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Uniqueness and Stability Results of Fractional Neutral Differential Equations with Infinite Delay

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Abstract: In this paper, some uniqueness and Ulam-Hyers stability results for neutral functional differential equations with fractional order and infinite delay are proved. The problem that is discussed includes the Caputo fractional derivative operator. The technique used is a variety of tools of fractional calculus properties and Banach's fixed point theorem. An example of the obtained results is given.

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1. Introduction

Fractional differential equations have grown as a new area of applied mathematics and many applications, such as physics, mechanics, chemistry, engineering, etc. For more details, see the monographs of Kilbas et al. [18], Miller and Ross [21], Podlubny [22], and Samko et al. [23]. There is also a lot of attention currently about these equations (see, as an instance [7, 12–14, 19, 20, 25] and the references cited therein. A few attractivity results for fractional functional differential equations and nonlinear functional integral equations are acquired by means of the use of the fixed point theory, see [2–6, 9, 11] and references therein. The stability of functional equations was first posed by Ulam [24]. Afterward, this kind of stability has been developed and expanded as a motivating field of research. The concept of stability of functional equations emerges when replacing inequality instead of equality which constitutes as a perturbation of the functional equation.

The recent development of stability results for fractional differential equations can be found in [1, 8, 10, 15] and the references referred to therein. The purpose of our paper is based on the Banach contraction principle to study the uniqueness and Ulam-Hyers stability of solutions for the fractional neutral functional differential equation with infinite delay

$$^{c}D^{\alpha}_{0^+}[u(t) - g(t, u_t)] = f(t, u_t), \qquad t \in [0, b] \tag{1}$$

$$u_0 = \phi \in \mathcal{B}, \tag{2}$$

where $0 < \alpha < 1$, $^{c}D^{\alpha}$ denotes the fractional derivative of order α in the sense of Caputo, $f, g : [0, b] \times \mathcal{B} \longrightarrow \mathbb{R}(b > 0)$ are given functions satisfying some assumptions that will be specified later, $\mathcal{B}$ the phase space of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, which will be specified in Section 2, and $u_t : (-\infty, 0] \to \mathbb{R}$ defined by $u_t(s) = u(t + s)$, for $-\infty < s \leq 0$.

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The rest of this paper is organized as follows, In Section 2, we provide some definitions, preliminary facts, and list the hypotheses as a good way to be used in this paper. Section 3 is devoted to the uniqueness of solutions of (1)–(2). The Ulam-Hyers stability of solution to such equations in the space $C([a,b])$ is discussed in Section 4. Finally, an example to illustrate the results is in Section 5.

2. Preliminaries

In this section, we introduce some definitions and preliminary facts which are used consistently thereafter throughout this paper. Let $C([0,b],\mathbb{R})$ be the space of all continuous real functions defined on $[0,b]$ and $C^1[0,b]$ we denote the set of a continuously differentable functions. Moreover for $u(t)\in C([0,b],\mathbb{R})$, we define $\|u\|_\infty = \sup\{|u(t)| : t\in[0,b]\}$ and for any function u defined on $(-\infty,b]$ and any $t\in[0,b]$, we denote by u_t the element of $\mathcal{B}$ defined by

$$u_t(s) = u(t+s), \text{ for } -\infty < s \le 0.$$

We consider the following space

$$\Lambda = \{u : (-\infty,b] \to \mathbb{R} :\ u\mid_{(-\infty,0]}\in \mathcal{B},\ u\mid_{[0,b]} \text{ is a continuous on } [0,b]\},$$

where $u\mid_{[0,b]}$is the restriction of u to $[0,b]$.

Definition 2.1 ([18]). *The left-sided Riemann-Liouville fractional integral operator of order $\alpha>0$ of the function $u : [0,b]\to\mathbb{R}$ is defined by*

$$I^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds, \quad t>0$$

where $\Gamma(.)$ is the Euler gamma function, note that $I^{\alpha}_{0+}u$ is exists for all $\alpha>0$, when $u\in C[0,b]\cap C^1[0,b]$.

Definition 2.2 ([18]). *The left-sided Riemann-Liouville fractional derivative of order α $(n-1<\alpha<n\in\mathbb{N})$ of the function $u\in C([0,b],\mathbb{R})$ at the point t is characterized as*

$$D^{\alpha}_{0+}u(t) = D^n I^{n-\alpha}_{0+}u(t), \quad t>0, D^n = \frac{d^n}{dt^n}.$$

Definition 2.3 ([26]). *The left-sided Caputo fractional derivative of order α $(n-1<\alpha<n\in\mathbb{N})$ of the function $u:[0,b]\to\mathbb{R}$ at the point t is determined as*

$${}^cD^{\alpha}_{0+}u(t) = D^{\alpha}_{0+}\left(u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)}(0)}{k!}t^k\right).$$

In particular, $0<\alpha<1$, we have ${}^cD^{\alpha}_{0+}u(t) = D^{\alpha}_{0+}(u(t)-u(0))$. Moreover, if ${}^cD^{\alpha}_{0+}u(t)\in C^1[0,b]$, then

$$I^{\alpha}_{0+}\ {}^cD^{\alpha}_{0+}u(t) = u(t)-u(0).$$

If $u(t)\in C^1[0,b]$, and $0<\alpha<1$, we have

$${}^cD^{\alpha}_{0+}u(t) = I^{1-\alpha}_{0+}\frac{d}{dt}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}u'(s)ds, \quad t>0.$$

Obviously, the Caputo derivative of a constant is equal to zero.

Lemma 2.4. *If $0<\alpha<1$, and $u(t)\in C^1[0,b]$. Then $D^{\alpha}_{0+}I^{\alpha}_{0+}u(t) = u(t)$, for $t>0$.*

Definition 2.5. *A function $u : (-\infty, b] \to \mathbb{R}$ is said to be a solution of (1)–(2) if u satisfies the equation ${}^cD^{\alpha}_{0+}[u(t) - g(t, u_t)] = f(t, u_t)$, $t \in [0, b]$, with initial condition $u_0 = \phi$, $u \mid_{[0,b]} \in C[0, b] \cap C^1[0, b]$ and $\frac{\partial g}{\partial t}$ is exists.*

In this paper, we suppose that the phase space $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is a seminormed linear of functions mapping $(-\infty, 0]$ into $\mathbb{R}$, and satisfying the following essential axioms which were presented by Hale and Kato in [16] and discussed in detail by Hino et al. in [17]:

(H1) If $u : (-\infty, b] \to \mathbb{R}$, such that $u \in C([0, b], \mathbb{R})$ and $u_0 \in \mathcal{B}$, then for every $t \in [0, b]$ the following statements hold:

(i). $u_t \in \mathcal{B}$;

(ii). $|u(t)| \leq H \|u_t\|_{\mathcal{B}}$ for some $H > 0$ which is equivalent to $\|\phi(0)\| \leq H \|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$;

(iii). $\|u_t\|_{\mathcal{B}} \leq K(t) \sup_{0 \leq s \leq t} |u(s)| + M(t) \|u_0\|_{\mathcal{B}}$; where $K, M : [0, +\infty) \to [0, +\infty)$ with K continuous and M locally bounded, such that K, M are independent of $u(.)$. Denote $K_b = \sup\{K(t) : t \in [0, b]\}$ and $M_b = \sup\{M(t) : t \in [0, b]\}$.

(H2) For the function $u(.)$ in (H1), u_t is a $\mathcal{B}-$ valued continuous function on $[0, b]$.

(H3) The space $\mathcal{B}$ is complete.

3. Main results

In this section, we prove the uniqueness of solution to problem (1)–(2). Firstly, we need the following auxiliary lemma and we also introduce the following hypotheses:

(A$_0$) **(i)** $f(t, u_t)$ is Lebesgue measurable with respect to t on $[0, b]$.

(ii) $f(t, \phi)$ is is continuous with respect to ϕ on $\mathcal{B}$.

(A$_1$) There exists $L_f > 0$ such that for any $u_t, v_t \in \mathcal{B}$,

$$|f(t, u_t) - f(t, v_t)| \leq L_f \|u_t - v_t\|_{\mathcal{B}}, \qquad t \in [0, b].$$

(A$_2$) g is continuous and for any $u_t, v_t \in \mathcal{B}$, $t \in [0, b]$,

$$|g(t, u_t) - g(t, v_t)| \leq L_g \|u_t - v_t\|_{\mathcal{B}}, \qquad L_g \in (0, 1).$$

Lemma 3.1. *A function $u \in \Lambda$ is a solution of the fractional integral equation*

$$u(t) = \phi(0) - g(0, \phi) + g(t, u_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s, u_s) ds, \quad t \in [0, b], \tag{3}$$

and $u(t) = \phi(t)$, $t \in (-\infty, 0]$, if and only if u is a solution of the fractional differential equation (1)–(2), provided that the integral in (3) exists.

Next, we prove the uniqueness result by means of the Banach contraction principle theorem.

Theorem 3.2. *Assume that the conditions $(A_0) - (A_2)$ hold. Then the problem (1)-(2) has a unique solution on $(-\infty, b]$, provided that*

$$K_b \left(L_g + \frac{b^{\alpha}}{\Gamma(\alpha + 1)} L_f \right) < 1. \tag{4}$$

Proof. According to Lemma 3.1, and transform the problem (1)-(2) into a fixed point problem, we can define the operator $Q:\Lambda\to\Lambda$ by

$$(Qu)(t)=\begin{cases}\phi(0)-g(0,\phi)+g(t,u_t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u_s)ds, & t\in[0,b],\\ \phi(t), & t\in(-\infty,0].\end{cases}$$

For $\phi\in\mathcal{B}$, let $w:(-\infty,b]\to\mathbb{R}$ be the function defined by

$$w(t)=\begin{cases}\phi(0), & t\in[0,b],\\ \phi(t), & t\in(-\infty,0].\end{cases} \tag{5}$$

Then, we get $w_0=\phi$. For each function $v\in C([0,b],\mathbb{R})$ with $v(0)=0$ we denote by $\overline{v}:(-\infty,b]\to\mathbb{R}$ the function defined by

$$\overline{v}(t)=\begin{cases}v(t), & t\in[0,b],\\ 0, & t\in(-\infty,0].\end{cases}$$

If $u(.)$ satisfies the integral equation

$$u(t)=\phi(0)-g(0,\phi)+g(t,u_t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u_s)ds,\quad t\in[0,b],$$

with $u_0=\phi$, then we can write $u(.)$ as follows $u(t)=w(t)+\overline{v}(t)$, $t\in(-\infty,b]$, which implies $u_t=w_t+\overline{v}_t$, for every $t\in[0,b]$ and the function $v(.)$ satisfies

$$v(t)=-g(0,\phi)+g(t,w_t+\overline{v}_t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,w_s+\overline{v}_s)ds,\quad t\in[0,b]. \tag{6}$$

with $\overline{v}_0=0$. Set $\Lambda_0=\{v\in\Lambda,\ v_0=0\}$. For $v\in\Lambda_0$, and let $\|.\|_{\Lambda_0}$ be seminorm in Λ_0 defined by

$$\|v\|_{\Lambda_0}=\|v_0\|_{\mathcal{B}}+\|v\|_C=\sup\{|v(t)|:t\in[0,b]\}. \tag{7}$$

Since $(\Lambda_0,\|v\|_{\Lambda_0})$ is a Banach space for $v\in\Lambda_0$, we shall show that the operator $P:\Lambda_0\to\Lambda_0$ defined by

$$(Pv)(t)=-g(0,\phi)+g(t,w_t+\overline{v}_t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,w_s+\overline{v}_s)ds,\quad t\in[0,b], \tag{8}$$

is a contraction map. It's far clear that the operator Q has a unique fixed point iff P has a unique fixed point. So we turn to prove that P has a unique fixed point. In order that, consider $v,v^*\in\Lambda_0$ and for each $t\in[0,b]$, we have

$$\begin{aligned}|(Pv)(t)-(Pv^*)(t)|&\le\left|g(t,w_t+\overline{v}_t)-g(t,w_t+\overline{v^*}_t)\right|+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left|f(s,w_s+\overline{v}_s)-f(s,w_s+\overline{v^*}_s)\right|ds\\&\le L_g\left\|\overline{v}_t-\overline{v^*}_t\right\|_{\mathcal{B}}+\frac{1}{\Gamma(\alpha)}L_f\left\|\overline{v}_t-\overline{v^*}_t\right\|_{\mathcal{B}}\int_0^t(t-s)^{\alpha-1}ds,\end{aligned}$$

which in according to (H1)-(iii) and the definition of the norm on Λ_0 yields

$$\begin{aligned}|(Pv)(t)-(Pv^*)(t)|&\le L_gK_b\|v-v^*\|_{\Lambda_0}+\frac{1}{\Gamma(\alpha)}L_fK_b\|v-v^*\|_{\Lambda_0}\int_0^t(t-s)^{\alpha-1}ds\\&\le K_b\left(L_g+\frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)\|v-v^*\|_{\Lambda_0},\end{aligned}$$

Consequently,

$$\|Pv-Pv^*\|_{\Lambda_0}\le K_b\left(L_g+\frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)\|v-v^*\|_{\Lambda_0}. \tag{9}$$

The inequalities (4) and (9) imply that the operator P is a contraction mapping. An application of Banach contraction principle shows that v is the fixed point of P, which is just the unique solution to the integral equation (6) on $[0,b]$. Set $u=w+\overline{v}$, then u is the unique solution to the fractional differential equation (1)-(2) on $(-\infty,b]$. □

4. Ulam-Hyers Stability

In this section, we discuss the Ulam-Hyers stability of solution to the problem (1)-(2).

Definition 4.1. *The problem (1)-(2) has the Ulam-Hyers stability if there exists a real number* $\lambda > 0$ *with the following property:*

For every $\epsilon > 0$, $u \in C[0,b]$, if

$$|{}^cD^{\alpha}_{0+}[u(t) - g(t,u_t)] - f(t,u_t)| \leq \epsilon, \tag{10}$$

then there exists some $\overline{u} \in C[0,b]$ satisfying

$$ {}^cD^{\alpha}_{0+}[\overline{u}(t) - g(t,\overline{u}_t)] = f(t,\overline{u}_t), \quad t \in [0,b], \tag{11}$$

$$\overline{u}_0 = \phi, \tag{12}$$

such that $|u(t) - \overline{u}(t)| \leq \lambda\epsilon, \quad t \in [0,b]$.

Definition 4.2. *The problem (1)-(2) is generalized Ulam-Hyers stable if there exists* $\psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ *with* $\psi(0) = 0$ *such that for each* $\epsilon > 0$ *and for each solution* $u \in C[0,b]$*, of inequality (10) there exists a solution* $\overline{u} \in C[0,b]$ *of problem (11)-(12) with*

$$|u(t) - \overline{u}(t)| \leq \psi(\epsilon), \quad t \in [0,b].$$

Theorem 4.3. *Let the assumptions of Theorem 3.2 hold. If* $\left(L_g + \frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)K_b < 1$*. Then the problem (1)-(2) has the Ulam-Hyers stability in* Λ.

Proof. Let $u \in C[0,b]$ be a solution of (1)-(2). Let us denote by $\overline{u} \in C[0,b]$ the unique solution of (11)-(12). By Lemma 3.1 and Theorem 3.2, we know that for every $\phi \in \mathcal{B}$, the equation (11) has solution is given by

$$\overline{u}(t) = \phi(0) - g(0,\phi) + g(t,\overline{u}_t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\overline{u}_s)ds, \quad t \in [0,b]. \tag{13}$$

Also by (10), we have

$$\left|u(t) - \phi(0) + g(0,\phi) - g(t,u_t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u_s)ds\right| \leq \epsilon. \tag{14}$$

From (13) and (14), we obtain

$$\begin{aligned}
|u(t) - \overline{u}(t)| &= \left|u(t) - \phi(0) + g(0,\phi) - g(t,\overline{u}_t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\overline{u}_s)ds\right| \\
&\leq \left|u(t) - \phi(0) + g(0,\phi) - g(t,u_t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u_s)ds\right| \\
&\quad + |g(t,u_t) - g(t,\overline{u}_t)| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} |f(s,u_s) - f(s,\overline{u}_s)|\, ds \\
&\leq \epsilon + \left(L_g + \frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)\|u_t - \overline{u}_t\|_{\mathcal{B}}.
\end{aligned}$$

Since

$$\begin{aligned}
\|u_t - \overline{u}_t\|_{\mathcal{B}} &\leq K(t)\sup_{0\leq\tau\leq t}|u(\tau) - \overline{u}(\tau)| + M(t)\|u_0 - \overline{u}_0\|_{\mathcal{B}} \\
&\leq K_b \sup_{0\leq\tau\leq t}|u(\tau) - \overline{u}(\tau)|,
\end{aligned}$$

we get

$$|u(t) - \overline{u}(t)| \leq \epsilon + \left(L_g + \frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)K_b \sup_{0\leq\tau\leq t}|u(\tau) - \overline{u}(\tau)|$$

From the fact that $\left(L_g + \frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)K_b < 1$ and take $\lambda = \frac{1}{1-\left(L_g + \frac{b^\alpha}{\Gamma(\alpha+1)}L_f\right)K_b}$, it follows that $|u(t) - \overline{u}(t)| \leq \lambda\epsilon$. □

Theorem 4.4. *Let the hypotheses of Theorem 4.3 hold. If there exists $\psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\psi(0) = 0$. Then the problem (1)-(2) has generalized Ulam-Hyers stability.*

Proof. In a manner similar to above Theorem 4.3, with putting $\psi(\epsilon) = \lambda\epsilon$ and $\psi(0) = 0$, we get $|u(t) - \overline{u}(t)| \leq \psi(\epsilon)$. □

5. An Example

Firstly, Let ρ be a positive real constant and we define the functional space $\mathbb{B}_\rho$ by

$$\mathbb{B}_\rho = \{u \in C((-\infty, 0] \to \mathbb{R} : \lim_{s \to -\infty} e^{\rho s} u(s) \text{ exist in } \mathbb{R}\},$$

endowed with the following norm $\|u\|_\rho = \sup \{e^{\rho s} |u(s)| : -\infty < s \leq 0\}$. Then $\mathbb{B}_\rho$ satisfies axioms (H1), (H2) and (H3) with $K(t) = M(t) = 1$ and $H = 1$. (see [7]). Next, we consider the fractional neutral functional differential equation

$$^cD_0^\alpha \left(u(t) - \frac{e^{-\rho t - t} \|u_t\|}{2(e^t + e^{-t})} \right) = \frac{e^{-\rho t}}{(3 + e^{-t})} \frac{\|u_t\|}{1 + \|u_t\|}, \quad t \in [0, b], \tag{15}$$

with initial condition

$$u(t) = \varphi(t), \quad t \in (-\infty, 0], \tag{16}$$

where $\varphi \in \mathbb{B}_\rho$ and $\rho > 0$. Next, we applying Theorem 3.2. Consider $f(t, x) = \frac{e^{-\rho t}}{(3+e^{-t})} \frac{x}{1+x}$ and $g(t, x) = \frac{e^{-\rho t - t} x}{2(e^t + e^{-t})}$, for $(t, x) \in [0, b] \times \mathbb{B}_\rho$. Let $x, y \in \mathbb{B}_\rho$, and $t \in [0, b]$. Then

$$\begin{aligned}
|f(t, x) - f(t, y)| &= \frac{e^{-\rho t}}{(3 + e^{-t})} \left| \frac{x}{1 + x} - \frac{y}{1 + y} \right| \\
&= \frac{e^{-\rho t}}{(3 + e^{-t})} \left| \frac{x - y}{(1 + x)(1 + y)} \right| \\
&\leq \frac{1}{(3 + \frac{1}{e^t})} e^{-\rho t} |x - y| \\
&\leq \frac{1}{3} \|x - y\|_\rho .
\end{aligned}$$

and

$$\begin{aligned}
|g(t, x) - g(t, y)| &= \left| \frac{e^{-\rho t - t} x}{2(e^t + e^{-t})} - \frac{e^{-\rho t - t} y}{2(e^t + e^{-t})} \right| \\
&= \frac{e^{-\rho t}}{2(e^{2t} + 1)} |x - y| \\
&\leq \frac{1}{(2e^{2t} + 2)} \|x - y\|_\rho \\
&\leq \frac{1}{2} \|x - y\|_\rho .
\end{aligned}$$

Hence the conditions $(A_0) - (A_2)$ hold with $L_f = \frac{1}{3}$ and $L_g = \frac{1}{2}$. It can be checked that condition (4) is satisfied with $K_b = 1$, $\alpha = \frac{1}{4}$ and $b = 1$. In fact, $K_b \left(L_g + \frac{b^\alpha}{\Gamma(\alpha+1)} L_f \right) = \left(\frac{1}{2} + \frac{1}{\Gamma(\frac{1}{4}+1)} \frac{1}{3} \right) \simeq 0.87 < 1$. According to Theorem 3.2, then the IVP (15)-(16) has a unique solution on $(-\infty, 1]$ and it has the Ulam-Hyers stability.

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