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Existence of Positive Periodic Solutions in a Non-selective Harvesting Impulsive Predator-prey System with Multiple Delays

R. Eswari[1,*] and V. Piramananantham[1]

1 Department of Mathematics, Bharathidasan University, Tiruchirappalli, Tamil Nadu, India.

Abstract: In this paper, we propose a non-selective harvesting impulsive predator-prey system with multiple delays. By applying the continuation theorem of coincidence degree theory, we establish a set of sufficient conditions on the existence of at least one positive periodic solution. The result not only improves but also generalizes that for the case with impulses. Some examples are provided to illustrate the result.

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1. Introduction

One of the important interactions among species is the predator-prey relationship and it has been extensively studied because of its universal existence. Predator-prey models are argubly the most fundamental building blocks of any bio and ecosystems as all biomasses are grown out of their resource masses. One of the familiar factors affecting the dynamics of predator-prey models is the functional response, which relates a single predator's prey consumption rate to prey population density. Lotka [1] and Volterra [2] introduced the first predator-prey model in 1925 and 1926 respectively. In recent years, predator-prey model takes the form:

$$\begin{cases} x' = rx\left(1 - \frac{x}{K}\right) - g(x)y, \\ y' = y(-d + \mu g(x)). \end{cases} \tag{1}$$

Here, x and y respectively stand for the densities of the prey and the predator and $g(x)$ is the functional response function, which reflects the capture ability of the predator to prey. For the detailed biological meaning, see for example [3–5] and the references contained therein. When we consider the exploit of human being should be added, the model (1) becomes:

$$\begin{cases} x' = rx\left(1 - \frac{x}{K}\right) - g(x)y - h_1, \\ y' = y(-d + \mu g(x)) - h_2. \end{cases} \tag{2}$$

* *E-mail: eswarimaths89@gmail.com*

For the study on systems with harvesting, one may consult [6–12] etc. In 1969, Hassell and Varley's [13] introduced a general predator-prey system, in which the functional response dependents on the predator density in different way. It is called a Hassell-Varley type functional response which takes the following form:

$$\begin{cases} x' = rx\Big(1-\frac{x}{k}\Big) - \frac{cxy}{my^{\gamma}+x}, \\ y' = y\Big(-d+\frac{fx}{my^{\gamma}+x}\Big), \ \gamma\in(0,1) \end{cases} \tag{3}$$

where γ is called the Hassell-Varley constant. In the typical predator-prey system interaction where predators do not form groups, one can assume that $\gamma = 1$, producing the so-called ratio dependent predator-prey system. For terrestrial predators that form a fixed number of tight groups, it is often reasonable to assume $\gamma = \frac{1}{2}$. For aquatic predators that form a fixed number of tight groups, $\gamma = \frac{1}{3}$ may be more appropriate. A special case of system (3) with $\gamma = 0$ and $\gamma = 1$ in ref [3, 14] cited therein. Hsu [15] studied system (3) and presented a systematic global qualitative analysis to it. Kai Wang [16] discussed the following delayed predator-prey model with Hassell-Varley type functional response:

$$\begin{cases} x'(t) = x(t)\Big(a(t)-b(t)x(t-\tau(t)) - \frac{c(t)y(t)}{my^{\gamma}(t)+x(t)}\Big), \\ y'(t) = y(t)\Big(-d(t)+\frac{r(t)x(t)}{my^{\gamma}(t)+x(t)}\Big), \ \gamma\in(0,1) \end{cases} \tag{4}$$

and established the existence of positive periodic solutions. In [17], the authors proposed a non-selective harvesting predator-prey model with multiple delays as follows:

$$\begin{cases} x'(t) = x(t)\Big(r_1(t)-b(t)x(t-\tau_1(t)) - \frac{a_1(t)y(t)}{my^{\gamma}(t)+x(t)}\Big) - c_1(t)x(t), \\ y'(t) = y(t)\Big(-r_2(t)+\frac{a_2(t)x(t-\tau_2(t))}{my^{\gamma}(t)+x(t-\tau_2(t))}\Big) - c_2(t)y(t). \end{cases} \tag{5}$$

They considered the periodic case and derived case of easily verifiable sufficient conditions on the existence of positive periodic solutions of (5).

Theorem 1.1 ([17]). *Suppose that the following assumptions hold.*

(I). $\tau_1'(t) < 1$ *and* $\tau_2'(t) < 1$, *for* $t \in R$,

(II). $\bar{a_2} > \bar{r_2} + \bar{c_2}$, $m(\bar{r_1} - \bar{c_1}) > \bar{a_1}$,

(III). The following algebraic equation set $\Gamma = \Big\{(x,y) : \bar{r_1} - \bar{c_1} - \bar{b}x - \frac{\bar{a_1}y}{my^{\gamma}+x} = 0, \ -\bar{r_2} - \bar{c_2} + \frac{\bar{a_2}x}{my^{\gamma}+x} = 0\Big\}$ *has a finite number of real-valued positive solution.*

Then system (5) has at least one positive periodic solution.

As we know, in population dynamics, many evolutionary processes experience short-time rapid change after undergoing relatively long smooth variation. Examples include annual harvest and stock of species as well as annual immigration. Incorporating the phenomena gives us impulsive differential systems. A lot of work has been done in this direction to namely a few see [18, 19] and the references therein. For the theory of impulsive differential equations we refer the reader to [20–22]. In this paper, we shall consider (5) with impulsive effect, precisely, we consider the following non-selective harvesting impulsive predator-prey system with multiple delays:

$$\begin{cases} \frac{dx(t)}{dt} = x(t)\Big(r_1(t)-b(t)x(t-\tau_1(t)) - \frac{a_1(t)y(t)}{my^{\gamma}(t)+x(t)}\Big) - c_1(t)x(t), \ t\neq t_k, \\ \frac{dy(t)}{dt} = y(t)\Big(-r_2(t)+\frac{a_2(t)x(t-\tau_2(t))}{my^{\gamma}(t)+x(t-\tau_2(t))}\Big) - c_2(t)y(t), \ t\neq t_k, \\ \Delta x(t) = x(t^+) - x(t) = d_{1k}x(t), \ t = t_k, \\ \Delta y(t) = y(t^+) - y(t) = d_{2k}y(t), \ t = t_k, \ k = 1,2,3,\cdots,q \end{cases} \tag{6}$$

where $a_1(t)$, $a_2(t)$, $r_1(t)$, $r_2(t)$, $b(t)$, $c_1(t)$, $c_2(t)$, $\tau_1(t)$, $\tau_2(t)$ are continuously nonnegative periodic functions with period ω, $m \geq 0$, $\gamma \in (0,1)$ and $d_{1k}, d_{2k} \in (-1,0]$ $(k \in N)$, $\{t_k\}$ is a strickly increasing sequence with $t_1 > 0$ and $\lim_{k\to\infty} t_k = \infty$, we further assume that there exists $q \in N$ such that $d_{1(k+q)} = d_{1k}$, $d_{2(k+q)} = d_{2k}$ and $t_{k+q} = t_k + \omega$ for $k \in N$. From the point of view of mathematical biology, we only consider the positive periodic solutions of (6). By applying the continuation theorem of coincidence degree theory [23]. We will derive sufficient conditons on the existence of positive periodic solutions. As we see, our result not only improve but also generalise Theorem (1) in Section 2. Then the result demonstrated by MATLAB simulation in Section 3.

2. Existence of positive periodic solutions

Before exploring the existence of periodic solutions of system, we will make some preparations. Let X, Y be real Banach spaces, $L : \mathrm{Dom}L \subset X \to Y$ be a linear mapping and $N : X \to Y$ be a continuous mapping. The mapping L is said to be a Fredholm mapping of index zero, if dimKerL=codimImL $<+\infty$ and ImL is closed in Y. If L is a Fredholm mapping of index zero, then there exists continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q)$. It follows that the restriction L_P of L to $\mathrm{Dom}L \cap \mathrm{Ker}P : (I-P)X \to \mathrm{Im}L$ is invertible. Denote the inverse of L_P by K_P.

Lemma 2.1 ([23]). *Let $\Omega \subset X$ be an open bounded set, L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Assume*

(a). for all $\lambda \in (0,1)$ and $x \in \partial\Omega \cap DomL$, $Lx \neq \lambda Nx$,

(b). for all $x \in \partial\Omega \cap DomL$, $QNx \neq 0$,

(c). $deg(JQN, \Omega \cap KerL, 0) \neq 0$.

Then $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap DomL$.

Lemma 2.2 ([19]). *The set $F \subset PC_\omega$ is relatively compact iff*

(a). F is bounded, that is $\|\phi\| = \sup_{t\in[0,\omega]} |\phi(t)| \leq M$ for each $\phi \in F$ and for some $M > 0$,

(b). F is quasi-equicontinuous.

Lemma 2.3. *Let $\phi \in PC'_\omega$. Then for $t \in [0,\ \omega]$*

$$\phi(t) \leq |\phi(s)| + \frac{1}{2}\left[\int_0^\omega |\phi'(s)|\mathrm{d}s + \left|\sum_{k=1}^{q} \Delta\phi(t_k)\right|\right],\ \phi(t) \geq |\phi(s)| - \frac{1}{2}\left[\int_0^\omega |\phi'(s)|\mathrm{d}s + \left|\sum_{k=1}^{q} \Delta\phi(t_k)\right|\right].$$

Lemma 2.4 ([16]). *If $\tau \in C'(R,R)$ with (i) $\tau(t+\omega) \equiv \tau(t)$ and (ii) $\frac{d\tau(t)}{dt} < 1$ for $t \in [0,\ \omega]$, then function $\mu(t) = t - \tau(t)$ has a unique inverse $\mu^{-1}(t)$ satisfying $\mu \in C(R,R)$ with $\mu^{-1}(s+\omega) \equiv \mu^{-1}(s) + \omega$ for $s \in [0,\ \omega]$.*

In what follows, for convenience, we shall use the notation $\bar{g} = \frac{1}{\omega}\int_0^\omega g(t)\mathrm{d}t$, $g^M = \max_{t\in[0,\ \omega]} g(t)$, $g^L = \min_{t\in[0,\ \omega]} g(t)$, where g is piecewise continuous function with period ω.

Theorem 2.5. *For system (6), assume that:*

(I). $\tau_1'(t) < 1$ and $\tau_2'(t) < 1$, for $t \in R$,

(II). $\bar{a_2} > \bar{r_2} + \bar{c_2} - \frac{d_2}{\omega}$, $m(\bar{r_1} - \bar{c_1} + \frac{d_1}{\omega}) > \bar{a_1}$,

(III). The following algebraic equation set $\Gamma = \left\{(x,y) : \frac{d_1}{\omega} + \bar{r_1} - \bar{c_1} - \bar{b}x - \frac{\bar{a_1} y}{my^{\gamma}+x} = 0,\ \frac{d_2}{\omega} - \bar{r}_2 - \bar{c}_2 + \frac{\bar{a}_2 x}{my^{\gamma}+x} = 0\right\}$ *has a finite number of real-valued positive solution.*

Then system (6) has at least one postive periodic solution.

Proof. By the biological meaning, we only focus on the positive periodic solutions to system (6). Let $x(t) = \mathrm{e}^{u_1(t)}$ and $y(t) = \mathrm{e}^{u_2(t)}$. Then system (6) becomes:

$$\begin{cases} u_1'(t) = r_1(t) - c_1(t) - b(t)\mathrm{e}^{u_1(t-\tau_1(t))} - \frac{a_1(t)e^{u_2(t)}}{me^{\gamma u_2(t)}+\mathrm{e}^{u_1(t)}}, \\ u_2'(t) = -(r_2(t)+c_2(t)) + \frac{a_2(t)\mathrm{e}^{u_1(t-\tau_2(t))}}{me^{\gamma u_2(t)}+\mathrm{e}^{u_1(t-\tau_2(t))}}, \\ \Delta u_1(t) = \log(1+d_{1k}), \\ \Delta u_2(t) = \log(1+d_{2k}). \end{cases} \tag{7}$$

In order to apply Lemma 2.1 to study the existence of positive periodic solutions to above system, we set

$$X = \{u(t) = (u_1(t), u_2(t))^T \in PC_\omega(R, R^2) : u_i(t+\omega) = u_i(t),\ i = 1,2\}, Y = X \times R^{2q}, q = 1,2,3\dots k.$$

and

$$\|u\|_X = \sup_{t\in[0,\ \omega]} |u_1(t)| + \sup_{t\in[0,\ \omega]} |u_2(t)|.$$

$$\|u\|_Y = \|u\|_X + \|y\|, u \in X, y \in R^{2q}.$$

Then both $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are Banach spaces. Define

$$\mathrm{Dom}L = \{u(t) = (u_1(t), u_2(t))^T \in PC_\omega(R, R^2) : u_i \in PC'_\omega,\ i = 1,2\}.$$

$$L : \mathrm{Dom}L \cap X \to Y, Lu(t) = (u_1'(t), u_2'(t), \Delta u_1(t_k), \Delta u_2(t_k))^T.$$

$$N : X \to Y,$$

$$N\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \left(\begin{pmatrix} r_1(t) - c_1(t) - b(t)\mathrm{e}^{u_1(t-\tau_1(t))} - \frac{a_1(t)e^{u_2(t)}}{me^{\gamma u_2(t)}+\mathrm{e}^{u_1(t)}} \\ -(r_2(t)+c_2(t)) + \frac{a_2(t)\mathrm{e}^{u_1(t-\tau_2(t))}}{me^{\gamma u_2(t)}+\mathrm{e}^{u_1(t-\tau_2(t))}} \end{pmatrix}, \left\{\begin{pmatrix} \log(1+d_{1k} \\ \log(1+d_{2k} \end{pmatrix}\right\}_{k=1}^{q}\right).$$

$$P : X \to X,\ P((u_1, u_2)^T) = (\bar{u_1}, \bar{u_2})^T.$$

$$Q : Y \to Y,$$

$$Q\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \left\{\begin{pmatrix} m_k \\ n_k \end{pmatrix}\right\}_{k=1}^{q}\right) = \left(\begin{pmatrix} \frac{1}{\omega}\int_0^\omega \mathrm{u}_1(t)\,\mathrm{d}t + \frac{1}{\omega}\sum_{k=1}^{q} m_k \\ \frac{1}{\omega}\int_0^\omega \mathrm{u}_2(t)\,\mathrm{d}t + \frac{1}{\omega}\sum_{k=1}^{q} n_k \end{pmatrix}, \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}_{k=1}^{q}\right).$$

It is not difficult to see that

$$\mathrm{Ker}L = \{u = (u_1, u_2)^T \in X : \exists\, c \in R^2,\ (u_1(t), u_2(t)) = c,\ for\ t \in R\}.$$

$$\mathrm{Im}L = \{y = (f, a_1, a_2, ..., a_q) \in Y : \exists\, f \in \mathrm{Dom}L,\ \int_0^\omega f(s)\mathrm{d}s + \sum_{k=1}^{q} a_k = 0\}.$$

Since ImL is closed in Y. ImP = Ker$L$, Ker$Q$ = ImL, and dimker$L$ = codim$L$ = 2. We know that L is a Fredholm mapping of index zero. Moreover, the generalized inverse (to L) K_p : Im$L \to$ Ker$P \cap$ Dom$L$ is

$$K_p(u) = \int_0^\omega \mathrm{u}(s)\,\mathrm{d}s + \sum_{0<t_k<t} a_k - \frac{1}{\omega}\int_0^\omega \int_0^t \mathrm{u}(s)\,\mathrm{d}s\mathrm{d}t - \sum_{k=1}^{q} a_k.$$

Then direct computation gives us

$$QN(u) = \left(\begin{pmatrix} \frac{1}{\omega}\int_0^\omega \Big[r_1(t) - c_1(t) - b(t)\mathrm{e}^{u_1(t-\tau_1(t))} - \frac{a_1(t)\mathrm{e}^{u_2(t)}}{m\mathrm{e}^{\gamma u_2(t)}+\mathrm{e}^{u_1(t)}}\Big]\mathrm{d}t + \frac{1}{\omega}\sum_{k=1}^{q}\log(1+d_{1k}) \\ \frac{1}{\omega}\int_0^\omega \Big[-(r_2(t)+c_2(t)) + \frac{a_2(t)\mathrm{e}^{u_1(t-\tau_2(t))}}{m\mathrm{e}^{\gamma u_2(t)}+\mathrm{e}^{u_1(t-\tau_2(t))}}\Big]\mathrm{d}t + \frac{1}{\omega}\sum_{k=1}^{q}\log(1+d_{2k}) \end{pmatrix}, \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}_{k=1}^{q}\right)$$

and

$$\begin{aligned} K_p(I-Q)Nu \;=\; & \begin{pmatrix} \int_0^t \Big[r_1(s) - c_1(s) - b(s)\mathrm{e}^{u_1(s-\tau_1(s))} - \frac{a_1(s)\mathrm{e}^{u_2(s)}}{m\mathrm{e}^{\gamma u_2(s)}+\mathrm{e}^{u_1(s)}}\Big]\mathrm{d}s + \sum_{o<t_k<t}\log(1+d_{1k}) \\ \int_0^t \Big[-(r_2(s)+c_2(s)) + \frac{a_2(s)\mathrm{e}^{u_1(s-\tau_2(s))}}{m\mathrm{e}^{\gamma u_2(s)}+\mathrm{e}^{u_1(s-\tau_2(s))}}\Big]\mathrm{d}s + \sum_{o<t_k<t}\log(1+d_{2k}) \end{pmatrix} \\ & - \begin{pmatrix} \frac{1}{\omega}\int_0^\omega\int_0^t \Big[r_1(s) - c_1(s) - b(s)\mathrm{e}^{u_1(s-\tau_1(s))} - \frac{a_1(s)\mathrm{e}^{u_2(s)}}{m\mathrm{e}^{\gamma u_2(s)}+\mathrm{e}^{u_1(s)}}\Big]\mathrm{d}s\mathrm{d}t + \sum_{k=1}^{q}\log(1+d_{1k}) \\ \frac{1}{\omega}\int_0^\omega\int_0^t \Big[-(r_2(s)+c_2(s)) + \frac{a_2(s)\mathrm{e}^{u_1(s-\tau_2(s))}}{m\mathrm{e}^{\gamma u_2(s)}+\mathrm{e}^{u_1(s-\tau_2(s))}}\Big]\mathrm{d}s\mathrm{d}t + \sum_{k=1}^{q}\log(1+d_{2k}) \end{pmatrix} \\ & - \begin{pmatrix} (\frac{t}{\omega}-\frac{1}{2})\int_0^\omega \Big[r_1(t) - c_1(t) - b(t)\mathrm{e}^{u_1(t-\tau_1(t))} - \frac{a_1(t)\mathrm{e}^{u_2(t)}}{m\mathrm{e}^{\gamma u_2(t)}+\mathrm{e}u_1(t)}\Big]\mathrm{d}t + \sum_{k=1}^{q}\log(1+d_{1k}) \\ (\frac{t}{\omega}-\frac{1}{2})\int_0^\omega \Big[-(r_2(t)+c_2(t)) + \frac{a_2(t)\mathrm{e}^{u_1(t-\tau_2(t))}}{m\mathrm{e}^{\gamma u_2(t)}+\mathrm{e}^{u_1(t-\tau_2(t))}}\Big]\mathrm{d}t + \sum_{k=1}^{q}\log(1+d_{2k}) \end{pmatrix}. \end{aligned}$$

By Lebesque covergence theorem, QN and $K_p(I-Q)N$ are continuous. Furthermore, it follows from Lemma 2.2 that $QN(\bar{\Omega})$ and $K_p(I-Q)N(\bar{\Omega})$ are relatively compact for any open bounded set $\Omega \subset X$. Therefore, N is L-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset X$. In the following, we consider the operator equation, $Lu = \lambda Nu$, $\lambda \in (0,1)$, that is,

$$\begin{cases} u_1'(t) = \lambda\Big[r_1(t) - c_1(t) - b(t)\mathrm{e}^{u_1(t-\tau_1(t))} - \frac{a_1(t)\mathrm{e}^{u_2(t)}}{m\mathrm{e}^{\gamma u_2(t)}+\mathrm{e}^{u_1(t)}}\Big], \\ u_2'(t) = \lambda\Big[-(r_2(t)+c_2(t)) + \frac{a_2(t)\mathrm{e}^{u_1(t-\tau_2(t))}}{m\mathrm{e}^{\gamma u_2(t)}+\mathrm{e}^{u_1(t-\tau_2(t))}}\Big], \\ \Delta u_1(t) = \lambda[\log(1+d_{1k})], \\ \Delta u_2(t) = \lambda[\log(1+d_{2k})]. \end{cases} \tag{8}$$

Integrating of both sides of of the first equation in equations (8) gives

$$\omega(\bar{r_1} - \bar{c_1}) + d_1 = \int_0^\omega \Big[b(t)\mathrm{e}^{u_1(t-\tau_1(t))} + \frac{a_1(t)\mathrm{e}^{u_2(t)}}{m\mathrm{e}^{\gamma u_2(t)} + \mathrm{e}^{u_1(t)}}\Big]\mathrm{d}t \tag{9}$$

In view of Lemma 2.4, conditions (i) and (ii), we obtain

$$\int_0^\omega b(t)\mathrm{e}^{u_1(t-\tau_1(t))}\mathrm{d}t = \int_0^\omega \frac{b(\mu^{-1}(t))\mathrm{e}^{u_1(t)}}{1-\tau_1'(\mu^{-1}(t))}\mathrm{d}t,$$

which together with (9) gives

$$\omega(\bar{r_1} - \bar{c_1}) + d_1 = \int_0^\omega \Big[\frac{b(\mu^{-1}(t))\mathrm{e}^{u_1(t)}}{1-\tau_1'(\mu^{-1}(t))} + \frac{a_1(t)\mathrm{e}^{u_2(t)}}{m\mathrm{e}^{\gamma u_2(t)} + \mathrm{e}^{u_1(t)}}\Big]\mathrm{d}t, \tag{10}$$

which yields,

$$\int_0^\omega \mathrm{e}^{u_1(t)}\mathrm{d}t \le \frac{\omega(\bar{r_1}-\bar{c_1}+\frac{d_1}{\omega})}{\Lambda_1^L} \triangleq \omega N_1, \tag{11}$$

where $\Lambda_1 = \frac{b(\mu^{-1}(t))}{1-\tau_1'(\mu^{-1}(t))}$. Multiplying both sides of the second equation of equations (8) by $e^{\gamma u_2(t)}$ and integrating them from 0 to ω, we have

$$\begin{aligned}\int_0^\omega (r_2(t)+c_2(t)-d_2)\mathrm{e}^{\gamma u_2(t)}\mathrm{d}t &= \int_0^\omega \frac{a_2(t)\mathrm{e}^{u_1(t-\tau_2(t))+\gamma u_2(t)}}{m\mathrm{e}^{\gamma u_2(t)}+\mathrm{e}^{u_1(t-\tau_2(t))}}\mathrm{d}t,\\ &< \frac{a_2^M}{m}\int_0^\omega \mathrm{e}^{u_1(t-\tau_2(t))}\mathrm{d}t,\\ &= \frac{a_2^M}{m}\int_0^\omega \frac{\mathrm{e}^{u_1(t)}}{1-\tau_2'(\mu^{-1}(t))}\mathrm{d}t,\\ &< \frac{a_2^M\Lambda_2^M}{m}\int_0^\omega \mathrm{e}^{u_1(t)}\mathrm{d}t,\end{aligned}$$

where $\Lambda_2 = \frac{1}{1-\tau_1'(\mu^{-1}(t))}$. Together with (11) gives

$$\begin{aligned}\int_0^\omega \mathrm{e}^{\gamma u_2(t)}\mathrm{d}t &< \frac{a_2^M\Lambda a_2^M}{m(r_2^L+c_2^L-d_2)}\int_0^\omega \mathrm{e}^{u_1(t)}\mathrm{d}t,\\ &\le \omega\Big[\frac{(\bar{r_1}-\bar{c_1}+\frac{d_1}{\omega})a_2^M\Lambda_2^M}{m\Lambda_1^L(r_2^L+c_2^L-d_2)}\Big],\\ &\triangleq \omega N_2.\end{aligned} \tag{12}$$

If $u_2(t)\ge 0$, then $\mathrm{e}^{\gamma u_2(t)}\ge 1$ and (11) gives

$$N_2 \ge \frac{1}{\omega}\int_0^\omega \mathrm{e}^{\gamma u_2(t)}\mathrm{d}t \ge 1,$$

which implies that there must be a constant $\beta_1\in[0,\ \omega]$ such that

$$u_2(\beta_1) \le \frac{\log N_2}{\gamma}, \tag{13}$$

which together with (11) and (12) yields

$$\begin{aligned}N_1 &\ge \frac{1}{\omega}\int_0^\omega \mathrm{e}^{u_1(t)}\mathrm{d}t,\\ &> \frac{m(r_2^L+c_2^L-d_2)}{\omega a_2^M\Lambda_2^M}\int_0^\omega \mathrm{e}^{\gamma u_2(t)}\mathrm{d}t,\\ &\ge \frac{m(r_2^L+c_2^L-d_2)}{a_2^M\Lambda_2^M},\end{aligned}$$

which yields that there is a constant $\alpha_1\in[0,\ \omega]$ such that

$$u_1(\alpha_1) \le \max\Big\{|\log N_1|, \Big|\log\frac{m(r_2^L+c_2^L-d_2)}{a_2^M\Lambda_2^M}\Big|\Big\}. \tag{14}$$

On the other hand, if $u_2(t)<0$ (10) yields

$$\omega(\bar{r_1}-\bar{c_1})+d_1 \le \Lambda_1^M\int_0^\omega \mathrm{e}^{u_1(t)}\mathrm{d}t + \frac{1}{m}\int_0^\omega a_1(t)\mathrm{e}^{(1-\gamma)u_2(t)}\mathrm{d}t, \tag{15}$$

and $0<\mathrm{e}^{(1-\gamma)u_2(t)}\le 1$, it follows from (15) that

$$\omega(\bar{r_1}-\bar{c_1})+d_1 \le \Lambda_1^M\int_0^\omega \mathrm{e}^{u_1(t)}\mathrm{d}t + \frac{\omega\bar{a_1}}{m},$$

together with condition (II), we have

$$\int_0^\omega e^{u_1(t)}\mathrm{d}t > \frac{\omega}{m\Lambda_1^M}\Big[m(\bar{r_1}-\bar{c_1}+\frac{d_1}{\omega})-\bar{a_1})\Big] \triangleq \omega k_1 > 0, \tag{16}$$

and by (11), we get

$$N_1 \geq \frac{1}{\omega}\int_0^\omega \mathrm{e}^{u_1(t)}\mathrm{d}t \geq k_1,$$

which yields that there is a constant $\alpha_2 \in [0,\ \omega]$ such that

$$u_1(\alpha_2) \leq \max\{|\log N_1|, |\log k_1|\}. \tag{17}$$

By (14) and (17), we know that there exists constant $\alpha \in [0,\ \omega]$ such that

$$u_1(\alpha) \leq \max\Big\{|\log N_1|, |\log k_1|, \Big|\log\frac{m(r_2^L+c_2^L-d_2)}{a_2^M\Lambda_2^M}\Big|\Big\} \triangleq H_1. \tag{18}$$

Meanwhile, from condition (II), and the second equation of (8) and (16) we obtain

$$\begin{aligned}
0 < \omega\Big[\bar{a_2}-\Big(\bar{r_2}+\bar{c_2}-\frac{d_2}{\omega}\Big)\Big] &= \int_0^\omega \frac{ma_2(t)\mathrm{e}^{\gamma u_2(t)}}{m\mathrm{e}^{\gamma u_2(t)}+\mathrm{e}^{u_1(t-\tau_2(t))}}\mathrm{d}t,\\
&\leq ma_2^M\int_0^\omega \frac{\mathrm{e}^{\gamma u_2(t)}}{\mathrm{e}^{u_1(t-\tau_2(t))}}\mathrm{d}t,\\
&\leq ma_2^M\frac{(\int_0^\omega \mathrm{e}^{2\gamma u_2(t)})^{\frac{1}{2}}}{(\int_0^\omega \mathrm{e}^{2u_1(t-\tau_2(t))})^{\frac{1}{2}}},\\
&= ma_2^M\frac{(\int_0^\omega \mathrm{e}^{2\gamma u_2(t)})^{\frac{1}{2}}}{(\int_0^\omega \Lambda_2\mathrm{e}^{2u_1(t)})^{\frac{1}{2}}},\\
&\leq ma_2^M\frac{(\int_0^\omega \mathrm{e}^{2\gamma u_2(t)})^{\frac{1}{2}}}{(\Lambda_2^L)^{\frac{1}{2}}(\int_0^\omega \mathrm{e}^{2\gamma u_1(t)})^{\frac{1}{2}}},\\
&\leq \frac{ma_2^M}{k_1(\omega\Lambda_2^L)^{\frac{1}{2}}}\Big(\int_0^\omega \mathrm{e}^{2\gamma u_2(t)}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Thus, we have

$$\frac{1}{\omega}\int_0^\omega \mathrm{e}^{2\gamma u_2(t)}\mathrm{d}t \geq \Lambda_2^L\left[\frac{\omega k_1\Big(\bar{a_2}-\Big(\bar{r_2}+\bar{c_2}-\frac{d_2}{\omega}\Big)\Big)}{ma_2^M}\right]^2 \triangleq k_2^2, \tag{19}$$

which yield that there is a constant $\beta_2 \in [0,\ \omega]$ such that

$$u_2(\beta_2) \geq \frac{\log k_2}{\gamma}. \tag{20}$$

Thus by (13) and (20), there must be a constant $\beta \in [0,\ \omega]$ such that

$$u_2(\beta) \leq \max\Big\{\Big|\frac{\log N_2}{\gamma}\Big|, \Big|\frac{\log k_2}{\gamma}\Big|\Big\} \triangleq H_2. \tag{21}$$

Now, by (8), (18) and (21), we obtain

$$\begin{aligned}
|u_1(t)| &\leq |u_1(\alpha)| + \frac{1}{2}\Big(\int_0^\omega |u_1'(t)|\mathrm{d}t + \Big|\sum_{k=1}^q \log(1+d_{1k}\Big|\Big)\\
&\leq H_1+\omega(\bar{r_1}-\bar{c_1})+\frac{1}{2}\Big(\sum_{k=1}^q\log(1+d_{1k})+\Big|\sum_{k=1}^q\log(1+d_{1k})\Big|\Big) \triangleq M_1 \qquad (22)\\
|u_2(t)| &\leq |u_1(\beta)| + \frac{1}{2}\Big(\int_0^\omega |u_2'(t)|\mathrm{d}t + \Big|\sum_{k=1}^q \log(1+d_{2k})\Big|\Big)\\
&\leq H_2+\omega(\bar{r_2}+\bar{c_2})-\frac{1}{2}\Big(\sum_{k=1}^q\log(1+d_{2k})-\Big|\sum_{k=1}^q\log(1+d_{2k})\Big|\Big) \triangleq M_2. \qquad (23)
\end{aligned}$$

From condition (II), we denote by $(u_{1i}^*(t)), (u_{2i}^*(t))$, $i = 1, 2, \ldots q$, all the real-valued solutions of the following algebraic solution set

$$\begin{cases} \frac{d_1}{\omega} + \bar{r_1} - \bar{c_1} - \bar{b}e^{u_1(t)} - \mu\left[\frac{\bar{a_1}e^{u_2(t)}}{me^{\gamma u_2(t)}+e^{u_1(t)}}\right] = 0, \\ \frac{d_2}{\omega} - \bar{r}_2 - \bar{c}_2 + \frac{\bar{a}_2 e^{u_1(t)}}{me^{\gamma u_2(t)}+e^{u_1(t)}} = 0. \end{cases} \tag{24}$$

Let use take

$$M_0 \triangleq e^{\left\{\max_{1\le i\le k, t\in[0,\omega]}\{(u_{1i}^*(t)), (u_{2i}^*(t)\}\right\}}. \tag{25}$$

And, now together with (22), (23) and (25) we can set $M = M_1 + M_2 + M_0$ and take $\Omega = \{u = (u_1(t), u_2(t))^T : \|u\| < D\}$. It is clear that Ω verifies the requirement in Lemma 2.1. If $u \in \partial\Omega \cap KerL = \partial\Omega \cap R^2$, then u is a constant vector in R^2 with $\|u\| = D$ satisfying

$$QN\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} \frac{d_1}{\omega} + \bar{r_1} - \bar{c_1} - \bar{b}e^{u_1(t)} - \frac{\bar{a_1}e^{u_2(t)}}{me^{\gamma u_2(t)}+e^{u_1(t)}} \\ \frac{d_2}{\omega} - \bar{r}_2 - \bar{c}_2 + \frac{\bar{a}_2 e^{u_1(t)}}{me^{\gamma u_2(t)}+e^{u_1(t)}} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Consider a homotopy $B_\mu((u_1, u_2)^T) = \mu JQN((u_1, u_2)^T) + (1-\mu)\phi((u_1, u_2)^T)$. By a direct computation and the invariance property of homotopy, one has $deg\{JQN, KerL \cap \partial\Omega, 0\} = deg\{\phi, KerL \cap \partial\Omega, 0\} = l > 0$. By now we have proved that Ω verifies all the requirements in Lemma 2.1. Then, we get that equations (8) have at least one periodic solution $(u_1^*(t)), (u_2^*(t))^T$ with period ω in $\mathrm{Dom}L \cap \bar{\Omega}$, which implies that at least one positive periodic solution $(e^{u_1^*(t)}, e^{u_2^*(t)})$ with period ω. This completes the proof of theorem. □

3. Examples

We illustrative the theorem with few examples.

Example 3.1. *In system (6)*

$$\begin{cases} \frac{dx(t)}{dt} = x(t)\Big[(4 - 0.2\cos t) - (2 - 0.1\cos t)x(t - 0.25) - \frac{(2+0.1\cos t)y(t)}{y^{\frac{1}{2}}(t)+x(t)} - (1 - 0.1\sin t)\Big], \ t \neq t_k, \\ \frac{dy(t)}{dt} = y(t)\Big[-(1 + 0.3\cos t) + \frac{(3+0.1\sin t)x(t-0.2)}{y^{\frac{1}{2}}(t)+x(t-0.2)} - (1 + 0.2\cos t)\Big], \ t \neq t_k, \\ \Delta x(t) = (\frac{1}{e} - 1)x(t), \ t = t_k, \\ \Delta y(t) = (\frac{1}{e} - 1)y(t), \ t = t_k, \ k = 1, 2, 3, \cdots, q. \end{cases} \tag{26}$$

It is obvious that conditons (i) and (ii) hold. In addition the algebraic set Γ has only positive real valued solution $(x, y) = (\frac{1003}{798}, \frac{5015^2}{13566^2})$, which together with Theorem (2.5) yields that System 26 has atleast one positive periodic solution. Take the initial value by $(f(0), g(0)) = (1, 1)$. Fig.1 shows the dynamic behaviors of the solution $(x(t), y(t))$, which is a positive periodic solution of System 26.

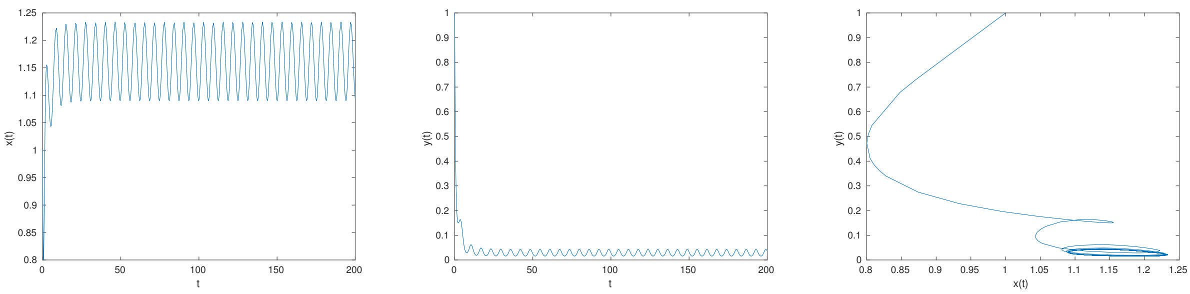

Figure 1. *Evolution of the positive periodic solution of System 26*

Example 3.2. *In system (6)*

$$\begin{cases} \frac{dx(t)}{dt} = x(t)\Big[(4-0.2\cos t) - (2-0.1\cos t)x(t-0.25) - \frac{(4+0.1\cos t)y(t)}{y^{\frac{1}{2}}(t)+x(t)} - (1-0.1\sin t)\Big], \ t \neq t_k, \\ \frac{dy(t)}{dt} = y(t)\Big[-(1+0.3\cos t) + \frac{(2.4+0.1\cos t)x(t-0.2)}{y^{\frac{1}{2}}(t)+x(t-0.2)} - (2+0.1\cos t)\Big], \ t \neq t_k, \\ \Delta x(t) = (\frac{1}{e}-1)x(t), \ t = t_k, \\ \Delta y(t) = (\frac{1}{e}-1)y(t), \ t = t_k, \ k = 1,2,3,\cdots,q. \end{cases} \tag{27}$$

In this system conditions (i) and (iii) are hold and condition (ii) fails. Fig.2 shows the details of the dynamic behaviour of the solution $(x(t), y(t))$ to System 27 with the same initial value as that in System 26, which shows that the integral curve of the prey is periodic, but the predator is finally extinct.

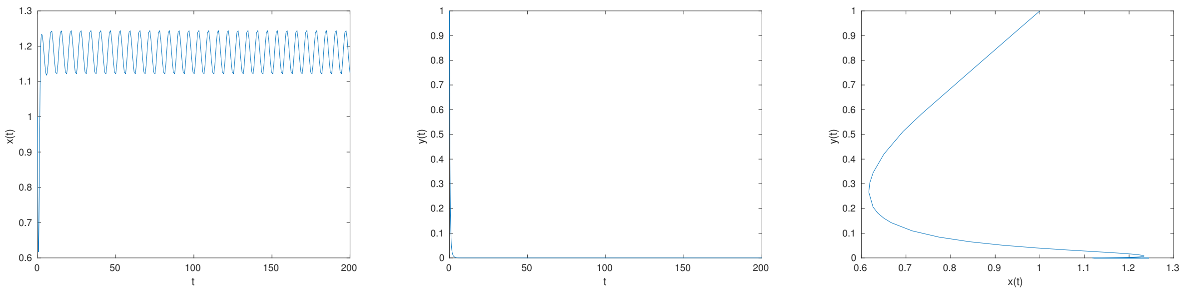

Figure 2. *Evolution of the solution of System 27*

Example 3.3. *In system (6)*

$$\begin{cases} \frac{dx(t)}{dt} = x(t)\Big[(4-0.2\cos t) - (2-0.1\cos t)x(t-0.25) - \frac{(\frac{59}{22}+0.1\cos t)y(t)}{y^{\frac{1}{2}}(t)+x(t)} - (1-0.1\sin t)\Big], \ t \neq t_k, \\ \frac{dy(t)}{dt} = y(t)\Big[-(1+0.3\cos t) + \frac{(\frac{51}{22}+0.1\sin t)x(t-0.2)}{y^{\frac{1}{2}}(t)+x(t-0.2)} - (1+0.2\cos t)\Big], \ t \neq t_k, \\ \Delta x(t) = (\frac{1}{e}-1)x(t), \ t = t_k, \\ \Delta y(t) = (\frac{1}{e}-1)y(t), \ t = t_k, \ k = 1,2,3,\cdots,q. \end{cases} \tag{28}$$

In this system condition (i) hold. But the algebraic set has no positive solution and condition (ii) satisfied for equality. Choosing the same initial value as in System 26 we can get the detail of its dynamic behaviour of the solutions are $(x(t), y(t))$ in fig.3, which shows that the solution of the prey is periodic and the extinction of the predator, that is there is no positive periodic solution of the System 28

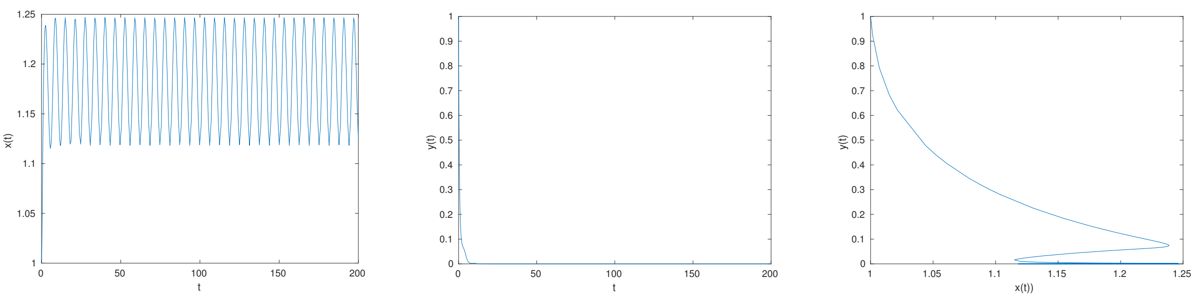

Figure 3. *Evolution the solution of System 28*

Example 3.4. *In system (6)*

$$\begin{cases} \frac{dx(t)}{dt} = x(t)\Big[(4-0.2\cos t)-(2-0.1\cos t)x(t-0.3)-\frac{(2+0.1\cos t)y(t)}{y^{\frac{1}{2}}(t)+x(t)}-(1-0.1\sin t)\Big],\ t\neq t_k,\\ \frac{dy(t)}{dt} = y(t)\Big[-(1+0.3\cos t)+\frac{(3+0.1\sin t)x(t-(1-0.25\cos 4t)}{y^{\frac{1}{2}}(t)+x(t-(1-0.25\cos 4t)}-(2+0.1\cos t)\Big],\ t\neq t_k,\\ \Delta x(t) = (\frac{1}{e}-1)x(t),\ t=t_k,\\ \Delta y(t) = (\frac{1}{e}-1)y(t),\ t=t_k,\ k=1,2,3,\cdots,q. \end{cases} \tag{29}$$

In this system does not satisfy condition (i) *but the conditions* (i) *and* (ii) *hold. From fig.4 shows that the solution of the prey and predator are both periodic.*

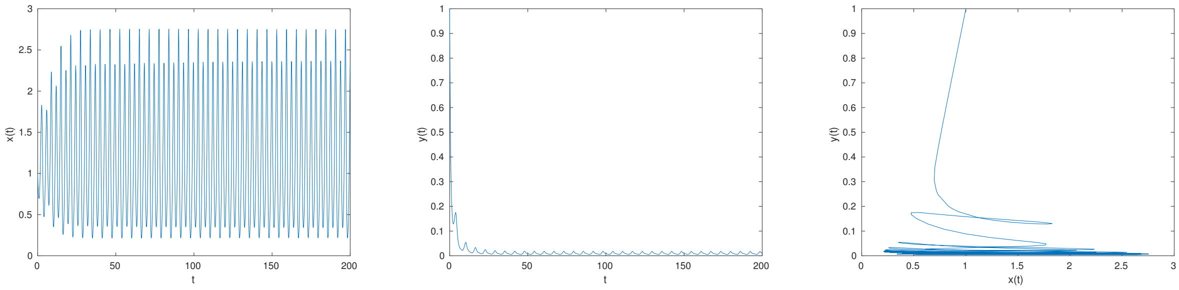

Figure 4. *Evolution of the positive periodic solution of System 29*

Acknowledgements

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