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Lie Classical Method for (2+1)-dimensional Rosenau Equation

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Abstract: In this paper, a (2+1)-dimensional Rosenau equation is subjected to Lie's classical method. Classification of its symmetry algebra into one- and two-dimensional subalgebras is carried out in order to facilitate its reduction systematically to (1+1)-dimensional PDE and then to first order ODE.

Keywords: Nonlinear PDE, Lie's Classical Method, Lies Algebra, Symmetry Reductions.

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1. Introduction

A simple model equation is the Korteweg-de Vries (KdV) equation [6]

$$v_t + 6vv_x + \delta v_{xxx} = 0, \tag{1}$$

which describe the long waves in shallow water. Its modified version is,

$$u_t - 6u^2 u_x + u_{xxx} = 0 \tag{2}$$

and again there is Miura transformation [5]

$$v = u^2 + u_x, \tag{3}$$

between the KdV equation (1) and its modified version (2). In 2002, Liu and Yang [4] studied the bifurcation properties of generalized KdV equation (GKdVE)

$$u_t + au^n u_x + u_{xxx} = 0, \quad a \in \mathbb{R}, \; n \in \mathbb{Z}^+. \tag{4}$$

Gungor and Winternitz [7] transformed the Generalized Kadomtsev-Petviashvili Equation (GKPE)

$$(u_t + p(t)uu_x + q(t)u_{xxx})_x + \sigma(y,t)u_{yy} + a(y,t)u_y + b(y,t)u_{xy} + c(y,t)u_{xx} + e(y,t)u_x + f(y,t)u + h(y,t) = 0, \tag{5}$$

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to its canonical form and established conditions on the coefficient functions under which (5) has an infinite dimensional symmetry group having a Kac-Moody-Virasoro structure. In [10], they carried out the symmetry analysis of Variable Coefficient Kadomtsev Petviashvili Equation (VCKP) in the form,

$$(u_t + f(x,y,t)uu_x + g(x,y,t)u_{xxx})_x + h(x,y,t)u_y = 0.$$

Burgers' equation $u_t + uu_x = \gamma u_{xx}$, is the simplest second order NLPDE which balances the effect of nonlinear convection and the linear diffusion. In this chapter, we discuss the symmetry reductions of the (2+1)-dimensional Rosenau equation as,

$$u_t + uu_x + u_x + u_y - u_{xxt} - u_{yyt}. \tag{6}$$

Our intention is to show that equation (6) admits a three-dimensional symmetry group and determine the corresponding Lie algebra, classify the one- and two-dimensional subalgebras of the symmetry algebra of (6), in order to reduce (6) to (1+1)-dimensional PDEs and then to ODEs. We shall establish that the symmetry generators form a closed Lie algebra and this allowed us to use the recent method due to Ahmad, Bokhari, Kara and Zaman [12] to successively reduce (6) to (1+1)-dimensional PDEs and ODEs with the help of two-dimensional Abelian solvable subalgebras. This chapter is organised as follows: First, we determine the symmetry group of (6) and write down the associated Lie algebra. secondly, we consider all one-dimensional subalgebras and obtain the corresponding reductions to (1+1)-dimensional PDEs. Next, we show that the generators form a closed Lie algebra and use this fact to reduce (6) successively to (1+1)- dimensional PDEs and ODEs. Finally, we summarises the conclusions of the present work.

2. The Symmetry Group and Lie Algebra of Rosenau Equation

If (6) is invariant under a one parameter Lie group of point transformations (Bluman and Kumei [3], Olver [11])

$$x^* = x + \epsilon\, \xi(x,y,t;u) + O(\epsilon^2), \tag{7}$$
$$y^* = y + \epsilon\, \eta(x,y,t;u) + O(\epsilon^2), \tag{8}$$
$$t^* = t + \epsilon\, \tau(x,y,t;u) + O(\epsilon^2), \tag{9}$$
$$u^* = u + \epsilon\, \phi(x,y,t;u) + O(\epsilon^2). \tag{10}$$

Then the third Prolongation $Pr^3(V)$ of the corresponding vector field

$$V = \xi(x,y,t;u)\frac{\partial}{\partial x} + \eta(x,y,t;u)\frac{\partial}{\partial y} + \tau(x,y,t;u)\frac{\partial}{\partial t} + \phi(x,y,t;u)\frac{\partial}{\partial u}, \tag{11}$$

satisfies

$$pr^3(V)\Omega(x,y,t;u)|_{\Omega(x,y,t;u=0} = 0. \tag{12}$$

The determining equations are obtained from (12) and solved for the infinitesimals ξ, η, τ and ϕ. They are as follows

$$\xi = k_1, \tag{13}$$
$$\eta = k_2, \tag{14}$$
$$\tau = k_3, \tag{15}$$
$$\phi = 0. \tag{16}$$

At this stage, we construct the symmetry generators corresponding to each of the constants involved. Totally there are three generators given by

$$\begin{aligned} V_1 &= \partial_x, \\ V_2 &= \partial_y, \\ V_3 &= \partial_t, \end{aligned} \tag{17}$$

The symmetry generators found in Equation (17) form a closed Lie Algebra whose commutation table is shown below.

$[V_i, V_j]$	V_1	V_2	V_3
V_1	0	0	0
V_2	0	0	0
V_3	0	0	0

Table 1. **Commutator Table**

The commutation relations of the Lie algebra, determined by V_1, V_2 and V_3 are shown in the above table. For this four-dimensional Lie algebra the commutator table for V_i is a $(3 \otimes 3)$ table whose $(i,j)^{th}$ entry expresses the Lie Bracket $[V_i, V_j]$ given by the above Lie algebra L. The table is skew-symmetric and the diagonal elements all vanish. The coefficient $C_{i,j,k}$ is the coefficient of V_i of the $(i,j)^{th}$ entry of the commutator table. The Lie algebra L is solvable. In the next section, we derive the reduction of (6) to PDEs with two independent variables and ODEs. These are four one-dimensional Lie subalgebras

$$L_{s,1} = \{V_1\}\ ,\ L_{s,2} = \{V_2\}\ ,\ L_{s,3} = \{V_3\},$$

and corresponding to each one-dimensional subalgebras we may reduce (6) to a PDE with two independent variables. Further reductions to ODEs are associated with two-dimensional subalgebras. It is evident from the commutator table that there is one two-dimensional solvable non-abelian subalgebras. And there are three two-dimensional Abelian subalgebras, namely,

$$L_{A,1} = \{V_1, V_2\},\ L_{A,2} = \{V_1, V_3\}\ ,\ L_{A,3} = \{V_2, V_3\}.$$

2.1. Reductions of (2+1)-dimensional Rosenau equation by One-Dimensional Subalgebras

Case 1: $V_1 = \partial_x$. The characteristic equation associated with this generator is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0}.$$

We integrate the characteristic equation to get three similarity variables,

$$y = r, \quad t = s \quad \text{and} \quad u = W(r,s). \tag{18}$$

Using these similarity variables in Equation (6) can be recast in the form

$$W_s + W_r - W_{rrs} = 0. \tag{19}$$

Case 2: $V_2 = \partial_y$. The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0}.$$

Following the standard procedure we integrate the characteristic equation to get three similarity variables,

$$x = r, \quad t = s \ \text{ and } \ u = W(r, s). \tag{20}$$

Using these similarity variables in Equation (6) can be recast in the form

$$W_s + W_r + WW_r - W_{rrs} = 0. \tag{21}$$

Case 3: $V_3 = \partial_t$. The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}.$$

Following standard procedure we integrate the characteristic equation to get three similarity variables,

$$x = r, \quad y = s \ \text{ and } \ u = W(r, s). \tag{22}$$

Using these similarity variables in Equation (6) can be recast in the form

$$W_r + W_s + WW_r = 0. \tag{23}$$

2.2. Reductions of (2+1)-dimensional Rosenau equation by Two-Dimensional Abelian Subalgebras

Case 1: Reduction under V_1 and V_2. From Table 1 we find that the given generators commute $[V_1, V_2] = 0$. Thus either of V_1 or V_2 can be used to start the reduction with. For our purpose we begin reduction with V_1. Therefore we get Equation (18) and Equation (19). At this stage, we express V_2 in terms of the similarity variables defined in (18). The transformed V_2 is

$$\tilde{V_2} = \partial r.$$

The characteristic equation for $\tilde{V_2}$ is

$$\frac{dr}{1} = \frac{ds}{0} = \frac{dW}{0}.$$

Integrating this equation as before leads to new variables

$$s = \zeta \ \text{ and } \ W = R(\zeta),$$

which reduce Equation (19) to

$$R_(\zeta) = 0. \tag{24}$$

Case 2: Reduction under V_1 and V_3. From Table 1 we find that the given generators commute $[V_1, V_3] = 0$. Thus either of V_1 or V_3 can be used to start the reduction with. For our convenience we begin reduction with V_1. At this stage, we express V_3 in terms of the similarity variables defined in Equation (18). The transformed V_3 is

$$\tilde{V_3} = \partial_s.$$

The characteristic equation for $\tilde{V_3}$ is

$$\frac{dr}{0} = \frac{ds}{1} = \frac{dW}{0}.$$

Integrating this equation as before leads to new variables

$$s = \zeta \ \text{ and } \ W = R(\zeta).$$

which reduce Equation (19) to

$$R_{(\zeta)} = 0. \tag{25}$$

Case 3: Reduction under V_2 and V_3. From Table 1 we find that the given generators commute $[V_2, V_3] = 0$. Thus either of V_2 or V_3 can be used to start the reduction with. For our convenience we begin reduction with V_2. Therefore we get Equation (20) and Equation (21). At this stage, we express V_3 in terms of the similarity variables defined in Equation (20). The transformed V_3 is

$$\tilde{V_3} = \partial_s.$$

The characteristic equation for $\tilde{V_3}$ is

$$\frac{dr}{0} = \frac{ds}{1} = \frac{dW}{0}.$$

Integrating this equation as before leads to new variables

$$r = \zeta \ \text{ and } \ W = R(\zeta).$$

which reduce Equation (21) to

$$R_{(\zeta)}(R(\zeta) + 1) = 0. \tag{26}$$

3. Conclusions

In this paper, A $(2+1)$-dimensional Rosenau equation, $u_t + u_x + u_y + uu_x - (u_{xxt} + u_{yyt}) = 0$ is subjected to Lie's classical method. Equation (6) admits a three-dimensional symmetry group. It is established that the symmetry generators form a closed Lie algebra. Classification of symmetry algebra of (6) into one- and two-dimensional subalgebras is carried out. Systematic reduction to (1+1)-dimensional PDE and then to first order ODEs are performed using one-dimensional and two-dimensional solvable Abelian subalgebras.

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