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The Upper Edge Fixed Steiner Number of a Graph

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Abstract: For a non-empty set W of vertices in a connected graph G, the Steiner distance $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each such subgraph is a tree and is called a Steiner tree with respect to W or a Steiner W-tree. $S(W)$ denotes the set of vertices that lies in Steiner W-trees. Let G be a connected graph with at least 2 vertices. A set $W \subseteq V(G)$ is called a Steiner set of G if $S(W) = V(G)$. The Steiner number $s(G)$ is the minimum cardinality of a Steiner set. Let G be a connected graph with at least 3 vertices. For an edge $e = xy$ in G, a set $W \subseteq V(G) - \{x, y\}$ is called an edge fixed Steiner set of G if $W' = W \cup \{x, y\}$ is a Steiner set of G. The minimum cardinality of an edge fixed Steiner set is called the edge fixed Steiner number of G and is denoted by $s_e(G)$. Also the Steiner W-tree necessarily contains the edge e and is called edge fixed Steiner W-tree. In this paper, the concept of upper edge fixed Steiner number of a graph G denoted by $s_e^+(G)$ is studied. Also the graphs in which the upper edge fixed Steiner number is equal to n or $n-1$ are characterized. It is shown that for every pair a, b of integers with $a \geq 2$ and $b \geq 2$, there exists a connected graph G with $s_e(G) = a$ and $s_e^+(G) = b$.

Keywords: Steiner set, edge fixed Steiner set, Steiner number, edge fixed Steiner number, minimal edge fixed Steiner set, upper edge fixed Steiner number.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. An $u - v$ path of length $d(u, v)$ is called an $u - v$ geodesic. It is known that the distance is a metric on the vertex set of G. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is called the radius and the maximum eccentricity is called the diameter of G and are denoted by rad G and diam G respectively. For basic graph theoretic terminology, we refer to Harary [5]. For a non-empty set W of vertices in a connected graph G, the Steiner distance $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each such subgraph is a tree and is called a Steiner tree with respect to W or a Steiner W-tree. A set $W \subseteq V$ of vertices in the graph G is called a Steiner set if every vertex in G lies in a Steiner-W-tree which is a minimum connected subgraph of G containing W. The Steiner number $s(G)$ is the minimum cardinality of a Steiner set. A Steiner set with minimum cardinality is denoted as an s-set. In this paper, we introduce the upper edge fixed Steiner number of a graph using edge fixed Steiner sets in a graph. We provide the upper edge fixed Steiner number of some standard graphs. Also,we provide the bounds for the upper edge fixed Steiner number through some distance related parameters. For positive integers r, d and $n \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G

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with rad $G = r$, diam $G = d$ and $s_e^+(G) =$ n or $n-1$ for any edge e in G. Also for any two positive integers a and b with $2 \le a \le b$, there exists a connected graph G with $s_e(G) = a$ and $s_e^+(G) = b$ for some edge e in G. The following theorems are used wherever required.

Theorem 1.1 ([7]). *Let T be any non trivial tree and k be the number of end vertices in T. Let $e = xy$ be any edge of T. Then*

$$s_e(T) = \begin{cases} k & \text{if neither } x \text{ nor } y \text{ is an end vertex of } T \\ k-1 & \text{if either } x \text{ or } y \text{ is an end vertex of } T \end{cases}$$

Theorem 1.2 ([7]). *For any complete graph K_p with $p \ge 3$, $s_e(K_p) = p - 2$ where e is any edge of K_p.*

Theorem 1.3 ([7]). *Let $e = xy$ be any edge of a connected graph G of order at least 3. Then every extreme vertex of G, other than the vertices x and y (whether x and y are extreme vertices or not) belongs to every edge fixed Steiner set in G. In particular, every end vertex of G other than x and y belongs to every edge fixed Steiner set of G.*

Theorem 1.4 ([7]). *For the path P_p with $p \ge 3$,*

$$s_e(P_p) = \begin{cases} 1 & \text{if } e \text{ is an end edge of } P_p \\ 2 & \text{otherwise} \end{cases}$$

Theorem 1.5 ([7]). *If C_p is a cycle of order p and e is any edge of C_p, then*

$$s_e(C_p) = \begin{cases} 1 & \text{if } p \text{ is odd} \\ 2 & \text{if } p \text{ is even} \end{cases}$$

Theorem 1.6 ([7]). *For any edge e in the complete bipartite graph $K_{m,n}$ with $m \le n$ and $m+n \ge 3$, $s_e(K_{m,n}) = m+n-2$.*

2. The Upper Edge Fixed Steiner Number

In this section, we define the upper edge fixed Steiner number of a graph and illustrate it with a suitable example.

Definition 2.1. *Let G be a simple connected graph with at least three vertices. For any edge $e = xy$ in G, an edge fixed Steiner set W is called a minimal edge fiexed Steiner set if no proper subset of W is an edge fixed Steiner set of G. The upper edge fixed Steiner number $s_e^+(G)$ is the maximum cardinality of a minimal edge fixed Steiner set of G for each edge e in G.*

Example 2.2. *Consider the graph G shown in Figure 1.*

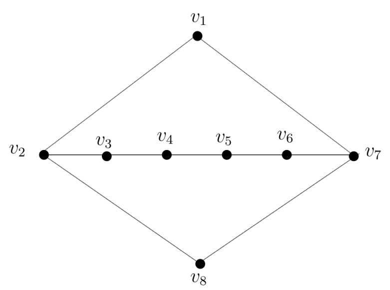

Figure 1.

The minimum edge fixed Steiner sets for each edge e of G, the edge fixed Steiner number for each edge e of G, the minimal edge fixed Steiner sets for each edge e of G and the upper edge fixed Steiner number for each edge e of G are given in the following table.

Edge e	s_e -sets	$s_e(G)$	minimal s_e -sets	$s_e^+(G)$
v_1v_2	$\{v_3,v_6\}$, $\{v_4, v_5\}$, $\{v_4, v_7\}$, $\{v_5,v_7\}$, $\{v_5, v_8\}$	2	$\{v_4, v_6, v_8\}$	3
v_1v_7	$\{v_2, v_4\}$, $\{v_2, v_5\}$,$\{v_2, v_8\}$, $\{v_3, v_6\}$, $\{v_4, v_5\}$	2	$\{v_4, v_6, v_8\}$	3
v_2v_3	$\{v_6\}$	1	$\{v_4, v_7\}$, $\{v_5, v_7\}$	2
v_2v_8	$\{v_1, v_5\}$, $\{v_2, v_5\}$, $\{v_3, v_6\}$, $\{v_4, v_5\}$, $\{v_4, v_7\}$	2	$\{v_1, v_4, v_6\}$	3
v_3v_4	$\{v_5\}$	1	$\{v_1, v_2\}$, $\{v_2, v_8\}$	2
v_4v_5	$\{v_1,v_3\}$, $\{v_1,v_6\}$, $\{v_1,v_8\}$, $\{v_3,v_8\}$, $\{v_6,v_8\}$	2	$\{v_1,v_2,v_7\}$, $\{v_2,v_7,v_8\}$	3
v_5v_6	$\{v_4\}$	1	$\{v_1,v_7\}$, $\{v_7,v_8\}$	2
v_6v_7	$\{v_3\}$	1	$\{v_2,v_5\}$, $\{v_2,v_4\}$	2
v_7v_8	$\{v_3,v_6\}$, $\{v_4,v_5\}$, $\{v_2,v_4\}$, $\{v_2,v_5\}$, $\{v_1,v_5\}$	2	$\{v_1,v_4,v_6\}$	3

Remark 2.3. *Every minimum edge fixed Steiner set for any edge e in G is a minimal edge fixed Steiner set for any edge e in G, but the converse is not true. For the graph G given in Figure 1, $\{v_4, v_6, v_8\}$ is a minimal edge fixed Steiner set but not a minimum edge fixed Steiner set for the edge v_1v_2 in G.*

Theorem 2.4. *If G is any connected graph of order $p \geq 3$ and e is an edge of G, $1 \leq s_e(G) \leq s_e^+(G) \leq p-2$.*

Proof. Let G be a connected graph with at least three vertices. Let $e = xy$ be an edge of G and let W be a minimum edge fixed Steiner set of G so that $s_e(G) = |W|$. Then $W \subseteq V(G) - \{x, y\}$. Therefore $s_e(G) \geq 1$. Since W is also a minimal edge fixed Steiner set of G, it is clear that $s_e^+(G) \geq s_e(G)) = |W|$. Hence $s_e(G) \leq s_e^+(G)$. Also since the vertices x and y do not belong to any edge fixed Steiner set W, it follows that $s_e^+(G) \leq p-2$. □

Theorem 2.5. *For any connected graph G of order $p \geq 3$, $s_e(G) = p-2$ if and only if $s_e^+(G) = p-2$ for any edge e in G.*

Proof. Let $e = xy$ be an edge of G and let $s_e^+(G) = p-2$. Then $W = V(G) - \{x, y\}$ is the unique minimal edge fixed Steiner set of G. Since no proper subset of W is an edge fixed Steiner set of G, it is clear that W is the unique minimum edge fixed Steiner set of G also. So $s_e(G) = p-2$. The converse part follows from Theorem 2.4. □

Corollary 2.6. *For any complete graph K_p with $p \geq 3$, $s_e^+(K_p) = p-2$ where e is any edge of K_p.*

Proof. It follows immediately from Theorem 2.5. □

Theorem 2.7. *For the path P_p with $p \geq 3$, $s_e^+(P_p) = s_e(P_p)$ for any edge e in P_p.*

Theorem 2.8. *If C_p is a cycle of order $p \geq 3$ and e is any edge of C_p, then $s_e^+(C_p) = s_e(C_p)$.*

Proof. Let $e = xy$ be any edge of C_p.

Case 1: Let p be odd. Choose a vertex u such that $d(x, u) = d(y, u) = \frac{p-1}{2}$. Then $\{u\}$ is a minimal edge fixed Steiner set of C_p. Therefore $s_e^+(C_p) = 1$.

Case 2: Let p be even. It is easily verified that no singleton set of C_p is an s_e^+-set of C_p and so $s_e^+(C_p) \geq 2$. Let a and b be the antipodal vertices of x and y respectively. Then $W = \{a, b\}$ is a minimal edge fixed Steiner set of G so that $s_e^+(C_p) = 2$. Thus $s_e^+(C_p) = s_e(C_p)$. □

Theorem 2.9. *Let $e = xy$ be any edge of a connected graph G of order at least 3. Then every extreme vertex of G, other than the vertices x and y (whether x and y are extreme vertices or not) belongs to every minimal edge fixed Steiner set of G. In particular, every end vertex of G other than x and y belongs to every minimal edge fixed Steiner set of G.*

Proof. Since every minimal edge fixed Steiner set of G is an edge fixed Steiner set of G for any edge e in G, the theorem follows from Theorem 1.3. □

Corollary 2.10. *If G is a connected graph of order p with k extreme vertices and $e = xy$ is any edge of G, then* *(i).* $\max\{1, k\} \le s_e^+(G) \le p - 2$ *if neither x nor y is an extreme vertex of G*

(ii). $\max\{1, k-1\} \le s_e^+(G) \le p - 2$ *if either x or y is an extreme vertex of G*

(iii). $\max\{1, k-2\} \le s_e^+(G) \le p - 2$ *if both x and y are extreme vertices of G.*

Theorem 2.11. *Let $e = xy$ be any edge of a connected graph G of order at least 3. Let v be a cut-vertex of G and W be a minimal edge fixed Steiner set of G.*

(i). If $v = x$ or y, then every component of $G - v$ contains an element of W.

(ii). If $v \neq x$ and $v \neq y$, then for each component C of $G - v$ with $x, y \notin C$, $W \cap C \neq \Phi$.

Proof.

(i). Let $v = x$ or y. Suppose there exists a component say G_1 of $G - v$ such that G_1 contains no vertex of W. By Theorem 2.9, W contains all the extreme vertices of G and hence G_1 does not have any extreme vertex of G. Thus G_1 contains at least one edge say uw distinct from xy. Since every minimal edge fixed Steiner W-tree T must have its end vertex in W and v is a cut vertex of G, it is clear that no minimal edge fixed Steiner W-tree would contain the edge uw. It contradicts that W is the minimal edge fixed Steiner set of G.

(ii). Let $v \neq x$ and $v \neq y$. Suppose there exists a component say C of $G - v$ with $x, y \notin C$, $W \cap C = \Phi$. Then proceeding as in (i), we arrive at a contradiction. □

Theorem 2.12. *If v is a cut-vertex of a connected graph G, e is any edge of G and W is a minimal edge fixed Steiner set of G, then v lies in every minimal edge fixed Steiner W-tree of G.*

Proof. Let v be a cut vertex of G and e be any edge in G. Let W be a minimal edge fixed Steiner set of G. Since every component of $G - v$ contains an element of W, it is clear that v lies in every minimal edge fixed Steiner W-tree of G. □

Theorem 2.13. *No cut-vertex of a connected graph G belongs to any minimal edge fixed Steiner set of G for any edge in G.*

Proof. Suppose that there exists a minimal edge fixed Steiner set W that contains a cut-vertex v of G. Let $G_1, G_2, ..., G_r$ $(r \ge 2)$ be the components of $G - v$. By Theorem 2.11, each component G_i $(1 \le i \le r)$ contains an element of W. We claim that $W = W - v$ is also an edge fixed Steiner set of G. Since v is a cut-vertex of G, by Corollary 2.12, each edge fixed Steiner W-tree contains v. Now, since $v \notin W$, it follows that each edge fixed Steiner W-tree is also a edge fixed Steiner W'-tree of G. Thus W' is a edge fixed Steiner set of G such that $W' \subseteq W$, which is a contradiction to W a minimal edge fixed Steiner set of G. Hence the theorem follows. □

Corollary 2.14. *Let T be any non trivial tree and k be the number of end vertices in T. Let $e = xy$ be any edge of T. Then*

$$s_e^+(T) = \begin{cases} k & \text{if neither } x \text{ nor } y \text{ is an end vertex of } T \\ k-1 & \text{if either } x \text{ or } y \text{ is an end vertex of } T \end{cases}$$

Proof. **Case 1:** Let $e = xy$ where neither x nor y is an end vertex of T. Let $W = \{u_1, u_2, ..., u_k\}$ be the set of all end vertices of T. Then by Corollary 2.10 (i), $s_e^+(T) \ge k$. Also it is clear that W is a minimal edge fixed Steiner set of T so that $s_e^+(T) \le k$. Therefore $s_e^+(T) = k$.

Case 2: Let $e = xy$ where either x or y is an end vertex of T. Suppose x is an end vertex of T. Then by Corollary 2.10 (ii), $s_e^+(T) \geq k-1$. Let $W = \{u_1, u_2, ..., u_k\}$ be the set of all end vertices of T. Then $x \in W$. Now let $W' = W - \{x\}$. Then W' is a minimal edge fixed Steiner set of G. Therefore $s_e^+(T) \leq k-1$. Thus $s_e^+(T) = k-1$. Similarly if y is an end vertex of T, then also $s_e^+(T) = k-1$. □

Theorem 2.15. *For any edge e in the complete bipartite graph $K_{m,n}$ with $m \leq n$ and $m+n \geq 3$, $s_e^+(K_{m,n}) = m+n-2 = s_e(K_{m,n})$.*

Proof. Let $U = \{u_1, u_2, ..., u_m\}$ and $V = \{v_1, v_2, ..., v_n\}$ be the bipartition of $K_{m,n}$. Let $e = u_iv_j$, $1 \leq i \leq m$, $1 \leq j \leq n$ be an edge in $K_{m,n}$. Let $W = V - \{u_i, v_j\}$. Since no proper subset of W is an edge fixed Steiner set of G, clearly W is the unique minimal edge fixed Steiner set of $K_{m,n}$. Hence $s_e^+(K_{m,n}) = m+n-2$. □

Theorem 2.16. *If G is a connected graph of order p, then $s_e^+(G) < p-2$ if and only if there exists a minimal edge fixed Steiner set W such that $\langle W \cup \{u, v\}\rangle$ is disconnected for any edge $e = uv$ in G.*

Proof. Let $e = uv$ be any edge in G. Assume that $s_e^+(G) < p-2$. Let W be a minimal edge fixed Steiner set of G. We claim that $\langle W \cup \{u, v\}\rangle$ is disconnected. Suppose that $\langle W \cup \{u, v\}\rangle$ is connected. Then the minimal edge fixed Steiner tree of G contains the members of $W \cup \{u, v\}$ only, which is a contradiction to W is a minimal edge fixed Steiner set of G. So $\langle W \cup \{u, v\}\rangle$ is disconnected.

Conversely, let $e = uv$ be any edge of G. Let W be a minimal edge fixed Steiner set of G such that $\langle W \cup \{x, y\}\rangle$ is disconnected. We claim that $s_e^+(G) < p-2$. Suppose that $s_e^+(G) = p-2$. Then G is complete and $W = V(G) - \{u, v\}$ is the unique minimal edge fixed Steiner set of G such that $\langle W \cup \{u, v\}\rangle$ is connected. This is a contradiction. So $s_e^+(G) < p-2$. □

Theorem 2.17. *For positive integers r, d and $n \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G with rad $G = r$, diam $G = d$ and $s_e^+(G) = n$ or $n-1$ for any edge $e = xy$ in G.*

Proof. Let $e = xy$ be any edge in G. When $r = 1$, let $G = K_{1,n+1}$. Then d = 2 and by the Theorem 2.15, $s_e^+(G) = n$ or $n-1$. Let $r \geq 2$. We construct a graph G with required properties as follows.

Case 1: Let $r = d$. For $n = 2$, let $G = C_{2r}$. Then we have done $s_e^+(G) = 2$ for all e in G. Now, let $n \geq 3$. Let $C_{2r} : \{u_1, u_2, ..., u_{2r}, u_1\}$ be a cycle of order 2r. Let G be the graph obtained by adding new vertices $\{x_1, x_2, ..., x_{n-2}\}$ and joining each x_i $(1 \leq i \leq n-2)$ with u_1 and u_2 of C_{2r}. Then the graph G thus obtained is shown in Figure 2.

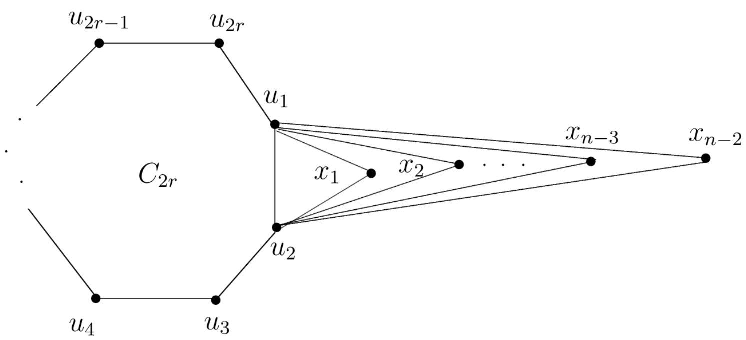

Figure 2. G

It is seen that the eccentricity of each vertex of G is r so that rad $G = diam$ $G = r$. Also $S = \{x_1, x_2, ..., x_{n-2}\}$ is the set of all extreme vertices of G with $|S| = n-2$. Let $U = \{u_1, u_2, ..., u_{2r}\}$. Let $e = xy$ where $x, y \in U$. By Theorem 2.9, S is contained in every minimal edge fixed Steiner set of G. It is clear that S is not a minimal edge fixed Steiner set of G and so $s_e^+(G) \geq n$. Let u and v be the antipodal vertices of x and y respectively in C_{2r}. Then it follows from Theorem 2.9 that $S \cup \{u, v\}$ is a minimal edge fixed Steiner set of G so that $s_e^+(G) = n$. Let $e = u_1x_i$ or u_2x_i where $x_i \in S$ for $i = 1, 2, ..., n-2$. Then by Theorem 2.9, $S - \{x_i\}$ for $i = 1, 2, ..., n-2$ is contained in every minimal edge fixed Steiner set of G. It is clear that $S - \{x_i\}$ is not a minimal edge fixed Steiner set of G. Let $u \in V(C_{2r})$, where $u \neq u_1, u_2$. Then $S - \{x_i\} \cup \{u\}$ is not a minimal edge fixed Steiner set of G and so $s_e^+(G) \geq n-1$. Let w and z be two antipodal vertices of x_i. Then $W = S - \{x_i\} \cup \{w, z\}$ is a minimal edge fixed Steiner set of G and hence $s_e^+(G) = n-1$.

Case 2: Suppose $r < d$. Let $C_{2r} : (v_1, v_2, ..., v_{2r}, v_1)$ be a cycle of order 2r and let $P_{d-r+1} : u_0, u_1, ..., u_{d-r}$ be the path of order $d - r + 1$. Let H be the graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. If $n = 2$, then let $G = H$. Let $e = xy$ be any edge in G. Clearly $s_e^+(G) = 1$ or 2 according as $xy \in \{v_rv_{r+1}, v_{r+1}v_{r+2}, u_{d-r-1}u_{d-r}\}$ or $xy \in \{v_1u_1, u_1u_2, u_2u_3, ..., u_{d-r-2}u_{d-r-1}, v_1v_2, v_2v_3, ..., v_{r-1}v_r, v_{r+2}v_{r+3}, ..., v_{2r}v_1\}$. If $n \geq 3$, then add $(n-2)$ new vertices $w_1, w_2, ..., w_{n-2}$ to H and joining each vertex w_i $(1 \leq i \leq n-2)$ to the vertex u_{d-r-1} and obtain the graph G, which is shown in Figure 3.

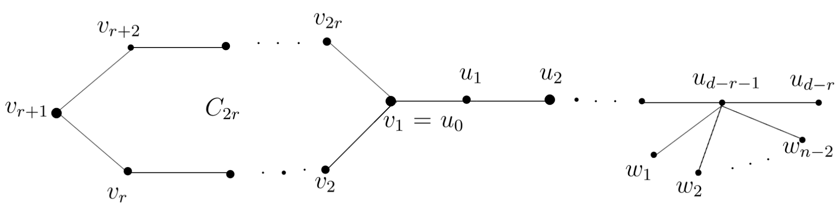

Figure 3. G

Now rad $G = r$ and *diam* $G = d$ and G has $n - 1$ end vertices. Clearly $s_e^+(G) = n$ or $n - 1$ according as $xy \in \{v_1u_1, u_1u_2, ..., u_{d-r-2}u_{d-r-1}, v_1v_2, v_2v_3, ..., v_{r-1}v_r, v_{r+2}v_{r+3}, ..., v_{2r}v_1\}$ or $xy \in \{v_rv_{r+1}, v_{r+1}v_{r+2}, u_{d-r-1}u_{d-r}, u_{d-r}w_1, u_{d-r}w_2, ..., u_{d-r}w_{n-2}\}$. □

Theorem 2.18. *For any two positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G with $s_e(G) = a$ and $s_e^+(G) = b$ for some edge $e = xy$ in G.*

Proof. **Case 1:** Let $a = b$. When $a = 2$, for any edge e in an even cycle G, we have $s_e^+(G) = 2$ by Theorem 2.9 and $s_e(G) = 2$ by Theorem 1.5. Now $a = b$ with $b \geq 3$ and $3 \leq a = b$. Let G be a tree with 'a' end vertices. Let $e = xy$ be an edge of G. Then by theorem 1.1, $s_e(G) = a$ and by Corollary 2.14 $s_e^+(G) = a$.

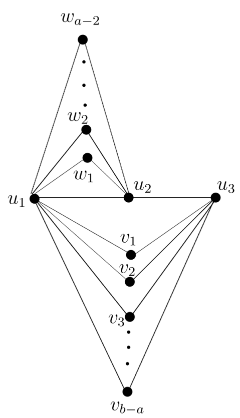

Figure 4.

Case 2: $3 < a < b$. Let G be the graph obtained in Figure 4 from the path P on the vertices: u_1, u_2, u_3, by adding the new vertices $v_1, v_2, ..., v_{b-a+2}$ and $w_1, w_2, ..., w_{a-2}$ and joining each v_i $(1 \leq i \leq b-a+2)$ with u_1 and u_3 and also joining each w_i $(1 \leq i \leq a-2)$ with u_1 and u_2. Let $W = \{w_1, w_2, ..., w_{a-2}\}$. Then W is the set of extreme vertices of G. Let S be any edge fixed Steiner set of G. Then by Theorem 1.3, $W \subseteq S$. It is clear that W is not an edge fixed Steiner set of G. Also it is verified that $W = W \cup \{v\}$, where $v \notin W$ is not an edge fixed Steiner set of G. But, it is seen that $W \cup \{u_1, u_3\}$ is an edge fixed Steiner set of G and so $s_e(G) = a$. Next, we prove that $s_e^+(G) = b$ for some $e \in G$. Let $e = u_3v_2$. By Theorem 2.9, W is a subset of every minimal edge fixed Steiner set of G. It is clear that W is not a minimal edge fixed Steiner set of G. Also, it is easily seen that every minimal edge fixed Steiner set contains each v_i $(1 \leq i \leq b-a+1)$ except the vertices u_3, v_2 and so $s_e^+(G) \geq a-2+b-a+1 = b-1$. It is easily verified that $Z = W \cup \{v_1, v_2, ..., v_{b-a+1}\}$ (except the vertices u_3, v_i) is not a minimal edge fixed Steiner set of G and so $s_e^+(G) \geq b$. Now $Z' = Z \cup \{u_1\}$ is a minimal edge fixed Steiner set of G so that $s_e^+(G) = b$. □

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