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A Step by Step Approach to Determination of Self-Adjointness and Conservation Laws for a General Burgers Equation

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Abstract: In this paper, the self-adjointness of a general Burgers equation that appears in a wide range of physical applications is investigated. The equation is shown to be quasi self-adjoint but not self-adjoint. Finally, the Ibragimov's theorem on conservation laws is used to construct some of the conservation laws of the equation.

Keywords: Differential equations, Self-adjointness, Lie symmetry generators, Conservation laws.

1. Introduction

Problems involving differential equations arise in various fields of science, mathematics and other related areas. Therefore the task of determining the self adjointness and constructing the conservation laws for the equations is of great significance as it plays an essential role in the study of nonlinear physical phenomena. In recent years, intense research has been conducted in order to find classes of equations that are self-adjoint, quasi-self adjoint or weak self-adjoint. A number of significant methods have been established for construction of conservation laws, one of them being the Noether's approach that yields conservation laws by using Lie symmetries of PDE's with variable principle [10]. Further, IBragimov [3, 4] established a theorem which is used to find conservation laws for PDE's that do not possess Lagrangian. For instance in [3], he introduced the general concept of nonlinear self-adjointness of differential equations by embracing the strict self-adjointness and quasi self-adjointness. Moreover, he did show that the equation possessing the nonlinear self-adjointness can be written equivalently in a strictly self-adjoint form by using appropriate multipliers and that all linear equations possess the property of nonlinear self-adjointness and hence can be re-written in nonlinear strictly self-adjoint. Furthermore he illustrated the construction of conservation laws using symmetry. In [6] , Igor and Julio applied the concept of self-adjoint equations formulated by Ibragimov and Gandariasto to a class of fifth order evolution equations and went further to establish the conservation laws for the generalized Kawahara equation, simplified Kawahara equation and Modified simplified Kawahara equation using Ibragimov's theorem on conservation laws. Other papers of Igor on self-adjointness and conservation laws can be found in [5, 7, 8].

Researchers such as Jaskiran [9] obtained the conservation laws of variable coefficient time fractional Kawahara equation.

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This was achieved by first deriving the similarity reduction and power series solutions of the equation using Lie symmetry method. Zhang [11] studied the nonlinear self-adjointness method for constructing conservation laws of partial differential equations(PDE's). He showed that any adjoint symmetry of PDE's is a differential substitution of nonlinear self-adjointness and vice versa and further illustrated that each symmetry of PDE's correspond to a conservation law via a formula if the system of PDE's is non-linearly self-adjoint with differential substitution. Finally, as a byproduct, he found that the set of differential substitutions include the set of conservation law multipliers as a subset.

In this paper, we determine the self-adjointness and conservation laws for the general Burgers equation given by:

$$u_t = au_x^2 + bu_{xx} \tag{1}$$

where $u(x,t)$ is the unknown function, $a, b \in R$ and $a, b \neq 0$. This equation represent the wave equation combining both the dissipative and nonlinear effect and thus appears in a wide variety of physical applications [1].

2. Adjoint Equations and Self-adjointness

2.1. Adjoint equation

Consider the system of m differential equations (linear or nonlinear) given by [3]:

$$F(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = 0, \qquad \alpha = 1, \ldots, m \tag{2}$$

with m dependent variables. The adjoint equation is written as

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = 0, \qquad \alpha = 1, \ldots, m \tag{3}$$

with F^* defined by

$$F^*(x, u, v, u_{(1)}, v_{(1)}, \ldots, u_{(s)}, v_{(s)}) = \frac{\delta \mathcal{L}}{\delta u} = 0 \tag{4}$$

where $\mathcal{L}$, the formal lagrangian for equations (2) is given by

$$\mathcal{L} = vF \tag{5}$$

such that $v = v(x)$ and

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{i=1}^{\infty} (-1)^i D_{i1} \ldots D_{is} \frac{\partial}{\partial u^{\alpha}_{i1 \ldots is}} \tag{6}$$

2.2. Self-adjointness

Definition 2.1 ([3]). *Equation (2) is said to be* self-adjoint *if the adjoint equation (3) becomes equivalent to the original equation (2) upon the substitution of $v = u$.*

Definition 2.2 ([6]). *Equation (2) is called* quasi self-adjoint *or nonlinear self-adjoint if the adjoint equation (3) become equivalent to (2) upon the substitution of the form $v = \phi(x, u)$ with $\phi_u \neq 0$.*

Thus equation (3) is said to be nonlinear self-adjoint if there exists a function $\phi = \phi(x, u)$ such that

$$F^*|_{v=\phi} = \lambda(x, u, \ldots)F \tag{7}$$

for some differential functions $\lambda = \lambda(x, u, \ldots)$

2.3. Determination of the self-adjointness of equation (1)

Let

$$F = u_t - au_x^2 - bu_{xx} \tag{8}$$

The corresponding lagrangian of (1) is given by

$$\mathcal{L} = vF = v(u_t - au_x^2 - bu_{xx}) \tag{9}$$

Substituting (9) into (4), we have

$$\begin{aligned} F^* &= \frac{\delta \mathcal{L}}{\delta u} = \frac{\delta}{\delta u}\Big[v(u_t - au_x^2 - bu_{xx})\Big] \\ &= \frac{\partial}{\partial u}\Big[v(u_t - au_x^2 - bu_{xx})\Big] - D_t\frac{\partial}{\partial u_t}\Big[v(u_t - au_x^2 - bu_{xx})\Big] \\ &\quad -D_x\frac{\partial}{\partial u_x}\Big[v(u_t - au_x^2 - bu_{xx})\Big] + D_x^2\frac{\partial}{\partial u_{xx}}\Big[v(u_t - au_x^2 - bu_{xx})\Big] \\ &= 0 - D_t(v) - D_x(-2avu_x) + D_x^2(-bv) \\ &= -v_t + 2av_xu_x + 2avu_{xx} - bv_{xx} = 0 \end{aligned} \tag{10}$$

Remark 2.3. *Note that equation (10) is not equivalent to equation (8) upon the substitution of $v = u$. Thus equation (1) is not self adjoint.*

Substituting $v = \phi(x, t, u)$ into (10) with its derivatives expressed as follows,

$$\begin{aligned} v_t &= \Big(\frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u}\Big)\phi \\ &= \frac{\partial \phi}{\partial t} + u_t\frac{\partial \phi}{\partial u} \\ &= \phi_t + \phi_u u_t \\ v_x &= \Big(\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u}\Big)\phi \\ &= \phi_x + \phi_u u_x \\ v_{xx} &= \Big(\frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + ..\Big)(\phi_x + \phi_u u_x) \\ &= \frac{\partial}{\partial x}\Big(\phi_x + \phi_u u_x\Big) + u_x\frac{\partial}{\partial u}\Big(\phi_x + \phi_u u_x\Big) \\ &= \phi_{xx} + \phi_{xu}u_x + \phi_u u_{xx} + u_x(\phi_{xu} + \phi_{uu}u_x) \\ &= \phi_{xx} + \phi_{xu}u_x + \phi_u u_{xx} + \phi_{xu}u_x + \phi_{uu}u_x^2 \\ &= \phi_{xx} + \phi_u u_{xx} + 2\phi_{xu}u_x + \phi_{uu}u_x^2 \end{aligned}$$

we obtain

$$\begin{aligned} F^* &= -\phi_t - \phi_u u_t + 2au_x(\phi_x + \phi_u u_x) + 2a\phi u_{xx} - b(\phi_{xx} + \phi_u u_{xx} + 2\phi_{xu}u_x + \phi_{uu}u_x^2) \\ &= -\phi_t - b\phi_{xx} - \phi_u u_t + (2a\phi_x - 2b\phi_{xu})u_x + (2a\phi_u - b\phi_{uu})u_x^2 + (2a\phi - b\phi_u)u_{xx} \end{aligned} \tag{11}$$

Substituting (8) and (11) into (7), we have

$$\begin{aligned} &- \phi_t - b\phi_{xx} - \phi_u u_t + (2a\phi_x - 2b\phi_{xu})u_x + (2a\phi_u - b\phi_{uu})u_x^2 + (2a\phi - b\phi_u)u_{xx} \\ &= \lambda(u_t - au_x^2 - bu_{xx}) \end{aligned} \tag{12}$$

Equating the coefficients of various monomials in the first and second partial derivatives of u, we obtain the following determining equations.

Monomial terms	Equation
u_t	$-\phi_u = \lambda$ (a)
u_{xx}	$2a\phi - b\phi_u = -\lambda b$ (b)
u_x^2	$2a\phi_u - b\phi_{uu} = -\lambda a$ (c)
u_x	$2a\phi_x - 2b\phi_{xu} = 0$ (d)
1	$-\phi_t - b\phi_{xx} = 0$ (e)

Substituting (a) into (b), we have

$$\begin{aligned} 2a\phi - b\phi_u &= b\phi_u \\ \Longrightarrow 2a\phi &= 2b\phi_u \\ \Longrightarrow \phi_u &= \frac{a}{b}\phi \\ \Longrightarrow \frac{d\phi}{du} &= \frac{a}{b}\phi \\ \Longrightarrow \frac{d\phi}{\phi} &= \frac{a}{b}du \\ \Longrightarrow \ln\phi &= \frac{a}{b}u + C \\ \phi &= Ce^{\frac{a}{b}u} \end{aligned}$$

where C is a constant of integration. Thus

$$v = \phi = Ce^{\frac{a}{b}u} \tag{13}$$

Inserting (13) into (10), with its derivatives given by

$$\begin{aligned} v_t &= \frac{a}{b}Ce^{\frac{a}{b}u}u_t \\ v_x &= \frac{a}{b}Ce^{\frac{a}{b}u}u_x \\ v_{xx} &= \frac{a^2}{b^2}Ce^{\frac{a}{b}u}u_x^2 + \frac{a}{b}Ce^{\frac{a}{b}u}u_{xx} \end{aligned}$$

we have

$$-\frac{a}{b}Ce_{\frac{a}{b}u}u_t + 2\frac{a^2}{b}Ce^{\frac{a}{b}u}u_x^2 + 2aCe^{\frac{a}{b}u}u_{xx} - \frac{a^2}{b}Ce^{\frac{a}{b}u}u_x^2 - aCe^{\frac{a}{b}u}u_{xx} = 0 \tag{14}$$

Diving equation (14) throughout by $aCe^{\frac{a}{b}u}$ and simplifying further, we obtain

$$\begin{aligned} -u_t + au_x^2 + bu_{xx} &= 0 \\ \Rightarrow u_t - au_x^2 - bu_{xx} &= 0 \end{aligned} \tag{15}$$

Remark 2.4. *Note that equation (10) becomes equivalent to equation (8) upon the substitution of (13). Therefore equation (1) is quasi self-adjoint.*

3. Conservation Laws for Equation (1)

Let

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u} \tag{16}$$

be a Lie point symmetry generator of equation (1) and the formal lagrangian $\mathcal{L}$ given by (5). The Ibragimov's theorem on conservation laws provides a conservation law of equation (1) written in the form [6].

$$D_tC^0 + D_xC^1 = 0 \tag{17}$$

where C^0, C^1 are called the conserved vectors if they satisfy the conservation equation (17) on all solutions of equation (1). The conserved vectors C^0 and C^1 are expressed as follows:

$$\begin{aligned} C^0 &= \tau\mathcal{L} + W\frac{\partial \mathcal{L}}{\partial u_t} \\ C^1 &= \xi\mathcal{L} + W\Big[\frac{\partial \mathcal{L}}{\partial u_x} - D_x\frac{\partial \mathcal{L}}{\partial u_{xx}}\Big] + D_x(W)\frac{\partial \mathcal{L}}{\partial u_{xx}} \end{aligned} \tag{18}$$

where $W = \eta - \tau u_t - \xi u_x$. The Lie point symmetry generators for equation (1) are spanned by the following vector fields [1].

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x} \\ V_2 &= \frac{\partial}{\partial t} \\ V_3 &= \frac{\partial}{\partial u} \\ V_4 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} \\ V_5 &= 2at\frac{\partial}{\partial x} - x\frac{\partial}{\partial u} \\ V_6 &= 4axt\frac{\partial}{\partial x} + 4at^2\frac{\partial}{\partial t} - (x^2 + 2bt)\frac{\partial}{\partial u} \end{aligned}$$

3.1. Case 1: $V_1 = \frac{\partial}{\partial x}$

Here

$$\xi = 1, \qquad \tau = 0, \qquad \eta = 0 \tag{19}$$

and thus the characteristic function W is of the form

$$W = -u_x \tag{20}$$

Substituting (9), (19) and (20) into (18), we have

$$\begin{aligned} C^0 &= 0(vu_t - avu_x^2 - bvu_{xx}) - u_x\frac{\partial}{\partial u_t}(vu_t - avu_x^2 - bvu_{xx}) \\ &= -vu_x \\ C^1 &= 1(vu_t - avu_x^2 - bvu_{xx}) - u_x\big[-2avu_x - D_x(-bv)\big] - bvD_x(-u_x) \\ &= vu_t - avu_x^2 - bvu_{xx} + 2avu_x^2 - bv_xu_x + bvu_{xx} \\ &= vu_t + avu_x^2 - bv_xu_x \end{aligned} \tag{21}$$

3.2. Case 2: $V_2 = \frac{\partial}{\partial t}$

Here

$$\xi = 0, \qquad \tau = 1, \qquad \eta = 0 \tag{22}$$

and thus the characteristic function W is of the form

$$W = -u_t \tag{23}$$

Substituting (9), (22) and (23) into (18), we have

$$C^0 = 0(vu_t - avu_x^2 - bvu_{xx}) - u_t\frac{\partial}{\partial u_t}(vu_t - avu_x^2 - bvu_{xx})$$

$$\begin{aligned}
&= vu_t - avu_x^2 - bvu_{xx} - vu_t \\
&= -avu_x^2 - bvu_{xx} \\
C^1 &= 0(vu_t - avu_x^2 - bvu_{xx}) - u_t\big[-2avu_x - D_x(-bv)\big] - bvD_x(-u_t) \\
&= -u_t(-2avu_x + bv_x) + bvu_{xt} \\
&= 2avu_tu_x - bv_xu_t + bvu_{xt} \qquad (24)
\end{aligned}$$

3.3. Case 3: $V_3 = \frac{\partial}{\partial u}$

In this case we have that:

$$\xi = 0, \qquad \tau = 0, \qquad \eta = 1 \qquad (25)$$

and the function W is of the form

$$W = 1 \qquad (26)$$

Substituting (9), (25) and (26) into (18), we have

$$\begin{aligned}
C^0 &= 0 + \frac{\partial}{\partial u_t}\big(vu_t - avu_x^2 - bvu_{xx}\big) \\
&= v \\
C^1 &= 0 + 1\big[-2avu_x - D_x(-bv)\big] + (-bv)D_x(1) \\
&= -2avu_x + bv_x \qquad (27)
\end{aligned}$$

3.4. Case 4: $V_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$

Here

$$\xi = x, \qquad \tau = 2t, \qquad \eta = 0 \qquad (28)$$

and the function W takes the form

$$W = -xu_x - 2tu_t \qquad (29)$$

Substituting (9), (28) and (29) into (18), we have

$$\begin{aligned}
C^0 &= 2t(vu_t - avu_x^2 - bvu_{xx}) + (-xu_x - 2tu_t)v \\
&= 2tvu_t - 2atv^2u_x - 2btvu_{xx} - xvu_x - 2tvu_t \\
&= -2atvu_x^2 - 2btvu_{xx} - xvu_x \\
C^1 &= x(vu_t - avu_x^2 - bvu_{xx}) + (-xu_x - 2tu_t)[-2avu_x - D_x(-bv)] + (-bv)D_x(-xu_x - 2tu_t) \\
&= xvu_t - axvu_x^2 - bxvu_{xx} + (-xu_x - 2tu_t)(-2avu_x + bv_x) + bv(u_x + xu_{xx} + 2tu_{xt}) \\
&= xvu_t - axvu_x^2 - bxvu_{xx} + 2axvu_x^2 - bxv_xu_x + 4atvu_tu_x - 2btv_xu_x + bvu_x + bxvu_{xx} + 2btvu_{xt} \\
&= xvu_t + axvu_x^2 - bxv_xu_x + 4atvu_tu_x - 2btv_xu_t + bvu_x + 2btvu_{xt} \qquad (30)
\end{aligned}$$

3.5. Case 5: $V_5 = 2at\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}$

Here

$$\xi = 2at, \qquad \tau = 0, \qquad \eta = -x \qquad (31)$$

and the function W takes the form

$$W = -x - 2atu_x \tag{32}$$

Substituting (9), (31) and (32) into (18), we have

$$\begin{aligned}
C^0 &= 0 + (-x - 2atu_x)v \\
&= -xv - 2atvu_x \\
C^1 &= 2at(vu_t - avu_x^2 - bvu_{xx}) + (-x - 2atu_x)[-2avu_x - D_x(-bv)] + (-bv)D_x(-x - 2atu_x) \\
&= 2atvu_t - 2a^2tvu_x^2 - 2abtvu_{xx} + (-x - 2atu_x)(-2avu_x + bv_x) + bv(1 + 2atu_{xx}) \\
&= 2atvu_t - 2a^2tvu_x^2 - 2abtvu_{xx} + 2axvu_x - bxv_x + 4a^2tvu_x^2 - 2abtv_xu_x + bv + 2abtvu_{xx} \\
&= 2atvu_t + 2a^2tvu_x^2 + 2axvu_x - bxv_x - 2abtv_xu_x + bv
\end{aligned} \tag{33}$$

3.6. Case 6: $V_6 = 4axt\frac{\partial}{\partial x} + 4at^2\frac{\partial}{\partial u} - (x^2 + 2bt)\frac{\partial}{\partial u}$

Here

$$\xi = 4axt, \qquad \tau = 4at^2, \qquad \eta = -x^2 - 2bt \tag{34}$$

and the function W takes the form

$$W = -x^2 - 2bt - 4at^2u_t - 4axtu_x \tag{35}$$

Substituting (9), (33) and (35) into (18), we obtain

$$\begin{aligned}
C^0 &= 4at^2(vu_t - avu_x^2 - bvu_{xx}) + (-x^2 - 2bt - 4at^2u_t - 4axtu_x)v \\
&= 4at^2vu_t - 4a^2t^2vu_x^2 - 4abt^2vu_{xx} - x^2v - 2btv - 4at^2vu_t - 4axtvu_x \\
&= -4a^2t^2vu_x^2 - 4abt^2vu_{xx} - x^2v - 2btv - 4axtvu_x \\
C^1 &= 4axt(vu_t - avu_x^2 - bvu_{xx}) + (-x^2 - 2bt - 4at^2u_t - 4axtu_x)[-2avu_x - D_x(-bv)] \\
&\quad +(-bv)D_x(-x^2 - 2bt - 4at^2u_t - 4axtu_x) \\
&= 4axtvu_t - 4a^2xtvu_x^2 - 4abxtvu_{xx}(-x^2 - 2bt - 4at^2u_t - 4axtu_x)(-2avu_x + bv_x) \\
&\quad +bv(2x + 4at^2u_{xt} + 4atu_x + 4axtu_{xx}) \\
&= 4axtvu_t - 4a^2xtvu_x^2 - 4abxtvu_{xx} + 2ax^2vu_x + 4abtvu_x + 8a^2t^2vu_tu_x + 8a^2xtv_xu^2 \\
&\quad -bx^2v_x - 2b^2tv_x - 4abt^2v_xu_t - 4abxtv_xu_x + 2bxv + 4abt^2vu_{xt} + 4abtvu_x + 4abxtvu_{xx} \\
&= 4axtvu_t + 4a^2xtvu_x^2 + 2ax^2vu_x + 8abtvu_x + 8a^2t^2vu_tu_x - bx^2v_x - 2b^2tv_x \\
&\quad -4abt^2v_xu_t - 4abxtv_xu_x + 2bxv + 4abt^2vu_{xt}
\end{aligned} \tag{36}$$

Using the value of v in (13) and setting $C = 1$, the conserved vectors (21), (24), (27), (30), (33) and (36) can, respectively, be expressed as follows:

$$\begin{aligned}
C^0 &= -e^{\frac{a}{b}u}u_x \\
C^1 &= e^{\frac{a}{b}u}u_t
\end{aligned} \tag{37}$$

$$\begin{aligned}
C^0 &= -e^{\frac{a}{b}u}(au_x^2 + bu_{xx}) \\
C^1 &= e^{\frac{a}{b}u}(au_tu_x + bu_{xt})
\end{aligned} \tag{38}$$

$$\begin{aligned} C^0 &= e^{\frac{a}{b}u} \\ C^1 &= -ae^{\frac{a}{b}u}u_x \end{aligned} \tag{39}$$

$$\begin{aligned} C^0 &= -e^{\frac{a}{b}u}\left(2atu_x^2 + 2btu_{xx} + xu_x\right) \\ C^1 &= e^{\frac{a}{b}u}\left(xu_x + 2atu_tu_x + bu_x + 2btu_{xt}\right) \end{aligned} \tag{40}$$

$$\begin{aligned} C^0 &= -e^{\frac{a}{b}u}\left(x + 2atu_x\right) \\ C^1 &= e^{\frac{a}{b}u}\left(2atu_t + axu_x + b\right) \end{aligned} \tag{41}$$

$$\begin{aligned} C^0 &= -e^{\frac{a}{b}u}\left(4a^2t^2u_x^2 + 4abt^2u_{xx} + x^2 + 2bt + 4xtu_x\right) \\ C^1 &= e^{\frac{a}{b}u}\left(4axtu_t + ax^2u_x + 6abtu_x + 4a^2t^2u_xu_t + 2bx + 4at^2u_{xt}\right) \end{aligned} \tag{42}$$

4. Conclusion

In this paper, we have investigated the self-adjointness of a general Burgers equation and thereafter used the Ibragimov's theorem on conservation laws to construct some of the conservation laws of the equation.

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