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Fixed Point Results for F-contractive Mappings

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Abstract: In this paper, we present a common fixed point theorem for a pair of self mappings, using F-contraction condition, in the setting of metric space.Further a fixed point theorem for a self map and application in dynamic programming are given. Our result extend the theorems of Wardoski, Cosentino and Vetro, Choudhury etc.

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1. Introduction

In the beginning of fixed point theory on complete metric space, one of the most important result is known as Banach cotraction principle, published in 1922.

Theorem 1.1. *Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be a contraction mapping if there exists a contant $L \in [0, 1)$ called a contraction such that $d(Tx, Ty) \leq Ld(x, y) \forall x, y \in X$.*

Banach Contraction Principle says that any contraction self-mappings on a complete metric space has a unique fixed point.Because of its importance Banach Contraction Principle has been extended and generalized in many directions([1,2,4,5,6,7,8] etc.). Recently, Wardoski [16] introduced a new concept of contraction and proved a fixed point theorem which generalizes Banach contraction principle. Further Cosentino and Vetro [10] proved some fixed point results of Hardy-Rogers Type for a self map on a complete metric space.Motivated by these results we prove some fixed point theorems for a self map.

2. Preliminaries

Throughout in this article we denote by $\mathbb{R}$, the set of all real numbers, by $\mathbb{R}^+$ the set of all positive real numbers and by $\mathbb{N}$ the set of all positive integers.

Definition 2.1 ([16])**.** *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:*

(F_1). *F is strictly increasing;*

(F_2). *for each sequence $\alpha_n \subset \mathbb{R}^+$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to F(\alpha_n)} = -\infty$;*

(F_3). *there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

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Definition 2.2 ([16]). *Let (X,d) be a metric space and $T: X \to X$ be a mapping. Then T is said to be an F-contraction if $F \in F$ and there exists $\tau > 0$ such that*

$$\forall x, y \in X[d(Tx,Ty) > 0 \implies \tau + F(d(Tx,Ty)) \leq F(d(x,y))]. \tag{1}$$

Example 2.3 ([16]). *Let $F_1 : (0,\infty) \to \mathbb{R}$ be given by $F_1(\alpha) = ln\alpha$. It is clear that $F_1 \in F$. Then each self mappings T on a metric space (X,d) satisfying is an F_1-contraction such that*

$$d(Tx,Ty) \leq e^{-\tau} d(x,y) \forall x, y \in X, Tx \neq Ty. \tag{2}$$

It is clear that for $x, y \in X$ such that $Tx = Ty$ the inequality $d(Tx,Ty) \leq e^{-\tau} d(x,y)$ also holds. Therefore T satisfies () with $L = e^{-\tau}$, thus T is a contraction.

The following theorem is generalization of Banach Contraction Principle given by Wardowski:

Theorem 2.4 ([16]). *Let (X,d) be a complete metric space and let $T: X \to X$ be an F-contraction.Then T has a unique fixed point in X.*

Lemma 2.5 ([1]). *Let f and g be weakly compatible self maps on a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is unique common fixed point of f and g.*

3. Main Results

Theorem 3.1. *Let (X,d) be a metric space and let S and T be self maps on X. Assume that there exist $F \in F$ and $\tau \in \mathbb{R}^+$ such that*

$$d(Sx,Sy) > 0 \implies \tau + F(d(Sx,Sy)) \leq F(max\{d(Tx,Ty), d(Sx,Tx), d(Sy,Ty)\}) \tag{3}$$

for all $x, y \in X$. If $S(X) \subseteq T(X)$ and $T(X)$ is a complete subspace of X. Then S and T have a unique point of coincidence in X. Moreover if S and T are weakly compatible, then S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point of X. Since $S(X) \subseteq T(X)$, we construct a sequence $\{x_n\}$ in X such that $Sx_n = Tx_{n+1}$ for all $n \geq 0$. If there exists an integer $N \geq 0$ such that $Sx_N = Sx_{N+1}$ then $Tx_{N+1} = Sx_{N+1}$ that is S and T have a point of coincidence. Hence we shall assume that $Sx_n \neq Sx_{n+1}$ for all $n \geq 0$. By (3), we have for all $n \geq 0$

$$\tau + F(d(Sx_n, S_{n+1})) \leq F(max\{d(Tx_n, Tx_{n+1}), d(Sx_n, Tx_n), d(Sx_{n+1}, Tx_{n+1})\})$$

that is

$$\begin{aligned} \tau + F(d(Sx_n, S_{n+1})) &\leq F(max\{d(Sx_{n-1}, Sx_n), d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1})\}) \\ &= F(max\{d(Sx_{n-1}, Sx_n), d(Sx_n, Sx_{n+1})\}) \end{aligned} \tag{4}$$

If $d(Sx_{n-1}, Sx_n) < d(Sx_n, Sx_{n+1})$ for some $n \in \mathbb{N}$, then we have

$$\tau + F(d(Sx_n, S_{n+1})) \leq F(d(Sx_n, Sx_{n+1}),$$

a contradiction.Thus $d(Sx_{n-1}, Sx_n) > d(Sx_n, Sx_{n+1})$ for all $n \in \mathbb{N}$, from (4) we have

$$\tau + F(d(Sx_n, S_{n+1})) \ \leq F(d(Sx_{n-1}, Sx_n)$$

implies

$$F(d(Sx_n, S_{n+1})) \ \leq F(d(Sx_{n-1}, Sx_n) - \tau.$$

Therefore, we have

$$F(d(Sx_n, S_{n+1})) \leq F(d(Sx_{n-1}, Sx_n) - \tau \leq F(d(Sx_{n-2}, Sx_{n-1})) - 2\tau \leq .... \leq F(d(Sx_0, Sx_1)) - n\tau \tag{5}$$

From (5), we get $\lim_{n\to\infty} F(d(Sx_n, Sx_{n+1})) = -\infty$.Thus from $(F2)$, we have $\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0$. From $(F3)$, there exists $k \in (0, 1)$ such that $\lim_{n\to\infty} d^k(Sx_n, Sx_{n+1})F(d(Sx_n, Sx_{n+1})) = 0$. By (5), the following holds for all $n \in \mathbb{N}$

$$d^k(Sx_n, Sx_{n+1})F(d(Sx_n, Sx_{n+1})) - d^k(Sx_n, Sx_{n+1})F(d(Sx_0, Sx_1)) \leq d^k(Sx_n, Sx_{n+1})n\tau \leq 0 \tag{6}$$

Letting $n \to \infty$ in (6), we obtain that

$$lim_{n\to\infty} nd^k(Sx_n, Sx_{n+1}) = 0 \tag{7}$$

From (7), there exists $n_1 \in \mathbb{N}$ such that $nd^k(Sx_n, Sx_{n+1}) \leq 1$ for all $n \geq n_1$. So, we have for all $n \geq n_1$

$$d(Sx_n, Sx_{n+1}) = \frac{1}{n^k}. \tag{8}$$

In order to show that $\{Sx_n\}$ is Cauchy sequence. Consider $m, n \in \mathbb{N}$ such that $m > n > n_1$. Using triangular inequality for metric and from (8), we have

$$\begin{aligned} d(Sx_n, Sx_m) \ &\leq d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, Sx_{n+2}) + ... + d(Sx_{m-1}, Sx_m) \\ &= \sum_{i=n}^{m-1} d(Sx_i, Sx_{i+1}) \leq \sum_{i=n}^{\infty} d(Sx_i, Sx_{i+1}) \leq \sum_{i=n}^{\infty} \frac{1}{n^{\frac{1}{k}}}. \end{aligned}$$

By the convergence of the series $\sum_{i=n}^{\infty} \frac{1}{n^{\frac{1}{k}}}$,as limit n tends to ∞ we get $d(Sx_n, Sx_m) \to 0$. This yields that $\{Sx_n\}$ is a Cauchy sequence in $T(x)$, since $T(X)$ is complete, there exists a $z \in T(X)$ such that $Sx_n \to z$ as $n \to \infty$. Since $z \in T(X)$, we can find $p \in X$ such that $z = Tp$. Now we will prove that $z = Sp$, on contrary suppose $z \neq Sp$. Putting $x = x_{n+1}$ and $y = p$ in (3), we have

$$\tau + F(d(Sx_{n+1}, Sp)) \leq F(max\{d(Tx_{n+1}, Tp), d(Sx_{n+1}, Tx_{n+1}), d(Sp, Tp)\})$$

that is

$$\tau + F(d(Sx_{n+1}, Sp)) \leq F(max\{d(Sx_n, Tp), d(Sx_{n+1}, Sx_n), d(Sp, Tp)\})$$

Taking the limit as $n \to \infty$ and using $z = Tp$, we have

$$\tau + F(d(z, Sp)) \leq F(max\{d(z, z), d(z, z), d(Sp, z)\})$$

$$= F(d(z, Sp)),$$

implies that

$$F(d(z, Sp)) \leq F(d(z, Sp)) - \tau,$$

a contradiction. Thus $z = Sp$. Therefore, we have that $z = Tp = Sp$. Hence p is a coincidence point and z is a point of coincidence of S and T. We next establish that the point of coincidence is unique. For this, assume that there exists another point q in X such that $z_1 = Sq = Tq$ and suppose that $z \neq z_1$. Then for $x = p$ and $y = q$, we have

$$\tau + F(d(Sp, Sq)) \leq F(max\{d(Tp, Tq), d(Sp, Tp), d(Sq, Tq)\})$$

that is

$$\tau + F(d(z, z_1)) \leq F(max\{d(z_1, z_1), d(z, z), d(z, z_1)\})$$

implies that $F(d(z, z_1)) \leq F(d(z, z_1)) - \tau$, a contradiction, which implies that $d(z, z_1) = 0$ i.e., $z = z_1$. Therefore z is the unique point of coincidence of S and T.Now if S and T are weakly compatible, then by Lemma 2.5, z is unique common fixed point of S and T. □

Corollary 3.2. *Let (X, d) be complete metric space and S and T be self map on X. Assume that there exist $F \in F$ and $\tau \in R^+$ such that*

$$d(Sx, Sy) > 0 \implies \tau + F(d(Sx, Sy)) \leq F(d(Tx, Ty)) \tag{9}$$

for all $x, y \in X$. If $S(X) \subseteq T(X)$ and $T(X)$ is a complete subspace of X. Then S and T have unique point of coincidence in X. Moreover if S and T are weakly compatible then S and T have unique common fixed point in X.

Corollary 3.3. *Let (X, d) be a complete metric space and S be a self map on X. Assume that there exist $F \in F$ and $\tau \in \mathbb{R}^+$ such that*

$$d(Sx, Sy) > 0 \implies \tau + F(d(Sx, Sy)) \leq F(d(x, y)) \tag{10}$$

for all $x, y \in X$. Then S have a fixed point in X.

Theorem 3.4. *Let (X, d) be a complete metric space and T be a self map on X also T satisfying following F-contraction condition if $F \in F$ and there exists $\tau > 0$ such that $\forall x, y \in X$ $d(Tx, Ty) > 0$ implies*

$$\tau + F(d(Tx, Ty)) \leq F(k_1 \max\{d(x, y), d(x, Tx)\} + k_2 \max\{d(x, y), d(y, Ty)\} + k_3 \max\{d(x, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\}) \tag{11}$$

for all $x, y \in X$ and $k_1 + k_2 + 2k_3 = 1$. If T or F is continuous, then T has a unique fixed point in X.

Proof. Let x_0 be arbitrary point and define sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for $n \in \mathbb{N}$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$, then $x_{n_0} = Tx_{n_0}$ and so T has a fixed point. Now let $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup 0$ and let $\gamma_n = d(x_n, x_{n+1})$ for $n \in \mathbb{N} \cup 0$, then $\gamma_n > 0$ for all $n \in N \cup 0$.

$$\tau + F(d(Tx_{n-1}, Tx_n)) \leq F(k_1 max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1})\} + k_2 max\{d(x_{n-1}, x_n), d(x_n, Tx_n)\}$$

$$+ k_3max\{d(x_{n-1},Tx_n), \frac{d(x_{n-1},Tx_n)+d(x_n,Tx_{n-1})}{2}\})$$
$$\leq F(k_1max\{d(x_{n-1},x_n),d(x_{n-1},x_n)\} + k_2max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}$$
$$+ k_3max\{d(x_{n-1},x_{n+1}), \frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2}\})$$
$$\leq F(k_1max\{d(x_{n-1},x_n),d(x_{n-1},x_n)\} + k_2max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}$$
$$+ k_3max\{d(x_{n-1},x_n)+d(x_n,x_{n+1}), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\})$$
$$= F(k_1max\{\gamma_{n-1},\gamma_{n-1}\} + k_2max\{\gamma_{n-1},\gamma_n\} + k_3max\{\gamma_{n-1}+\gamma_n, \frac{\gamma_{n-1}+\gamma_n}{2}\}) \tag{12}$$
$$= F(k_1\gamma_{n-1} + k_2max\{\gamma_{n-1},\gamma_n\} + k_3(\gamma_{n-1}+\gamma_n)). \tag{13}$$

If $\gamma_n \geq \gamma_{n-1}$ for some $n \in \mathbb{N}$, then from (12) we have $F(\gamma_n) \leq F(k_1\gamma_{n-1} + k_2\gamma_n + 2k_3\gamma_n) - \tau$ this implies that $\gamma_n \leq \frac{k_1}{1-k_2-2k_3}\gamma_{n-1} = \gamma_{n-1}$. Since $k_1+k_2+2k_3 = 1$ So $\gamma_n \leq \gamma_{n-1}$, a contradiction. Thus $\gamma_n < \gamma_{n-1}$ for all $n \in \mathbb{N}$ and we from (12) we have

$$\tau + F(\gamma_n) \leq F((k_1+k_2+2k_3)\gamma_{n-1})$$
$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \forall n \in \mathbb{N}.$$

This implies

$$F(\gamma_n) \leq F(\gamma_{n-1}) - \tau \leq F(\gamma_{n-1}) - 2\tau..... \leq F(\gamma_0) - n\tau, \forall n \in \mathbb{N}. \tag{14}$$

From (13) $lim_{n\to\infty}F(\gamma_n) = -\infty$. Thus (F2), we have $lim_{n\to\infty}\gamma_n = 0$. From (F3) there exists $k \in (0,1)$ such that $lim_{n\to\infty}\gamma_n^kF(\gamma_n) = 0$. By (3) the following hold $\forall\ n \in \mathbb{N}$

$$\gamma_n^kF(\gamma_n) - \gamma_n^kF(\gamma_0) \leq -\gamma_n^kn\tau \leq 0. \tag{15}$$

Letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty} n\gamma_n^k = 0. \tag{16}$$

From (15) there exists $n_1 \in N$ such that $n\gamma_n^k \leq 1\ \forall\ n \geq n_1$. So, we have for all $n \geq n_1$

$$\gamma_n \leq \frac{1}{n^{\frac{1}{k}}} \tag{17}$$

In order to show that $\{x_n\}$ is Cauchy sequence.Consider $m,n \in \mathbb{N}$ such that $m > n > n_1$.Using triangular inequality and from (16) we have

$$d(x_n,x_m) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + ... + d(x_{m-1},x_m)$$
$$= \gamma_n + \gamma_{n+1} + ... + \gamma_{m-1}$$
$$= \sum_{i=n}^{m-1}\gamma_i \leq \sum_{i=n}^{\infty}\gamma_i \leq \sum_{i=n}^{\infty}\frac{1}{n^{\frac{1}{k}}}$$

By convergence of series $\sum_{i=n}^{\infty}\frac{1}{n^{\frac{1}{k}}}$ as $n \to \infty$ we get $d(x_n,x_m) \to 0$.This means $\{x_n\}$ is Cauchy in (X,d).Since (X,d) is complete, so $\{x_n\}$ is converges to some $x \in X$ that is $lim_{n\to\infty}x_n = z$. Now if T is continuous, then we have

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty}) = Tz. \tag{18}$$

So, z is a fixed point of T. Now, Suppose that F is continuous we claim that $z = Tz$. Assume the contrary that is $z \neq Tz$. In this case, there exists an $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $d(Tx_{n_k}, Tz) > 0$ for all $n_k \geq n_0$.(otherwise there exists $n_1 \in \mathbb{N}$ such that $x_n = Tz$ for all $n \geq n_1$, a contraction).Since $d(Tx_{n_k}, Tz) > 0$ for all $n_k \geq n_0$, then from (11), we have

$$\tau + F(d(Tx_{n_k}, Tz)) \leq F(k_1 \max\{d(x_{n_k}, z), d(x_{n_k}, Tx_{n_k})\} + k_2 \max\{d(x_{n_k}, z), d(z, Tz)\}$$
$$+ k_3 \max\{d(x_{n_k}, Tz), \frac{d(x_{n_k}, Tz) + d(z, Tx_{n_k})}{2}\}).$$

Taking $k \to \infty$ and using continuity of F we have

$$\tau + F(d(z, Tz)) \leq F(k_1 \max\{d(z, z), d(z, Tz)\} + k_2 \max\{d(z, z), d(z, Tz)\} + k_3 \max\{d(z, Tz), \frac{d(z, Tz) + d(z, Tz)}{2}\})$$
$$= F((k_1 + k_2 + k_3)d(z, Tz)).$$

This implies $\tau + F(d(z, Tz)) \leq F(d(z, Tz))$, a contradiction.Hence $z = Tz$. Now, we prove the uniqueness of the fixed point. Assume that $z^* \in X$ be another fixed point of T different from z. This means that $d(z, z^*) > 0$. Taking $x = z$ and $y = z^*$ in (11), we have

$$\tau + F(d(Tz, Tz^*)) \leq F(k_1 \max\{d(z, z^*), d(z, Tz)\} + k_2 \max\{d(z, z^*), d(z^*, Tz^*)\} + k_3 \max\{d(z, Tz^*), \frac{d(z, Tz^*) + d(z^*, Tz)}{2}\})$$

this implies that $\tau + F(d(z, z^*)) \leq F((k_1 + k_2 + k_3)d(z, z^*))$, which is a contradiction, since $k_1 + k_2 + k_3 < 1$ and hence $z = z^*$. Hence z is unique fixed point of T. □

Corollary 3.5. *Let (X, d) be a complete metric space and T be a self map on X.Assume that there exist $F \in F$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(d(Tx, Ty)) \leq F(k_1 d(x, y) + k_2 d(x, Tx) + k_3 d(y, Ty) + k_4 d(x, Ty) + k_5 d(y, Tx)),$$

for all $x, y \in X$, $Tx \neq Ty$, where $k_1 + k_2 + k_3 + 2k_4 = 1$, $k_3 \neq 1$ and $L \geq 0$. Then T has a fixed point.Moreover, if $k_1 + k_4 + k_5 \leq 1$, then the fixed point of T is unique.

4. Applications

In this section, we present an application on dynamic programming. The existence of solution of functional equations and system of functional equations arising in dynamic programming which have been studied by using various fixed point theorems.We will prove the extension of a common solution for class of functional equations. Here we assume that U and V are Banach spaces, $W \subseteq U$ is a state space and $D \subseteq V$ is a decision space. In particular, we are interested in solving the following two functional equations arising in dynamic programming:

$$r(x) = \sup_{y \in D}\{f(x, y) + G(x, y, r(\tau(x, y)))\}, x \in W \tag{19}$$

$$r(x) = \sup_{y \in D}\{f(x, y) + Q(x, y, r(\tau(x, y)))\}, x \in W \tag{20}$$

Where $\tau : W \times D \to W$, $f : W \times D \to \mathbb{R}$, and $G, Q : W \times D \times \mathbb{R} \to \mathbb{R}$.Here we study the existence and uniqueness of $h^* \in B(W)$ a common solution of the functional equations (18) & (19). Let $B(W)$ denote the set of all bounded real-valued functions on W. We know that $B(W)$ endowed with metric $d(h,k) = sup|h(x) - k(x)|$, $h, k \in B(W)$ is a complete metric space. Consider $S, T : B(W) \to B(W)$ such that

$$S(h)(x) = \sup_{y \in D}\{f(x,y) + G(x,y,h(\tau(x,y)))\}, x \in W \tag{21}$$

$$T(h)(x) = \sup_{y \in D}\{f(x,y) + Q(x,y,h(\tau(x,y)))\}, x \in W \tag{22}$$

It is clear that if f, G and Q are bounded, then the operation S and T are well defined.

Theorem 4.1. *Suppose that there exists $k \in (0,1)$ such that for all $(x,y) \in W \times D$ and $h_1, h_2 \in B(W)$, we have*

$$|G(x,y,h(\tau(x,y))) - Q(x,y,h_2(\tau(x,y)))| \leq kM(h_1,h_2), \tag{23}$$

where $M(h_1,h_2) = max\{d(Th_1,Th_2), d(Sh_1,Th_1), d(Sh_2,Th_2)\}$. Then S and T have a unique common fixed point in $B(W)$.

Proof. Let $\lambda > 0$ be arbitrary positive real number $x \in W$, $h_1, h_2 \in B(W)$. Then by (20) and (21), there exist $y_1, y_2 \in D$ such that

$$S(h_1)(x) < f(x,y_1) + G(x,y_1,h_1(\tau(x,y_1))) + \lambda \tag{24}$$

$$T(h_2)(x) < f(x,y_2) + Q(x,y_2,h_2(\tau(x,y_2))) + \lambda \tag{25}$$

$$S(h_1)(x) \geq f(x,y_2) + G(x,y_2,h_1(\tau(x,y_2))) \tag{26}$$

$$T(h_2)(x) \geq f(x,y_1) + Q(x,y_1,h_2(\tau(x,y_1)))\} \tag{27}$$

Form (23) and (26), it follows that

$$\begin{aligned} S(h_1)(x) - T(h_2)(x) &\leq G(x,y_1,h_1(\tau(x,y_1))) - Q(x,y_1,h_2(\tau(x,y_1))) + \lambda \\ &\leq |G(x,y_1,h_1(\tau(x,y_1))) - Q(x,y_1,h_2(\tau(x,y_1)))| + \lambda \ \leq kM(h_1,h_2) + \lambda \end{aligned}$$

Similarly $T(h_2)(x) - S(h_1)(x) \leq kM(h_1,h_2) + \lambda$. Consequently

$$|S(h_1)(x) - T(h_2)(x)| \leq kM(h_1,h_2) + \lambda \tag{28}$$

Since the inequality (27) is true for any $x \in W$, we get

$$d(S(h_1), T(h_2)) \leq kM(h_1,h_2) + \lambda \tag{29}$$

Finally, λ is arbitrary, so

$$d(S(h_1), T(h_2)) \leq kM(h_1,h_2) \tag{30}$$

that is (29) hold by taking $\tau = -ln(k)$ and $F(t) = ln(t)$. Applying the Theorem 3.1, the mappings S and T have a unique common fixed point that is the functional equation (18) and (19) have unique common solution $h^* \in B(W)$. ☐

Competing Interests

The authors declare that they have no competing interests.

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References

[1] M. Abbas and G. Jungck, *Common fixed point results for noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl., 341(2008), 416-420.

[2] R. P. Agarwal, D. ORegan and N. Shahzad, *Fixed point theorems for generalized contractive maps of Mei-Keeler type*, Mathematische Nachrichten, 276(2004), 312.

[3] . R. Bellman and E.S. Lee, *Functional equations in dynamic programming*, Aequationes Math., 17(1978), 118.

[4] V. Berinde, *On the approximation of fixed points of weak contractive mappings*, Carpathian Journal of Mathematics, 19(1)(2003),722.

[5] V. Berinde, *Iterative Approximation of Fixed Points*, Springer-Verlag, Berlin Heidelberg, (2007).

[6] . V. Berinde and F. Vetro, *Common fixed points of mappings satisfying implicit contractive conditions*, Fixed Point Theory Appl., 2012(2012).

[7] D. W. Boyd and J. S. W. Wong, *On nonlinear contractions*, Proceedings of the American Mathematical Society, 20(1969), 458464.

[8] Lj. B. Ciric, *A generalization of Banachs contraction principle*, Proceedings of the American Mathematical Society, 45(1974), 267273.

[9] Lj. B. Ciric, *Fixed Point Theory, Contraction Mapping Principle*, Faculty of Mechanical Engineearing, University of Belgrade, Beograd, (2003).

[10] M. Cosentino and P. Vetro, *Fixed point results for F-contractive mappings of Hardy-Rogers-Type*, Filomat, 28(4)(2014), 715-722.

[11] B. S. Choudhury and N. Metiya, *The point of coincidence and common fixed point for a pair of mappings in cone metric spaces*, Computers and Mathematics with Applications, 60(2010), 1686-1695.

[12] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math, Sci., 9(1986), 771-779.

[13] G. Jungck, *Compatible mappings and common fixed points (2)*, Int. J. Math. Math, Sci., 11(1988), 285-288.

[14] . H. Piri and P. Kumam, *Some fixed point theorems concerning F-contraction in complete metric spaces*, Fixed point Theory Appl., 2014(2014), 11 pages.

[15] N. A. Secelean, *Iterated function systems consisting of F-contractions*, Fixed Point Theory Appl., 2013(2013).

[16] D. Wardowski, *Fixed point theory of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., 2012(2012).