we follow the following equation

$$\widehat{\widehat{\theta}} = \sqrt{\frac{s^2}{\widehat{\widehat{p}}}}, \quad s^2 = \frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n-1} \tag{49}$$

and moment estimates for location parameter is

$$\widehat{\widehat{\mu}} = \overline{x} - \widehat{\widehat{p}}\ \widehat{\widehat{\theta}} \tag{50}$$

Moment estimates are efficient when p is fairly large. We can use the moment estimates as the starting values for the iterative solutions of the likelihood equations.

4.2. Maximum Likelihood Estimation

Maximum likelihood Estimation can be done by taking log-likelihood of Pdf of new failure model by equation (3)

$$f(x,\mu,p,\theta) = p\ \frac{(x-\mu)^{p-1}}{\theta^p}\exp\left(-\frac{(x-\mu)^p}{\theta^p}\right),\ \ where\ \ 0<x<\infty,\ \mu\geq 0,\ \theta>0$$

Applying log-likelihood both sides we get,

$$\log\left[\prod_{i=1}^{n} f(x,\mu,p,\theta)\right] = \log\left[\prod_{i=1}^{n}\left[p\ \frac{(x-\mu)^{p-1}}{\theta^p}\exp\left(-\frac{(x-\mu)^p}{\theta^p}\right)\right]\right]$$

$$Lf(x) = \log\left[\frac{p}{\theta^p}\right]^n + \prod_{i=1}^{n}\left[(x_i-\mu)^{p-1}\right] + \exp\left[-\sum_{i=1}^{n}\frac{(x_i-\mu)^p}{\theta^p}\right]$$

$$Lf(x) = n\log p - np\ \log\theta + (p-1)\sum_{i=1}^{n}\ \log(x_i-\mu) - \sum_{i=1}^{n}\frac{(x_i-\mu)^p}{\theta^p} \tag{51}$$

differentiating equation (51) w.r.t. p, θ and μ and equation to zero to get MLE's of those parameters,respectively

$$\frac{\partial L}{\partial p} = \frac{n}{p} - n\log\theta + \sum_{i=1}^{n}\ \log(x_i-\mu) - \sum_{i=1}^{n}(x_i-\mu)^p\theta^{-p}(\log(x_i-\mu)-\log\theta) \tag{52}$$

$$\frac{\partial L}{\partial \theta} = -\frac{n}{\theta} + \widehat{p}\frac{\sum_{i=1}^{n}(x-\mu)_i^p}{\theta^{p+1}} \tag{53}$$

$$\frac{\partial L}{\partial \mu} = -\sum_{i=1}^{n}\frac{(\widehat{p}-1)}{(x_i-\mu)} - p\sum_{i=1}^{n}\frac{(x_i-\mu)^{p-1}}{\theta^p} \tag{54}$$

equating equation (52), (53) and (54) to zero ,in order to get MLE's

$$g(\widehat{p}) \approx \frac{n}{\widehat{p}} - n\log\theta + \sum_{i=1}^{n}\ \log(x_i-\mu) - \sum_{i=1}^{n}(x_i-\mu)^{\widehat{p}}\ \ \theta^{-\widehat{p}}(\log(x_i-\mu)-\log\theta)$$

$$\widehat{\theta} = \left[\frac{\sum_{i=1}^{n}(x_i-\mu)^{\widehat{p}}}{n}\right]^{\frac{1}{\widehat{p}}}$$

$$\sum_{i=1}^{n}(x_i-\widehat{\mu})^{\widehat{p}} = \frac{(\widehat{p}-1)\theta^{\widehat{p}}}{\widehat{p}}$$

$$\tau(\widehat{\mu}) \approx \sum_{j=1}^{\widehat{p}}(-1)^j\frac{\Gamma(\widehat{p})}{j!\Gamma(\widehat{p}-j)}\mu^j = \frac{(\widehat{p}-1)\ \ \theta^{\widehat{p}}\sum_{i=1}^{n}\sum_{j=1}^{\widehat{p}}(-1)^j\frac{\Gamma(\widehat{p})}{j!\Gamma(\widehat{p}-j)}x_i^j}{\widehat{p}\ \sum_{i=1}^{n}x_i^p}$$

if location parameter $\mu = 0$ then MLE's of shape and scale parameter reduces to

$$g(\widehat{p}) \approx \frac{n}{\widehat{p}} - n\log\theta + \sum_{i=1}^{n}\ \log x_i - \sum_{i=1}^{n}x_i^{\widehat{p}}\ \ \theta^{-\widehat{p}}(\log x_i - \log\theta) \tag{55}$$

$$\widehat{\theta} = \left[\frac{\sum_{i=1}^{n}x_i^{\widehat{p}}}{n}\right]^{\frac{1}{\widehat{p}}} \tag{56}$$

equation (55) may be solved for $\widehat{p}$ by Newton-Rampson method or other suitable iterative method and this value substituted in equation (56) to get $\widehat{\theta}$.

Example 4.1. *A random sample of 25 observation was generated from the new exponential failure model with* $\mu = 0$, $p = 1$, $\theta = 3$.

1.8487	0.3761	0.7500	3.0530	1.3545
1.8802	1.5700	1.7708	1.3592	3.0466
1.7961	1.5319	0.5903	0.6288	0.6461
1.6560	1.7172	1.9310	1.0509	1.6173
1.3162	0.7705	1.8889	1.8889	4.1505

Table 1. **Sample**

By Table 2 it is clear that $g(p) = 0.098965 (\approx 0)$ *at* $p = 0.93007$ *so,* $\widehat{p} = 0.93007$.

p	g(p)
0.9	0.997045226
0.95	-0.464943078
1	-1.780732552
1.05	-2.971208742
1.025	-2.390488649
0.002	12473.21927
0.899312	1.018296039
1.034375	-2.611548262
0.935937	-0.069532973
0.924535	0.259887868
0.912	0.63154815
0.934961	-0.041649317
0.935059	-0.04445174
1.035107	-2.628640063
1.035083	-2.628080059
0.93007	0.098964675
0.93	0.100987878
0.73508	7.229171132
0.835081	3.15648312

Table 2. **Estimation of p**

5. Conclusion

Above study describes three parameter exponential model in a wide perspectives model.It gives maximum characteristic details of three parameter exponential model which can be used for the real life data.

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