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Multi-variable Hermite-Tricomi Functions: Generating Relations and Applications

Rehana Khan[1,*] and Q. H. Khan[1]

1 Department of Mathematics, Aligarh Muslim University, Aligarh, Uttar Pradesh, India.

Abstract: In this paper, the authors uses some operational methods combining with the principle of monomiality to introduce the Hermite Tricomi function of four variables and three parameters. Series definition, differential equation and certain other properties of are derived. Generating relations of these functions are also derived by using Lie-algebraic method. Further,as applications of these functions, certain new and known generating relations involving other forms of Hermite Tricomi functions are established.

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1. Introduction

The study of the properties of multi-variable generalized special functions has provided new means of analysis for the derivation of the solution of large classes of partial differential equations often encountered in physical problems. The relevance of the special functions in physics is well established. Most of the special functions of mathematical physics as well as their generalizations have been suggested by physical problems. The Tricomi function

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\ x^r}{r!\ (n+r)!}, \tag{1}$$

is a Bessel like function and is characterized by the following link with the ordinary Bessel function $J_n(x)$ [5]:

$$C_n(x) = x^{-n/2} J_n\Big(2\sqrt{x}\Big). \tag{2}$$

or

$$J_n(x) = \Big(\frac{x}{2}\Big)^n\ C_n\Big(\frac{x^2}{4}\Big). \tag{3}$$

The Tricomi function $C_n(x)$ satisfies the generating function

$$\exp\Big(t - \frac{x}{t}\Big) = \sum_{n=-\infty}^{\infty} C_n(x)\ t^n, \tag{4}$$

* *E-mail: rehanakhan97@gmail.com*

The zeroth-order Tricomi function $C_0(x)$,is an eigenfunction of the operator

$$\hat{D}_t = -\frac{\partial}{\partial t}\ t\ \frac{\partial}{\partial t} \tag{5}$$

indeed we have

$$\hat{D}_t C_0(\alpha t) = \alpha C_0(\alpha t). \tag{6}$$

The operator given in (5) plays the role of derivative operator for 2-varible laguerre polynomials $L_n(x,y)$ within the context of the monomiality [1, 6]. According to the principle of monomiality the polynomials $p_n(x)$ $(n \in \mathbb{N},\ x \in \mathbb{C})$ are called quasi-monomials, if two operators $\hat{M}$ and $\hat{P}$, can be defined in such a way that

$$\begin{aligned} \hat{M}\{p_n(x)\} &= p_{n+1}(x), \\ \hat{P}\{p_n(x)\} &= np_{n-1}(x). \end{aligned} \tag{7}$$

The operators $\hat{M}$ and $\hat{P}$ are called multiplicative and derivative operators and can be recognized as raising and lowering operators acting on the polynomials $p_n(x)$. Obviously $\hat{M}$ and $\hat{P}$ satisfy the commutative relation

$$[\hat{P},\ \hat{M}] = 1 \tag{8}$$

and thus display a Weyl group structure. Further consequence of Equation (7) is the eigen property of $\hat{M}\hat{P}$

$$\hat{M}\hat{P}\,p_n(x) = np_n(x). \tag{9}$$

The polynomials $p_n(x)$ are obtained by taking the action of $\hat{M}$ on $p_0(x)$

$$p_n(x) = \hat{M}^n\,p_0(x), \tag{10}$$

(in the following we shall always set $p_0(x) = 1$) and consequently the exponential generating function of $p_n(x)$ is

$$G(t,x) = \sum_{n=0}^{\infty} p_n(x)\,\frac{t^n}{n!} = \exp(t\hat{M})\{1\}. \tag{11}$$

Recently, Subuhi Khan and Rehana Khan [9] studied the properties of 3-variable 2-parameter Hermite-Tricomi function (3V2PHTF) ${}_HC_n(x,y,z;\tau_1,\tau_2)$which are constructed in terms of quasi-monomials. To extend there work, we define 4-variable 3-parameter Hermite-Tricomi function (4V3PHTF)${}_HC_n(x,y,z,w;\tau_1,\tau_2,\tau_3)$ as follows:

$$\exp\left(t - \frac{x}{t} + \frac{y\tau_1}{t^2} - \frac{z\tau_2}{t^3} + \frac{w\tau_3}{t^4}\right) = \sum_{n=-\infty}^{\infty}\ {}_HC_n(x,y,z,w;\tau_1,\tau_2,\tau_3)\ t^n, \tag{12}$$

In this paper,we derive multi variables Hermite Tricomi functions and drive its generating relation. In Section 2, we derive Hermite Tricomi function of 4-variables and 3-parameters (4V3PHTF) ${}_HC_n(x,y,z,w;\tau_1,\tau_2,\tau_3)$. In Section 3, we give properties of 4V3PHTF ${}_HC_n(x,y,z,w;\tau_1,\tau_2,\tau_3)$ and their special cases. In Section 4, we derive generating relations involving 4V3PHTF by using the representation $Q(w,m_0)$ of the Lie algebra $\mathcal{T}_3$. In Section 5, we obtain certain new and known generating relations involving various forms of Hermite- Tricomi,Tricomi and Bessel functions as applications.

2. Generating function of 4V3PHTF ${}_HC_n(x,y,z,w;\tau_1,\tau_2,\tau_3)$

In order to obtain the generating function of 4-variable 3-parameter Hermite-Tricomi function (4V3PHTF) ${}_HC_n(x,y,z,w;\tau_1,\tau_2,\tau_3)$, let us recall the definition of 3-variable 2-parameter Hermite functions $H_n(x,y,z;\tau_1,\tau_2)$ which is given by [9]:

$$H_n(x,y,z;\tau_1,\tau_2) = n!\sum_{r=0}^{\left[\frac{n}{3}\right]} \frac{(z\tau_2\ )^r H_{n-3r}(x,y;\tau_1)}{r!(n-3r)!}. \tag{13}$$

The above definition is equivalent to the following generating function for $H_n(x,y,z;\tau_1\ ,\tau_2)$

$$\exp\left(xt + y\tau_1\ t^2 + z\tau_2\ t^3\right) = \sum_{n=0}^{\infty} H_n(x,y,z;\tau_1\ ,\tau_2)\ t^n. \tag{14}$$

We define the 4-variable 3-parameter analogue of 3V2PHP $H_n(x,y,z;\tau_1\ ,\tau_2)$ as follows:

$$H_n(x,y,z,w;\tau_1,\tau_2,\tau_3) = n!\sum_{r=0}^{\left[\frac{n}{4}\right]} \frac{(w\tau_3\ )^r H_{n-4r}(x,y,z;\tau_1,\tau_2)}{r!(n-4r)!}, \tag{15}$$

which is equivalent to the following generating function for $H_n(x,y,z,w;\tau_1\ ,\tau_2,\tau_3)$

$$\exp\left(xt + y\tau_1\ t^2 + z\tau_2\ t^3 + w\tau_3\ t^4\right) = \sum_{n=0}^{\infty} H_n(x,y,z,w;\tau_1\ ,\tau_2,\tau_3)\ t^n. \tag{16}$$

These polynomials are quasi-monomials under the action of the operators

$$\begin{aligned} \hat{M} &= x + 2y\tau_1\frac{\partial}{\partial x} + 3z\tau_2\frac{\partial^2}{\partial x^2} + 4w\tau_3\frac{\partial^3}{\partial x^3}, \\ \hat{P} &= \frac{\partial}{\partial x}. \end{aligned} \tag{17}$$

which play the role of multiplicative and derivative operators respectively in the sense that

$$\begin{aligned} \hat{M}H_n(x,y,z,w;\tau_1\ ,\tau_2,\tau_3) &= H_{n+1}(x,y,z,w;\tau_1\ ,\tau_2,\tau_3), \\ \hat{P}H_n(x,y,z,w;\tau_1\ ,\tau_2,\tau_3) &= nH_{n-1}(x,y,z,w;\tau_1\ ,\tau_2,\tau_3). \end{aligned} \tag{18}$$

we can explicitly write the polynomials $H_n(x,y,z,w;\tau_1\ ,\tau_2,\tau_3)$ in terms of the operators (17) as follows

$$H_n(x,y,z,w;\tau_1\ ,\tau_2,\tau_3) = (x + 2y\tau_1\frac{\partial}{\partial x} + 3z\tau_2\frac{\partial^2}{\partial x^2} + 4w\tau_3\frac{\partial^3}{\partial x^3})^n. \tag{19}$$

To generate Hermite-Tricomi functions associated with the polynomials $H_n(x,y,z,w;\tau_1\ ,\tau_2,\tau_3)$, we introduce the generating function

$$\begin{aligned} G(x,y,z,w;\tau_1\ ,\tau_2,\tau_3,t) &= \exp\left(t - \frac{\hat{M}}{t}\right) \\ &= \exp\left(t - (x + 2y\tau_1\frac{\partial}{\partial x} + 3z\tau_2\frac{\partial^2}{\partial x^2} + 4w\tau_3\frac{\partial^3}{\partial x^3})\frac{1}{t}\right). \end{aligned} \tag{20}$$

Now by decoupling the exponential on the r.h.s. of Equation (20) by means of the rule [7]

$$e^{\hat{A}+\hat{B}} = e^{m^2/12}e^{-m/2\hat{A}^{1/2+\hat{A}}}e^{\hat{B}}; \qquad [\hat{A},\hat{B}] = m\hat{A}^1/2, \tag{21}$$

with

$$m = (-1)^{-1/2} 2\sqrt{3} t^{-3/2} z^{1/2} \tau_2^{1/2} w^{1/3} \tau_3^{1/3},$$

and again decoupling the exponential by means of the rule [7]

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}+\hat{B}} e^{-m/2}; \qquad [\hat{A}, \hat{B}] = m,$$

where $\hat{A}$ and $\hat{B}$ are the operators and m a complex number,we obtain

$$G(x, y, z; \tau_1 \, \tau_2, t) = \exp(t) \exp\left(\frac{-z\tau_2}{t^3}\right) \exp\left(\frac{w\tau_3}{t^4}\right) \exp\left(-\frac{x}{t} + \frac{y\tau_1}{t^2}\right)$$
$$\exp\left(-\frac{2y\tau_1}{t}\frac{\partial}{\partial x} - \frac{3z\tau_2}{t^2}\frac{\partial}{\partial x} - \frac{3z\tau_2}{t}\frac{\partial^2}{\partial x^2} - \frac{4w\tau_3}{t^3}\frac{\partial}{\partial x} - \frac{4w\tau_3}{t^2}\frac{\partial^2}{\partial x^2} - \frac{4w\tau_3}{t}\frac{\partial^3}{\partial x^3}\right). \tag{22}$$

If the r.h.s. of Equation (22) is not considered as an operator, the last exponential can be replaced by 1, thus we get

$$\exp\left(t - \frac{x}{t} + \frac{y\tau_1}{t^2} - \frac{z\tau_2}{t^3} + \frac{w\tau_3}{t^4}\right) = \sum_{n=-\infty}^{\infty} {}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3)\, t^n, \tag{23}$$

where ${}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3)$ defined by

$${}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3) = \sum_{r=0}^{\infty} \frac{(-1)^r H_r(x, y, z, w; \tau_1 \ , \tau_2, \tau_3)}{r! \ (n+r)!}. \tag{24}$$

is the 4V3PHTF

3. Properties and Special Cases of 4V3PHTF ${}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3)$

The 4V3PHTF ${}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3)$ defined by Equation (**??**) satisfy the following differential and pure recurrence relations:

$$\begin{aligned}
\frac{\partial}{\partial x} {}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3) &= -{}_HC_{n+1}(x, y, z, w; \tau_1, \tau_2, \tau_3), \\
\frac{\partial}{\partial y} {}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3) &= \tau_1 \, {}_HC_{n+2}(x, y, z, w; \tau_1, \tau_2, \tau_3), \\
\frac{\partial}{\partial z} {}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3) &= -\tau_2 \, {}_HC_{n+3}(x, y, z, w; \tau_1, \tau_2, \tau_3), \\
\frac{\partial}{\partial w} {}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3) &= \tau_3 \, {}_HC_{n+4}(x, y, z, w; \tau_1, \tau_2, \tau_3), \\
\frac{\partial}{\partial \tau_1} {}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3) &= y {}_HC_{n+2}(x, y, z, w; \tau_1, \tau_2, \tau_3), \\
\frac{\partial}{\partial \tau_2} {}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3) &= -z {}_HC_{n+3}(x, y, z, w; \tau_1, \tau_2, \tau_3), \\
\frac{\partial}{\partial \tau_3} {}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3) &= w {}_HC_{n+4}(x, y, z, w; \tau_1, \tau_2, \tau_3),
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
n {}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3) =& {}_H C_{n-1}(x, y, z, w; \tau_1, \tau_2, \tau_3,) + x {}_HC_{n+1}(x, y, z, w; \tau_1, \tau_2, \tau_3) - 2y\tau_1 {}_HC_{n+2}(x, y, z, w; \tau_1, \tau_2, \tau_3) \\
&+ 3z\tau_2 \, {}_HC_{n+3}(x, y, z, w; \tau_1, \tau_2, \tau_3) - 4w\tau_3 \, {}_HC_{n+4}(x, y, z, w; \tau_1, \tau_2, \tau_3).
\end{aligned} \tag{26}$$

The differential equation satisfied by 4V3PHTF ${}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3)$ is

$$\left(-x\frac{\partial^2}{\partial x^2} - (1+n)\frac{\partial}{\partial x} - 2y\frac{\partial^2}{\partial x \partial \tau_1} - 3z\frac{\partial^2}{\partial x \partial \tau_2} - 4w\frac{\partial^2}{\partial x \partial \tau_3} - 1\right) {}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3) = 0. \tag{27}$$

We note the following special cases of 4V3PHTF ${}_HC_n(x, y, z, w; \tau_1, \tau_2, \tau_3)$:

(1).

$$ {}_HC_n(x,y,z,0;\tau_1,\tau_2,\tau_3) = {}_H C_n(x,y,z;\tau_1,\tau_2) \tag{28}$$

where ${}_HC_n(x,y,z;\tau_1,\tau_2)$ denotes 3-variable 2-parameter Hermite-Tricomi functions(3V2PHTF)defined by [9].

$$ {}_HC_n(x,y,z;\tau_1,\tau_2) = \sum_{r=0}^{\infty} \frac{(-1)^r H_r(x,y,z;\tau_1\,\tau_2)}{r!\ (n+r)!}. \tag{29}$$

(2).

$$ {}_HC_n(x,y,z,0;1,1,\tau_3) = {}_H C_n(x,y,z) \tag{30}$$

where ${}_HC_n(x,y,z)$ denotes 3-variable Hermite-Tricomi functions(3VHTF)defined by [4],

$$ {}_HC_n(x,y,z) = \sum_{r=0}^{\infty} \frac{(-1)^r H_r(x,y,z)}{r!\ (n+r)!}. \tag{31}$$

(3).

$$ {}_HC_n(x,y,0,0;1,\tau_2,\tau_3) = {}_H C_n(x,y), \tag{32}$$

where ${}_HC_n(x,y)$ denotes 2-variable Hermite-Tricomi function (2VHTF) defined by the generating function [3]

$$ \sum_{n=-\infty}^{\infty} {}_HC_n(x,y)t^n = \exp\left(t - \frac{x}{t} + \frac{y}{t^2}\right). \tag{33}$$

(4).

$$ {}_HC_n\,(x,0,0,0;\tau_1,\tau_2,\tau_3) = C_n(x), \tag{34}$$

where $C_n(x)$ is given by Equation (1.4).

(5).

$$ C_n\left(\frac{x^2}{4},0,0,0;\tau_1,\tau_2,\tau_3\right) = \left(\frac{x}{2}\right)^{-n} J_n(x), \tag{35}$$

where $J_n(x)$ denotes the ordinary Bessel function.

4. Generating Relations of ${}_HC_n(x,y,z,w;\tau_1,\tau_2,\tau_3)$ using Representation $Q(w,m_0)$ of $\mathcal{T}_3$

The theory of special functions from the group-theoretic point of view is a well established topic, providing a unifying formalism to deal with the immense aggregate of the special functions and a collection of formulae such as the relevant differential equations, integral representations, recurrence formulae, composition theorems, etc., see for example [10, 11]. The first significant advance in the direction of obtaining generating relations by Lie-theoretic method is made by Weisner [12–14] and Miller [8]. Miller [8] have determined realizations of the irreducible representation $Q(\omega,m_0)$ of $\mathcal{T}_3$ where $\omega, m_0 \in \mathbb{C}$ such that $\omega \neq 0$ and $0 \leq \text{Re } m_0 < 1$. The spectrum S of this representation is the set $\{m_0+k:\ \ k \text{ an integer}\}$ and the representation space V has a basis $(f_m)_{m\in S}$, such that

$$ \begin{aligned} &J^3 f_m = m f_m, \quad J^+ f_m = \omega f_{m+1}, \quad J^- f_m = \omega f_{m-1}, \\ &C_{0,0} f_m = (J^+ J^-) f_m = \omega^2 f_m, \qquad \omega \neq 0. \end{aligned} \tag{36}$$

The commutation relations satisfied by the operators $J^3,\ J^{\pm}$ are

$$[J^3, J^+] = J^+, \quad [J^3, J^-] = -J^-, \quad [J^+, J^-] = 0. \tag{37}$$

In order to find the realizations of this representation on spaces of functions of two complex variables x and y, Miller [8] has taken the functions $f_m(x,y) = Z_m(x)e^{my}$, such that relations (36) are satisfied for all $m \in S$, where the differential operators $J^3, J^{\pm}$ are given by

$$\begin{aligned} J^3 &= \frac{\partial}{\partial y}, \\ J^+ &= e^y\left[\frac{\partial}{\partial x} - \frac{1}{x}\frac{\partial}{\partial y}\right], \\ J^- &= e^{-y}\left[-\frac{\partial}{\partial x} - \frac{1}{x}\frac{\partial}{\partial y}\right]. \end{aligned} \tag{38}$$

In particular, we look for the functions

$$f_m(x,y,z,t;\tau_1,\tau_2) = Z_m(x,y,z;\tau_1,\tau_2)t^m, \tag{39}$$

such that

$$\begin{aligned} &K^3 f_m = m f_m, \quad K^+ f_m = \omega f_{m+1}, \quad K^- f_m = \omega f_{m-1}, \\ &C_{0,0} f_m = (K^+K^-) f_m = \omega^2 f_m, \qquad (\omega \neq 0; \quad m \in S). \end{aligned} \tag{40}$$

The set of operators $\{K^3, K^+, K^-\}$ satisfy the commutation relations identical to (37). There are numerous possible solutions of Equation (40). We assume that the set of linear differential operators $\{K^3, K^+, K^-\}$ takes the form

$$\begin{aligned} K^3 &= t\,\frac{\partial}{\partial t}, \\ K^+ &= t\,\frac{\partial}{\partial x}, \\ K^- &= -\frac{x}{t}\frac{\partial}{\partial x} - \frac{2y}{t}\frac{\partial}{\partial y} - \frac{3z}{t}\frac{\partial}{\partial z} - \frac{4w}{t}\frac{\partial}{\partial w} - \frac{\partial}{\partial t}. \end{aligned} \tag{41}$$

The operators in Equation (41) satisfy the commutation relations (37). In terms of the functions $Z_m(x,y,z,w;\tau_1,\tau_2,\tau_3)$ and using operators (41), relations (40) reduce to

(i). $\frac{\partial}{\partial x} Z_m(x,y,z,w;\tau_1,\tau_2,\tau_3) = \omega Z_{m+1}(x,y,z,w;\tau_1,\tau_2,\tau_3)$,

(ii). $\left[-x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} - 3z\frac{\partial}{\partial z} - 4w\frac{\partial}{\partial w} - m\right] Z_m(x,y,z,w;\tau_1,\tau_2,\tau_3) = \omega Z_{m-1}(x,y,z,w;\tau_1,\tau_2,\tau_3)$,

(iii).

$$\left[x\frac{\partial^2}{\partial x^2} + 2y\frac{\partial^2}{\partial x \partial y} + 3z\frac{\partial^2}{\partial x \partial z} + 4w\frac{\partial^2}{\partial x \partial w} + (m+1)\frac{\partial}{\partial x}\right] Z_m(x,y,z,w;\tau_1,\tau_2,\tau_3) = \ \omega^2 Z_m(x,y,z,w;\tau_1,\tau_2,\tau_3). \tag{42}$$

We see that for $\omega = -1$, (iii) of Equation (42) coincides with the differential equation (27) of 4V3PHTF ${}_HC_n(x,y,z,w;\tau_1,\tau_2,\tau_3)$. In fact, for all $m \in S$ the choice for $Z_m(x,y,z,w;\tau_1,\tau_2,\tau_3) =_H C_m(x,y,z,w;\tau_1,\tau_2,\tau_3)$ satisfy Equation (42). Thus we conclude that the functions $f_m(x,y,z,w,t;\tau_1,\tau_2,\tau_3) =_H C_m(x,y,z,w;\tau_1,\tau_2,\tau_3)t^m$, $m \in S$ form a basis for a realization of the representation $Q(-1,m_0)$ of $\mathcal{T}_3$. By using ([8]), this representation of $\mathcal{T}_3$ can be extended to a local multiplier representation T ([8]) of $\mathcal{T}_3$. Using operators (41), the local multiplier representation $T(g)$, $g \in T_3$ defined on $\mathcal{F}$, the space of all functions analytic in a neighbourhood of the point $(x^0, y^0, z^0, w^0, t^0, \tau_1^0, \tau_2^0, \tau_3^0) = (1, 1, 1, 1, 1, 0, 0, 0)$, takes the form

$$[T(\exp b\mathcal{J}^+)f](x, y, z, w, t; \tau_1, \tau_2, \tau_3) = f\left(x\left(1 + \frac{bt}{x}\right), y, z, w, t; \tau_1, \tau_2, \tau_3\right).$$
$$[T(\exp c\mathcal{J}^-)f](x, y, z, w, t; \tau_1, \tau_2, \tau_3) = f\left(x\left(1 - \frac{c}{t}\right), y, z, w, t\left(1 - \frac{c}{t}\right); \tau_1\left(1 - \frac{c}{t}\right)^2, \tau_2\left(1 - \frac{c}{t}\right)^3, \tau_3\left(1 - \frac{c}{t}\right)^4\right),$$
$$[T(\exp a\mathcal{J}^3)f](x, y, z, w, t; \tau_1, \tau_2, \tau_3) = f(x, y, z, w, te^a; \tau_1, \tau_2, \tau_3), \tag{43}$$

If $g \in T_3$, we find

$$T(g) = T(\exp b\mathcal{J}^+)T(\exp c\mathcal{J}^-)T(\exp a\mathcal{J}^3)$$

and therefore we obtain

$$[T(g)f](x, y, z, w, t; \tau_1, \tau_2, \tau_3) = f\left(x\left(1 + \frac{bt}{x}\right)\left(1 - \frac{c}{t}\right), y\left(1 - \frac{c}{t}\right)^2, z\left(1 - \frac{c}{t}\right)^3, w\left(1 - \frac{c}{t}\right)^4, te^a\left(1 - \frac{c}{t}\right); \tau_1, \tau_2, \tau_3\right), \tag{44}$$

$\left|\frac{bt}{x}\right| < 1$, $\left|\frac{c}{t}\right| < 1$. The matrix elements of $T(g)$ with respect to the analytic basis $(f_m)_{m\in S}$ are the functions $A_{lk}(g)$ uniquely determined by $Q(-1, m_0)$ of $\mathcal{T}_3$ and are defined by

$$[T(g)f_{m_0+k}](x, y, z, w, t; \tau_1, \tau_2, \tau_3) = \sum_{l=-\infty}^{\infty} A_{lk}(g)f_{m_0+l}(x, y, z, w, t; \tau_1, \tau_2, \tau_3), \quad k = 0, \pm1, \pm2, \pm3\cdots. \tag{45}$$

Therefore, we prove the following result:

Theorem 4.1. *The following generating relation holds*

$$\left(1 - \frac{c}{t}\right)^m {}_HC_m\left(x\left(1 + \frac{bt}{x}\right)\left(1 - \frac{c}{t}\right), y\left(1 - \frac{c}{t}\right)^2, z\left(1 - \frac{c}{t}\right)^3, w\left(1 - \frac{c}{t}\right)^4; \tau_1, \tau_2, \tau_3\right)$$
$$= \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!}\, b^{(p+|p|)/2}\, c^{(-p+|p|)/2}\, {}_0F_1[-; |p| + 1; bc]\, {}_HC_{m+p}(x, y, z, w; \tau_1, \tau_2, \tau_3)t^p, \left|\frac{bt}{x}\right| < 1, \quad \left|\frac{c}{t}\right| < 1. \tag{46}$$

Proof. Using (44) and (45) we obtain

$$\exp(ma)\left(1 - \frac{c}{t}\right)^m {}_HC_m\left(x\left(1 + \frac{bt}{x}\right)\left(1 - \frac{c}{t}\right), y\left(1 - \frac{c}{t}\right)^2, z\left(1 - \frac{c}{t}\right)^3, w\left(1 - \frac{c}{t}\right)^4; \tau_1, \tau_2, \tau_3\right)$$
$$= \sum_{l=-\infty}^{\infty} A_{l,m-m_0}(g)\, {}_HC_{m_0+l}(x, y, z, w; \tau_1, \tau_2, \tau_3)t^{m_0+l-m} \tag{47}$$

and the matrix elements $A_{lk}(g)$ are given by ([8]),

$$A_{lk}(g) = \exp((m_0 + k)a)\,\frac{(-1)^{|k-l|}}{|k - l|!}\, b^{(l-k+|k-l|)/2}\, c^{(k-l+|k-l|)/2}\, {}_0F_1[-; |k - l| + 1; bc], \tag{48}$$

valid for all integral values of l, k and where ${}_0F_1$ denotes confluent hypergeometric function. Substituting the value of $A_{lk}(g)$ given by (48) into (47) and simplifying we obtain result (46). □

Corollary 4.2. *The following generating relation holds*

$$\left(1 + \frac{r}{2\nu t}\right)^m {}_HC_m\left(x\left(1 + \frac{r\nu t}{2x}\right)\left(1 + \frac{r}{2\nu t}\right), y\left(1 + \frac{r}{2\nu t}\right)^2, z\left(1 + \frac{r}{2\nu t}\right)^3, w\left(1 + \frac{r}{2\nu t}\right)^4; \tau_1, \tau_2, \tau_3\right)$$
$$= \sum_{p=-\infty}^{\infty} (-\nu)^p\, J_p(r)\, {}_HC_{m+p}(x, y, z, w; \tau_1, \tau_2, \tau_3)t^p, \qquad \left|\frac{r\nu t}{2x}\right| < 1, \quad \left|\frac{r}{2\nu t}\right| < 1. \tag{49}$$

Proof. If $bc \neq 0$, we can introduce the co-ordinates r, ν such that $b = \frac{r\nu}{2}$ and $c = -\left(\frac{r}{2\nu}\right)$, with these new co-ordinates the matrix elements (48) can be expressed as

$$A_{lk}(g) = \exp((m_0 + k)a)\,(-\nu)^{l-k}\,J_{l-k}(r), \quad k = 0, \pm 1, \pm 2 \cdots \tag{50}$$

and generating relation (46) yields (49). □

5. Applications

We discuss some applications of the generating relations obtained in the preceding section.

I. Taking $c = 0$ and $t = 1$ in generating relation (46), we get

$$ {}_HC_m((x+b), y, z, w; \tau_1, \tau_2, \tau_3) = \sum_{p=0}^{\infty} \frac{(b)^p}{p!}\, {}_HC_{m+p}(x, y, z, w; \tau_1, \tau_2, \tau_3), \qquad \left|\frac{b}{x}\right| < 1. \tag{51}$$

Again, taking $b = 0$ and $t = 1$ in generating relation (46), we get

$$(1-c)^m\, {}_HC_m(x(1-c), y(1-c)^2, z(1-c)^3, w(1-c)^4; \tau_1\ , \tau_2\ , \tau_3) = \sum_{p=0}^{\infty} \frac{c^p}{p!}\, {}_HC_{m-p}(x, y, z, w; \tau_1, \tau_2, \tau_3), \quad |c| < 1. \tag{52}$$

II. Taking $w = 0, \tau_3 = 1$, in generating relation (46) and using Equation (29), we obtain [9]

$$\begin{aligned}\left(1-\frac{c}{t}\right)^m {}_HC_m\left(x\left(1+\frac{bt}{x}\right)\left(1-\frac{c}{t}\right), y\left(1-\frac{c}{t}\right)^2, z\left(1-\frac{c}{t}\right)^3; \tau_1\ , \tau_2\right) &= \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!}\, b^{(p+|p|)/2} \\ \times\ c^{(-p+|p|)/2}\, {}_0F_1[-; |p|+1; bc]\, {}_HC_{m+p}(x, y, z; \tau_1\ , \tau_2)t^p, &\qquad \left|\frac{bt}{x}\right| < 1, \quad \left|\frac{c}{t}\right| < 1,\end{aligned} \tag{53}$$

where ${}_HC_m(x, y, z; \tau_1\ , \tau_2)$ is given by Equation (29). Similar results can be obtained from generating relations (49).

III. Taking $w = 0, \tau_1 = \tau_2 = 1$, in generating relation (46) and using Equation (30), we obtain [4]

$$\begin{aligned}\left(1-\frac{c}{t}\right)^m {}_HC_m\left(x\left(1+\frac{bt}{x}\right)\left(1-\frac{c}{t}\right), y\left(1-\frac{c}{t}\right)^2, z\left(1-\frac{c}{t}\right)^3\right) &= \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!}\, b^{(p+|p|)/2} \\ \times\ c^{(-p+|p|)/2}\, {}_0F_1[-; |p|+1; bc]\, {}_HC_{m+p}(x, y, z)t^p, &\qquad \left|\frac{bt}{x}\right| < 1, \quad \left|\frac{c}{t}\right| < 1,\end{aligned} \tag{54}$$

where ${}_HC_m(x, y, z)$ is given by Equation (31). Similar results can be obtained from generating relations (49).

IV. Taking $\tau_1 = 1$ and $z = w = 0$, in generating relation (46) and using Equation (32), we get ([4])

$$\begin{aligned}\left(1-\frac{c}{t}\right)^m {}_HC_m\left(x\left(1+\frac{bt}{x}\right)\left(1-\frac{c}{t}\right), y\left(1-\frac{c}{t}\right)^2\right) &= \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!}\, b^{(p+|p|)/2} \\ \times\ c^{(-p+|p|)/2}\, {}_0F_1[-; |p|+1; bc]\, {}_HC_{m+p}(x, y)t^p, &\qquad \left|\frac{bt}{x}\right| < 1, \quad \left|\frac{c}{t}\right| < 1,\end{aligned} \tag{55}$$

where ${}_HC_m(x, y)$ is given by Equation (33). Similar results can be obtained from generating relations (49).

V. Taking $y = z = w = 0$ and $\tau_1 = \tau_2 = \tau_3 = 1$, in generating relation (46) and (49), we obtain the generating relation of tricomi function

$$\left(1-\frac{c}{t}\right)^m C_m\left(x\left(1+\frac{bt}{x}\right)\left(1-\frac{c}{t}\right)\right) = \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!}\, b^{(p+|p|)/2}$$

$$\times\ c^{(-p+|p|)/2}\ {}_0F_1[-;|p|+1;bc]\,C_{m+p}(x)t^p, \qquad \left|\frac{bt}{x}\right|<1, \quad \left|\frac{c}{t}\right|<1, \tag{56}$$

and

$$\left(1+\frac{r}{2\nu t}\right)^m C_m\left(x\left(1+\frac{r\nu t}{2x}\right)\left(1+\frac{r}{2\nu t}\right)\right)=\sum_{p=-\infty}^{\infty}(-\nu)^p\, J_p(r)\,C_{m+p}(x)t^p, \qquad \left|\frac{r\nu t}{2x}\right|<1, \quad \left|\frac{r}{2\nu t}\right|<1. \tag{57}$$

VI. Replacing x by $z^2/4$, t by $zt/2$ in generating relation (56) and using Equation (3), we obtain ([8]), for $Z_m=J_m$)

$$\left(\frac{\left(1-\frac{2c}{zt}\right)}{\left(1+\frac{2bt}{z}\right)}\right)^{m/2} J_m\left(z\left(1+\frac{2bt}{z}\right)^{1/2}\left(1-\frac{2c}{zt}\right)^{1/2}\right)=\sum_{p=-\infty}^{\infty}\frac{(-1)^{|p|}}{|p|!}\, b^{(p+|p|)/2}$$

$$\times\ c^{(-p+|p|)/2}\ {}_0F_1[-;|p|+1;bc]\,J_{m+p}(z)t^p, \qquad \left|\frac{2bt}{z}\right|<1, \quad \left|\frac{2c}{zt}\right|<1. \tag{58}$$

Several of the fundamental identities for cylindrical functions are special cases of generating relation (58). Also, for $c=0$, $t=1$ and $b=0$, $t=1$, relation (58) gives the formulas of Lommel ([8], for $Z_m=J_m$).

Again, replacing x by $z^2/4$, t by $z/2$ in generating relation (57) and using Equation (3), we obtain a generalization of Graf's addition theorem ([8], for $Z_m=J_m$)

$$\left(\frac{\left(1+\frac{r}{\nu z}\right)}{\left(1+\frac{r\nu}{z}\right)}\right)^{m/2} J_m\left(z\left(1+\frac{r\nu}{z}\right)^{1/2}\left(1+\frac{r}{\nu z}\right)^{1/2}\right)=\sum_{p=-\infty}^{\infty}(-\nu)^p\, J_p(r)J_{m+p}(z), \qquad \left|\frac{r\nu}{z}\right|<1, \quad \left|\frac{r}{\nu z}\right|<1. \tag{59}$$

References

[1] G. Dattoli, *Hermite-Bessel and Laguerre-Bessel functions: a by-product of the monomiality principle, Advanced special functions and applications*, (Melfi, 1999), Proc. Melfi Sch. Adv. Top. Math. Phys., 1(2000), 147-164.

[2] G. Dattoli and A. Torre, *Theory and Applications of Generalized Bessel Functions*, ARACNE, Rome, Italy, (1996).

[3] G. Dattoli and A. Torre, *Exponential operators, quasi-monomials and generalized polynomials*, Radiat. Phys. Chem., 57(2000), 21-26.

[4] G. Dattoli, Subuhi Khan and Ghazala Yasmin, *generating relations of Hermite-Tricomi functions using a representation of Lie algebra T_3*, Georgian Math. J., 14(1)(2007), 99-107.

[5] G. Dattoli, M. Migliorati and H. M. Srivastava, *Some families of generating functions foe the Bessels and related functions*, Georgian Math. J., 11(2)(2004), 219-228.

[6] G. Dattoli, A. Torre and G. Mazzacurati, *Quasi-monomials and isospectral problems*, Nuov. Cim. B, 112(1997), 133-138.

[7] G. Dattoli, P. L. Ottavini, A. Torre and L. Vazquez, *Evolution operator equations; integration with algebraic and finite difference methods, applications to physical problem in classical and quantum mechanics and quantum field theory*, La Revista Del Nuovo Cimento, 20(2)(1997), 1-133.

[8] W. Jr. Miller, *Lie Theory and Special Functions*, Academic Press, New York and London, (1968).

[9] Subuhi Khan and Rehana Khan, *Lie- theoretic generating relations involving multi-variable Hermite-Tricomi functions*, Integral Transform and Special Functions, 20(5)(2009), 365-375.

[10] N. Ja. Vilenkin, *Special Functions and the Theory of Group Representations*, Amer. Math. Soc. Trans. Providence, Rhode Island, (1968).

[11] A. Wawrzyñczyk, *Group Representation and Special Functions*, PWN-Polish Scientific Publ., Warszawa, (1984).

[12] L. Weisner, *Group-theoretic origin of certain generating functions*, Pacific J. Math., 5(1955), 1033-1039.

[13] L. Weisner, *Generating functions for Hermite functions*, Canad. J. Math., 11(1959), 141-147.

[14] L. Weisner, *Generating functions for Bessel functions*, Canad. J. Math., 11(1959), 148-155.