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PD-Divisor Cordial Labeling of Graphs

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Abstract: Let $G = (V(G), E(G))$ be a simple, finite and undirected graph of order n. Given a bijection $f : V(G) \to \{1, 2, ..., |V(G)|\}$, we associate two integers $P = f(u)f(v)$ and $D = |f(u) - f(v)|$ with every edge uv in $E(G)$. The labeling f induces on edge labeling $f' : E(G) \to \{0, 1\}$ such that for any edge uv in $E(G)$, $f'(uv) = 1$ if $D \mid P$ and $f'(uv) = 0$ if $D \nmid P$. Let $e_{f'}(i)$ be the number of edges labeled with $i \in \{0, 1\}$. We say f is an PD-divisor labeling if $f'(uv) = 1$ for all $uv \in E(G)$. Moreover, G is PD-divisor if it admits an PD-divisor labeling. We say f is an PD-divisor cordial labeling if $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Moreover, G is PD-divisor cordial if it admits an PD-divisor cordial labeling. In this paper, we are dealing in PD-divisor cordial labeling of some standard graphs.

MSC: 05C78.

Keywords: Divisor cordial labeling, PD-divisor cordial labeling, PD-divisor cordial graph.

1. Introduction

Let $G = (V(G), E(G))$ (or $G = (V, E)$) be a simple, finite and undirected graph of order $|V(G)| = n$ and size $|E(G)| = m$. All notations not defined in this paper can be found in [4].

Definition 1.1 ([2]). *Let a and b be two integers. If a divides b means that there is a positive integer k such that $b = ka$. It is denoted by $a \mid b$. If a does not divide b , then we denote $a \nmid b$.*

Definition 1.2 ([1]). *Let $G = (V, E)$ be a graph. A mapping $f : V(G) \to \{0, 1\}$ is called binary vertex labeling of G and $f(v)$ is called the label of the vertex v of G under f. For an edge $e = uv$, the induced edge labeling $f' : E(G) \to \{0, 1\}$ is given by $f'(e) = |f(u) - f(v)|$. Let $v_f(0), v_f(1)$ be the number of vertices of G having labels 0 and 1 respectively under f and $e_{f'}(0), e_{f'}(1)$ be the number of edges having labels 0 and 1 respectively under f'. This labeling is called cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_{f'}(0) - e_{f'}(1)| \leq 1$. A graph G is cordial if it admits cordial labeling.*

Definition 1.3 ([9]). *A bijection $f : V \to \{1, 2, ..., n\}$ induces an edge labeling $f' : E \to \{0, 1\}$ such that for any edge uv in G, $f'(uv) = 1$ if $gcd(f(u), f(v)) = 1$, and $f'(uv) = 0$ otherwise. We say that f is a prime cordial labeling if $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Moreover, G is prime cordial if it admits a prime cordial labeling.*

Definition 1.4 ([10]). *Let $G = (V, E)$ be a simple graph and $f : V \to \{1, 2, ..., n\}$ be a bijection. For each edge uv, assign the label 1 if either $f(u) \mid f(v)$ or $f(v) \mid f(u)$ and the label 0 otherwise. We say that f is a divisor cordial labeling if $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Moreover, G is divisor cordial if it admits a divisor cordial labeling.*

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Given a bijection $f : V \to \{1, 2, ..., n\}$, we associate two integers $S = f(u) + f(v)$ and $D = |f(u) - f(v)|$ with every edge uv in E.

Definition 1.5 ([7]). *A bijection $f : V \to \{1, 2, ..., n\}$ induces an edge labeling $f' : E \to \{0, 1\}$ such that for any edge uv in G, $f'(uv) = 1$ if $gcd(S, D) = 1$, and $f'(uv) = 0$ otherwise. We say f is an SD-prime labeling if $f'(uv) = 1$ for all $uv \in E$. Moreover, G is SD-prime if it admits an SD-prime labeling.*

Definition 1.6 ([6]). *A bijection $f : V \to \{1, 2, ..., n\}$ induces an edge labeling $f' : E \to \{0, 1\}$ such that for any edge uv in G, $f'(uv) = 1$ if $gcd(S, D) = 1$, and $f'(uv) = 0$ otherwise. The labeling f is called an SD-prime cordial labeling if $|e_{f'}(0) - e_{f'}(1)| \leq 1$. We say that G is SD-prime cordial if it admits an SD-prime cordial labeling.*

Definition 1.7 ([5]). *Let $G = (V(G), E(G))$ be a simple graph and a bijection $f : V(G) \to \{1, 2, 3, ..., |V(G)|\}$ induces an edge labeling $f' : E(G) \to \{0, 1\}$ such that for any edge uv in $E(G)$, $f'(uv) = 1$ if $D \mid S$ and $f'(uv) = 0$ if $D \nmid S$. We say f is an SD-divisor labeling if $f'(uv) = 1$ for all $uv \in E(G)$. Moreover, G is SD-divisor if it admits an SD-divisor labeling.*

Definition 1.8 ([5]). *Let $G = (V(G), E(G))$ be a simple graph and a bijection $f : V(G) \to \{1, 2, 3, ..., |V(G)|\}$ induces an edge labeling $f' : E(G) \to \{0, 1\}$ such that for any edge uv in $E(G)$, $f'(uv) = 1$ if $D \mid S$ and $f'(uv) = 0$ if $D \nmid S$. The labeling f is called an SD-divisor cordial labeling if $|e_{f'}(0) - e_{f'}(1)| \leq 1$. We say that G is SD-divisor cordial if it admits an SD-divisor cordial labeling.*

In [5], we introduced two new types of labeling called SD-divisor and SD-divisor cordial labeling. Also, we proved some graphs are SD-divisor. Motivated by the concepts of SD-divisor and SD-divisor cordial labeling, we introduce PD-divisor cordial labeling. In this paper, we are dealing in PD-divisor cordial labeling of some standard graphs.

2. PD-divisor Cordial Labeling of Graphs

Given a bijection $f : V \to \{1, 2, 3, ..., n\}$, we associate two integers $P = f(u)f(v)$ and $D = |f(u) - f(v)|$ with every edge uv in E.

Definition 2.1. *Let $G = (V(G), E(G))$ be a simple graph and a bijection $f : V(G) \to \{1, 2, 3, ..., |V(G)|\}$ induces an edge labeling $f' : E(G) \to \{0, 1\}$ such that for any edge uv in $E(G)$, $f'(uv) = 1$ if $D \mid P$ and $f'(uv) = 0$ if $D \nmid P$. The labeling f is called an PD-divisor cordial labeling if $|e_{f'}(0) - e_{f'}(1)| \leq 1$. We say that G is PD-divisor cordial if it admits an PD-divisor cordial labeling.*

Example 2.2. *Consider the following graph G.*

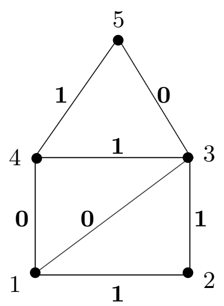

Figure 1. *Graph G*

We see that $e_{f'}(0) = 3$ and $e_{f'}(1) = 4$. Thus $|e_{f'}(0) - e_{f'}(1)| \leq 1$ and hence G is PD-divisor cordial.

Theorem 2.3. *If G is PD-divisor cordial of size q, then $G - e$ is also PD-divisor cordial* *(i) for all $e \in E(G)$ when q is even.*

(ii) for some $e \in E(G)$ when q is odd.

Proof. **Case (i)**: when q is even.

Let G be the PD-divisor cordial graph of size q, where q is an even number. It follows that $e_{f'}(0) = e_{f'}(1) = \frac{q}{2}$. Let e be any edge in G which is labeled either 0 or 1. Then in $G - e$, we have either $e_{f'}(0) = e_{f'}(1) + 1$ or $e_{f'}(1) = e_{f'}(0) + 1$ and hence $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Thus $G - e$ is PD-divisor cordial for all $e \in E(G)$.

Case (ii): when q is odd.

Let G be the PD-divisor cordial graph of size q, where q is an odd number. It follows that either $e_{f'}(0) = e_{f'}(1) + 1$ or $e_{f'}(1) = e_{f'}(0) + 1$. If $e_{f'}(0) = e_{f'}(1) + 1$ then remove on edge e which is labeled as 0 and if $e_{f'}(1) = e_{f'}(0) + 1$ then remove on edge e which is labeled as 1 from G. It follows that $e_{f'}(0) = e_{f'}(1)$. Thus, $G - e$ is PD-divisor cordial for some $e \in E(G)$. □

Corollary 2.4. *The graph $G + e$ is PD-divisor cordial if G is PD-divisor cordial having even size.*

Theorem 2.5. *The path P_n is PD-divisor cordial for all $n \geq 2$.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of path P_n. Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$. Therefore, P_n is of order n and size $n - 1$. Define $f : V(P_n) \to \{1, 2, 3, ..., n\}$ as follows:

$$f(v_i) = 2i - 1, \quad 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil;$$
$$f(v_{n+1-i}) = 2i, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

From the above labeling pattern we get, $e_{f'}(1) = \frac{n}{2}$ and $e_{f'}(0) = \frac{n-2}{2}$ if n is even and $e_{f'}(1) = e_{f'}(0) = \frac{n-1}{2}$ if n is odd. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, P_n is PD-divisor cordial. □

Example 2.6. *Consider P_{10}.*

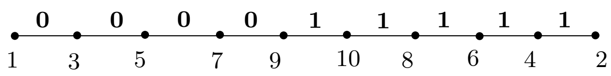

Fig. 2. *Path P_{10}*

Here $e_{f'}(0) = 4$ and $e_{f'}(1) = 5$. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, P_{10} is PD-divisor cordial.

Theorem 2.7. *The cycle C_n is PD-divisor cordial for all $n \geq 3$.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of cycle C_n. Let $V(C_n) = \{v_i : 1 \leq i \leq n\}$ and $E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \bigcup \{v_n v_1\}$. Therefore, C_n is of order n and size n. Define $f : V(C_n) \to \{1, 2, 3, ..., n\}$ as follows:

$$f(v_i) = 2i - 1, \quad 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil;$$
$$f(v_{n+1-i}) = 2i, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

From the above labeling pattern we get, $e_{f'}(1) = \frac{n+1}{2}$ and $e_{f'}(0) = \frac{n-1}{2}$ if n is odd and $e_{f'}(1) = \frac{n+2}{2}$ and $e_{f'}(0) = \frac{n-2}{2}$ if n is even. Then, $e_{f'}(1) - e_{f'}(0) = 1$ if n is odd and $e_{f'}(1) - e_{f'}(0) = 2$ if n is even. Now switch the vertex label of 2 and 4 if n is even. Then, we get $e_{f'}(1) = e_{f'}(0) = \frac{n}{2}$ if n is even. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, C_n is PD-divisor cordial. □

Example 2.8. *Consider C_{10}.*

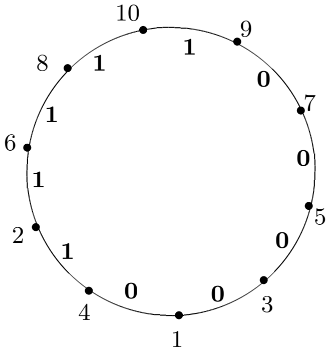

Fig. 3. *Cycle C_{10}*

Here $e_{f'}(0) = 5$ and $e_{f'}(1) = 5$. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, C_{10} is PD-divisor cordial.

Theorem 2.9. *The wheel graph W_n is PD-divisor cordial for all $n \geq 5$.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of wheel W_n. Let $V(W_n) = \{v_i : 1 \leq i \leq n\}$ and $E(W_n) = \{v_1 v_i : 2 \leq i \leq n\} \bigcup \{v_i v_{i+1} : 2 \leq i \leq n-1\} \bigcup \{v_2 v_n\}$. Therefore, W_n is of order n and size $2n-2$.

Define $f : V(W_n) \to \{1, 2, 3, ..., n\}$ by $f(v_i) = i$ for $1 \leq i \leq n$.

From the above labeling pattern we get, $e_{f'}(1) = n$ and $e_{f'}(0) = n-2$ if $n = 6, 8$ and $e_{f'}(1) = e_{f'}(0) = n-1$ otherwise. Then, $e_{f'}(1) - e_{f'}(0) = 2$ if $n = 6, 8$ and $e_{f'}(1) - e_{f'}(0) = 0$ otherwise. Now switch the vertex label of 2 and 4 if $n = 6, 8$. Then, we get $e_{f'}(1) = e_{f'}(0) = n-1$. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, W_n is PD-divisor cordial. □

Example 2.10. *Consider W_5 and W_6.*

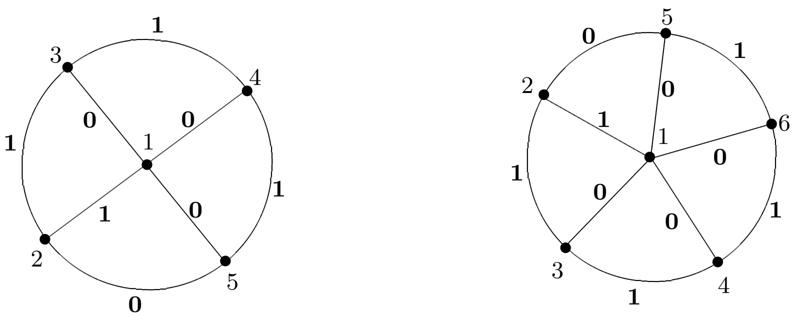

Fig. 4. *Wheel W_5 and Wheel W_6*

Here W_5 have $e_{f'}(0) = 4$ and $e_{f'}(1) = 4$ and W_6 have $e_{f'}(0) = 5$ and $e_{f'}(1) = 5$. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, W_5 and W_6 are PD-divisor cordial.

Theorem 2.11. *The graph $K_{1,n,n}$ is PD-divisor cordial for all $n \geq 1$.*

Proof. Let $V(K_{1,n,n}) = \{v, v_i, u_i : 1 \leq i \leq n\}$ and $E(K_{1,n,n}) = \{vv_i, v_i u_i : 1 \leq i \leq n\}$. Therefore, $K_{1,n,n}$ is of order $2n+1$ and size $2n$.

Define $f : V(K_{1,n,n}) \to \{1, 2, 3, ...2n+1\}$ as follows:

$$f(v) = 1;$$

$$f(v_i) = 2i + 1, \quad 1 \leq i \leq n;$$

$$f(u_i) = 2i, \qquad 1 \leq i \leq n.$$

From the above labelling pattern we get $e_{f'}(1) = e_{f'}(0) = n$. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, $K_{1,n,n}$ is PD-divisor cordial. □

Example 2.12. *Consider $K_{1,7,7}$.*

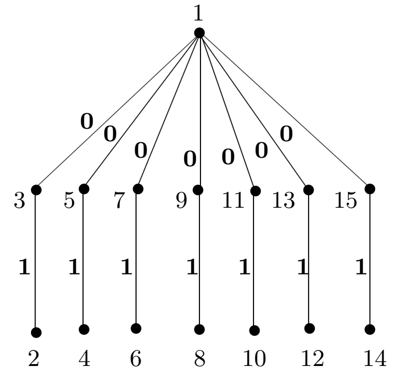

Fig. 5. *Graph $K_{1,7,7}$*

Here $e_{f'}(0) = 7$ and $e_{f'}(1) = 7$. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, $K_{1,7,7}$ is PD-divisor cordial.

Theorem 2.13. *The fan graph F_n is PD-divisor cordial for all $n \geq 5$.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of fan F_n. Let $V(F_n) = \{v_i : 1 \leq i \leq n\}$ and $E(F_n) = \{v_1v_i : 2 \leq i \leq n\} \bigcup \{v_iv_{i+1} : 2 \leq i \leq n-1\}$. Therefore, F_n is of order n and size $2n - 3$.

Define $f : V(F_n) \to \{1, 2, 3, ..., n\}$ by $f(v_i) = i$ for $1 \leq i \leq n$.

From the above labeling pattern we get $e_{f'}(1) = n - 1$ and $e_{f'}(0) = n - 2$. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, F_n is PD-divisor cordial. □

Example 2.14. *Consider F_{10}.*

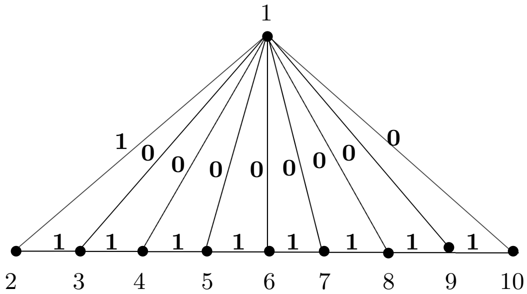

Fig. 6. *Fan Graph F_{10}*

Here $e_{f'}(0) = 8$ and $e_{f'}(1) = 9$. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, F_{10} is PD-divisor cordial.

Theorem 2.15. *The graph obtained by switching of an arbitrary vertex in cycle C_n admits PD-divisor cordial labeling for all $n \geq 4$.*

Proof. Let $v_1, v_2, ..., v_n$ be the vertices of cycle C_n and G_v denotes the graph obtained by switching of a vertex v. Without loss of generality let the switched vertex be v_1 and we initiate the labeling from the switched vertex v_1.

Let $V(G_{v_1}) = \{v_i : 1 \leq i \leq n\}$ and $E(G_{v_1}) = \{v_iv_{i+1} : 2 \leq i \leq n-1\} \bigcup \{v_1v_i : 3 \leq i \leq n-1\}$. Therefore, G_{v_1} is of order n and size $2n - 5$.

Define $f : V(G_{v_1}) \to \{1, 2, 3, ..., n\}$ by $f(v_i) = i$ for $1 \leq i \leq n$.

This labeling pattern gives $e_{f'}(1) = n - 2$ and $e_{f'}(0) = n - 3$. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, G_{v_1} is PD-divisor cordial. □

Example 2.16. *Consider switching of C_7.*

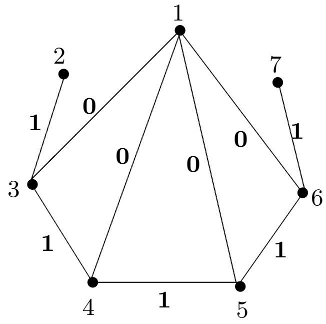

Fig. 7. *Switching of C_7*

Here $e_{f'}(0) = 4$ and $e_{f'}(1) = 5$. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, switching of C_7 (G_{v_1}) is PD-divisor cordial.

Theorem 2.17. *Every complete binary tree BT_n is PD-divisor cordial for all $n \geq 1$.*

Proof. Let $G = BT_n$ be a complete binary tree with level n. Let v be a root of BT_n, which is called a zero level vertex. Clearly, the i^{th} level of BT_n has 2^i vertices. Therefore, BT_n is of order $2^{n+1} - 1$ and size $2^{n+1} - 2$. Now assign the label 1 to the root v. Next, we assign the labels $2^i, 2^i + 1, 2^i + 2, ..., 2^{i+1} - 1$ to the p^{th} level vertices, where $1 \leq i \leq n$. This labeling pattern gives $e_{f'}(1) = e_{f'}(0)$. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, BT_n is PD-divisor cordial. □

Example 2.18. *Consider the following complete binary tree BT_3.*

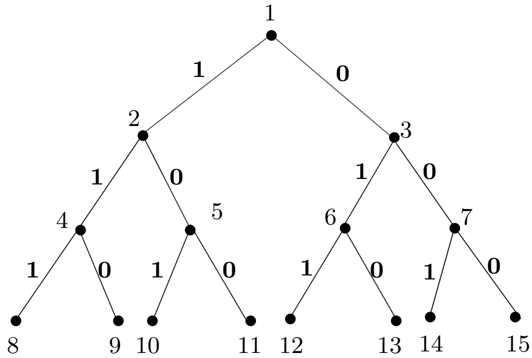

Fig. 8. *Complete Binary Tree BT_3*

Here $e_{f'}(0) = e_{f'}(1) = 7$. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, BT_3 is PD-divisor cordial.

Theorem 2.19. *The graph $C_4^{(n)}$ is PD-divisor cordial for all $n \geq 2$.*

Proof. Let $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, v_4^{(i)} (i = 1, 2, ..., n)$ be the vertices of $C_4^{(n)}$. Let $v_1^{(1)} = v_1^{(2)} = ... = v_1^{(n)} = v$. Let $G = C_4^{(n)}$. Therefore. G is of order $3n + 1$ and size $4n$. Define $f : V(G) \to \{1, 2, ..., 3n + 1\}$ as follows:

$$f(v) = 1;$$
$$f(v_2^{(i)}) = 3i - 1, \quad 1 \leq i \leq n;$$
$$f(v_3^{(i)}) = 3i, \qquad 1 \leq i \leq n;$$

$$f(v_4^{(i)}) = 3i + 1, \quad 1 \leq i \leq n.$$

Note that, this labeling pattern gives $e_{f'}(1) - e_{f'}(0) = 2$. Now switch the vertex label of 2 and 3. Then, we get $e_{f'}(1) = e_{f'}(0) = 2n$. Thus, $|e_{f'}(0) - e_{f'}(1)| \leq 1$. Hence, $C_4^{(n)}$ is PD-divisor cordial. □

Example 2.20. *Consider* $C_4^{(5)}$.

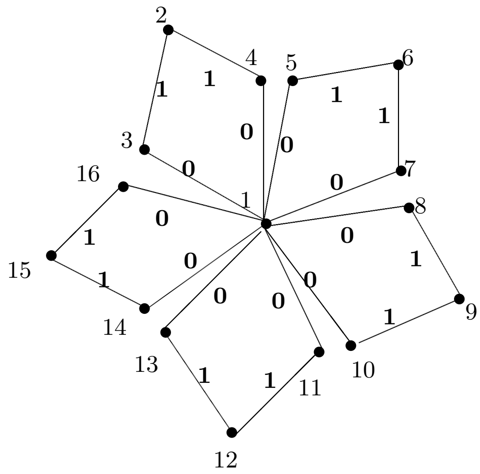

Fig. 9. *Graph* $C_4^{(5)}$

Here $e_{f'}(0) = e_{f'}(1) = 10$. *Thus,* $|e_{f'}(0) - e_{f'}(1)| \leq 1$. *Hence,* $C_4^{(5)}$ *is PD-divisor cordial.*

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