Int. J. Math. And Appl., 6(2–B)(2018), 1–9
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Monotone Iterative Technique for Nonlinear Volterra-Fredholm Integro-Differential Equations

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Abstract: This paper presents an efficient algorithm based on a monotone method for the solution of a class of nonlinear Volterra-Fredholm integro-differential equations of second order. This method is applied to derive two monotone sequences of upper and lower solutions which are uniformly convergent. Theoretical analysis of the existence and convergence of those sequences are discussed. The numerical results demonstrate reliability and efficiency of the proposed technique.

MSC: 34B15, 45J05.

Keywords: Monotone method, Lower and upper solutions, Volterra-Fredholm integro-differential equation.

 Accepted on: 21.03.2018

1. Introduction

The integro-differential equations have attracted more attention of physicists and mathematicians which provide an efficiency for the description of many practical dynamical arising in engineering and scientific disciplines such as, physics, biology, electrochemistry, electromagnetic, control theory and viscoelasticity [4, 10, 11]. Recently, many authors focus on the development of numerical and analytical techniques for integro-differential equations. For instance, the Homotopy perturbation method [8], the variational iteration method [10], the combined modified Laplace with Adomian decomposition method [9, 11–13], the spline collocation method [3], Taylor polynomials [6, 15], Tau method [7], and the method of upper and lower solutions [1, 14], and the references therein. In [3] is suggested the method of sub-solution and super-solution for a type of nonlinear Volterra integro-differential equation and proved their convergence. In this paper, we consider a class of boundary value problems for second-order nonlinear Volterra-Fredholm integro-differential equations of the form

$$Au := u''(x) + \int_0^x K(x,t)F(u(t))dt + \int_0^1 K^*(x,t)F^*(u(t))dt + h(x) = 0, \tag{1}$$

subject to,

$$u(0) = u_0, \quad u(1) = u_1, \tag{2}$$

where $F, F^* \in C[\mathbb{R}, \mathbb{R}]$ is a decreasing function, $K, K^* \in C[I \times I, \mathbb{R}^+]$ where $I = [0, 1]$, is a positive kernel, $h(x) \in C[I, \mathbb{R}]$ and $u_0, u_1 \in \mathbb{R}$ and $x \in I$. The purpose of this paper is to employ an efficient method based on the lower and upper

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solutions, to construct two sequences of decreasing upper solutions, $\{S_k\}$, and increasing lower solutions, $\{s_k\}$, which are uniformly convergent to the solution of Eqs.(1)-(2). Theoretical analysis of the existence and convergence of those sequences are discussed. The simplicity, reliability and efficiency of the proposed scheme are demonstrated by discussing two numerical examples. It should be noted that the present work is partially an extension to the approach described in [2] in order to solve a class of elliptic equations. The rest of the paper is organized as follows: In Section 2, some preliminaries and basic definitions related are recalled. In Section 3, the analytical example is presented to illustrate the accuracy of this method. Finally, the paper concludes with Section 4 which is the report of the study.

2. Preliminaries

The definitions of lower and upper solutions of the problem (1)-(2) are given by

Definition 2.1. *A function $w \in C^2[I, \mathbb{R}]$ is called a lower solution of (1)-(2) on I if*

$$Aw := w''(x) + \int_0^x K(x,t)F(w(t))dt + \int_0^1 K^*(x,t)F^*(w(t))dt + h(x) \geq 0,$$

and $w(0) \leq u_0$, $w(1) \leq u_1$, and $v \in C^2[I, \mathbb{R}]$ is called an upper solution of (1)-(2) on I if

$$Av := v''(x) + \int_0^x K(x,t)F(v(t))dt + \int_0^1 K^*(x,t)F^*(v(t))dt + h(x) \leq 0,$$

and $v(0) \geq u_0$, $v(1) \geq u_1$.

Definition 2.2. *If $w, v \in C^2[I, \mathbb{R}]$ are, respectively, lower and upper solutions of (1)-(2) on I with $w(x) \leq v(x)$ for all $x \leq I$, then we say that $w$and $v$are ordered lower and upper solutions. In the present study, we assume that an initial ordered lower and upper solutions w and v of (1)-(2) on I with $w(x) \leq v(x)$ for all $x \in I$ are known. The initials w and v can be constructed by several techniques such as polynomial bounds, eigenfunction expansion bounds or by linearizing the nonlinear part in the problem, for more details see [2, 14]. Now, we present some theoretical results that should be utilized to construct two monotone sequences of lower and upper solutions of problem (1)-(2) on I. In what follows, $[w, v] = \{y \in C^2(I) : w \leq y \leq v\}$.*

Lemma 2.3. *Consider the nonlinear Volterra-Fredholm integro-differential equation (1)-(2) with F and F^* are decreasing and $K, K^* \geq 0$ in D. If w and v are solutions to (1)-(2) with $w(x) \leq v(x)$ for all $x \in I$ then $w = v$ on I.*

Proof. We shall prove that $w(x) \geq v(x)$ for all $x \in I$. Since w and v are solutions to (1)-(2), we have

$$w''(x) + \int_0^x K(x,t)F(w(t))dt + \int_0^1 K^*(x,t)F^*(w(t))dt + h(x) = 0, \tag{3}$$

$$w(0) = u_0, \quad w(1) = u_1, \tag{4}$$

and

$$v''(x) + \int_0^x K(x,t)F(v(t))dt + \int_0^1 K^*(x,t)F^*(v(t))dt + h(x) = 0, \tag{5}$$

$$v(0) = u_0, \quad v(1) = u_1, \tag{6}$$

Subtracting Equation (3) from Equation (5), we obtain

$$(v-w)''(x)=\int_0^x K(x,t)(F(w(t))-F(v(t)))dt+\int_0^1 K^*(x,t)(F^*(w(t))-F^*(v(t)))dt. \tag{7}$$

Since $w \le v$ and F, F^* are decreasing, the integrand $K(x,t)(F(w(t))-F(v(t)))$ and $K^*(x,t)(F^*(w(t))-F^*(v(t)))$ must be positive on the interval $[0,x]$. If we set $Z=v-w$ then $Z''\ge 0$ with $Z(0)=Z(1)=0$. It follows from the Maximum Principle that $Z\le 0$ and, therefore, $v\le w$ in I as desired. □

Lemma 2.4. *Consider the nonlinear Volterra-Fredholm integro-differential equation (1)-(2) with F and F^* are decreasing and $K,K^*\ge 0$ in D. Let $g_1(y), g_1^*$ and $G_1(y), G_1^*(y)$ be two decreasing functions in the strip $[w,v]$ with $g_1(y)\le F(y)\le G_1(y)$, and $g_1^*(y)\le F^*(y)\le G_1^*(y)$. Let s_1 and S_1 be satisfying*

$$As_1 := s_1''(x)+\int_0^x K(x,t)g_1(v(t))dt+\int_0^1 K^*(x,t)g_1^*(v(t))dt+h(x)=0, \tag{8}$$

$$s_1(0)\le u_0, \quad s_1(1)\le u_1, \tag{9}$$

and

$$AS_1 := S_1''(x)+\int_0^x K(x,t)G_1(w(t))dt+\int_0^1 K^*(x,t)G_1^*(w(t))dt+h(x)=0, \tag{10}$$

$$S_1(0)\ge u_0, \quad S_1(1)\ge u_1. \tag{11}$$

If $w\le s_1$ and $S_1\le v$ in I, then

(i). $s_1, S_1\in[w,v]$.

(ii). s_1 and S_1 are, respectively, ordered lower and upper solutions of (1)-(2) on I.

Proof. (i) We shall prove that $S_1\ge w$ and $s_1\le v$ in I. Since w represents a lower solution of (1)-(2) it must satisfies

$$Aw := w''(x)+\int_0^x K(x,t)F(w(t))dt+\int_0^1 K^*(x,t)F^*(w(t))dt+h(x)\ge 0, \tag{12}$$

with $w(0)\le u_0$, $w(1)\le u_1$. Subtracting Equation (10) from Equation (12), we obtain

$$(w-S_1)''(x)\ge\int_0^x K(x,t)(G_1(w(t))-F(w(t)))dt+\int_0^1 K^*(x,t)(G_1^*(w(t))-F^*(w(t)))dt. \tag{13}$$

Since (by the assumptions $K,K^*\ge 0$ and $F(y)\le G_1(y), F^*(y)\le G_1^*(y), \quad \forall y\in[w,v]$) the integrand $K(x,t)(G_1(w(t))-F(w(t)))$ and $K^*(x,t)(G_1^*(w(t))-F^*(w(t)))$ are positive on the interval $[0,x]$, the right hand side of (13) should be positive and therefore $(w-S_1)''\ge 0$. Setting $Z=w-S_1$ gives $Z''\ge 0$ in I with $Z(0),Z(1)\le 0$. It follows from the Maximum Principle that $Z\le 0$ in I as desired. The proof of $s_1\le v$ in I is done in the same way and will not be presented.

(ii) In this part we have to show that

(ii–1) s_1 and S_2 are respectively, lower and upper solutions of (1)-(2) on I.

(ii–2) $s_1(x)\le S_1(x), \quad \forall x\in I$.

To show (ii–1), it suffices to prove that $As_1 \geq 0$ with $s_1(0) \leq u_0, s_1(1) \leq u_1$ because the proof of $AS_1 \leq 0$ with $S_1(0) \geq u_0, S_1(1) \geq u_1$ is similar. Recall

$$As_1 = s_1''(x) + \int_0^x K(x,t)F(s_1(t))dt + \int_0^1 K^*(x,t)F^*(s_1(t))dt + h(x). \tag{14}$$

Since $g_1 \leq F, g_1^* \leq F^*$ and g_1, g_1^* are decreasing, we have

$$\begin{aligned} As_1 &\geq s_1''(x) + \int_0^x K(x,t)g_1(s_1(t))dt + \int_0^1 K^*(x,t)g_1^*(s_1(t))dt + h(x) \\ &\geq s_1''(x) + \int_0^x K(x,t)g_1(v(t))dt + \int_0^1 K^*(x,t)g_1^*(v(t))dt + h(x). \end{aligned} \tag{15}$$

Equation (8)-(9) implies that $As_1 \geq 0$ with $s_1(0) \leq u_0, s_1(1) \leq u_1$ as desired. To prove (ii–2), we subtract Equation (10) from (8)

$$\begin{aligned} (s_1 - S_1)''(x) &= \int_0^x K(x,t)(G_1(w(t)) - g_1(v(t)))dt + \int_0^1 K^*(x,t)(G_1^*(w(t)) - g_1^*(v(t)))dt \\ &\geq \int_0^x K(x,t)(G_1(w(t)) - g_1(w(t)))dt + \int_0^1 K^*(x,t)(G_1^*(w(t)) - g_1^*(w(t)))dt \\ &\geq 0. \end{aligned} \tag{16}$$

Therefore the function $Z = s_1 - S_1$ is such that $Z'' \geq 0$ in I and $Z(0) \leq 0, Z(1) \leq 0$. Applying the Maximum Principle implies $Z \leq 0$ which completes the proof. □

The following theorem emphasizes the existence of two monotone sequences of lower and upper solutions to problems (1)-(2) on I. Also, it describes the construction of those sequences.

Theorem 2.5. *Consider the nonlinear Volterra-Fredholm integro-differential equation (1)-(2) with $F(u), F^*(u)$ are decreasing and $K, K^* \geq 0 in D$. Let $s_0 = w$ and $S_0 = v$ be an initial ordered lower and upper solutions of (1)-(2) on I. Let $g_k(y), g_k^*$ and $G_k(y), G_k^*(y), k \geq 1$, be decreasing functions in $[w, v]$ with*

$$g_1(y) \leq g_2(y) \leq \cdots \leq g_k(y) \cdots \leq F(y) \leq \cdots \leq G_k(y) \cdots \leq G_2(y) \leq G_1(y),$$

$$g_1^*(y) \leq g_2^*(y) \leq \cdots \leq g_k^*(y) \cdots \leq F^*(y) \leq \cdots \leq G_k^*(y) \cdots \leq G_2^*(y) \leq G^*1(y).$$

Let s_k and S_k for $k \geq 1$ be, respectively, the solutions of

$$\begin{aligned} &s_k'' + \int_0^x K(x,t)g_k(v(t))dt + \int_0^1 K^*(x,t)g_k^*(v(t))dt + h(x) = 0, \\ &s_k(0) = s_1(0) \leq u_0, \quad s_k(1) = s_1(1) \leq u_1, \quad \forall k \geq 1. \end{aligned} \tag{17}$$

$$\begin{aligned} &S_k'' + \int_0^x K(x,t)G_k(w(t))dt + \int_0^1 K^*(x,t)G_k^*(w(t))dt + h(x) = 0, \\ &S_k(0) = S_1(0) \geq u_0, \quad S_k(1) = S_1(1) \geq u_1, \quad \forall k \geq 1. \end{aligned} \tag{18}$$

Then we have [14, 16]

(i). $\{S_k\}$ is a decreasing sequence of upper solutions to (1)-(2) on I.

(ii). $\{s_k\}$ is an increasing sequence of lower solutions to (1)-(2) on I.

(iii). $s_k \leq S_k$, for $k \geq 1$.

Proof. (i) Since the proof of (ii) is similar to that of (i) we prove only part (i). To show that $\{Sk\}$ is a decreasing sequence, it is suffices to prove

$$S_k - S_{k-1} \leq 0, \quad \forall k \geq 1. \tag{19}$$

To this end, Equation (19) gives

$$(S_k - S_{k-1})'' + \int_0^x K(x,t)(G_k(w(t)) - G_{k-1}(w(t)))dt + \int_0^1 K^*(x,t)(G_k^*(w(t)) - G_{k-1}^*(w(t)))dt = 0,$$

or

$$(S_k - S_{k-1})'' = \int_0^x K(x,t)(G_{k-1}(w(t)) - G_k(w(t)))dt + \int_0^1 K^*(x,t)(G_{k-1}^*(w(t)) - G_k^*(w(t)))dt.$$

Recalling that $K, K^* \geq 0$ and $G_{k-1} \geq G_k$ and $G_{k-1}^* \geq G_k^*$ ensures the positivity of the integrand $K(x,t)(G_{k-1}(w) - G_k(w)), K^*(x,t)(G_{k-1}^*(w) - G_k^*(w))$ on the interval $[0,x]$ and therefore the integral in (20) should be positive. Therefore the function $Z = S_k - S_{k-1}$ is such that $Z'' \geq 0$ in I and $Z(0) = Z(1) = 0$. The Maximum Principle implies $Z \geq 0; i.e. S_k \geq S_{k-1}, \quad \forall k \geq 1$. Now, we show that S_k is an upper solution to (1)-(2) on I; hence

$$\begin{aligned} AS_k \quad &:= S_k'' + \int_0^x K(x,t)F(S_k(t))dt + \int_0^1 K^*(x,t)F^*(S_k(t))dt + h(x) \\ &\leq S_k'' + \int_0^x K(x,t)G_k(S_k(t))dt + \int_0^1 K^*(x,t)G_k^*(S_k(t))dt + h(x) \\ &\leq S_k'' + \int_0^x K(x,t)G_k(w(t))dt + \int_0^1 K^*(x,t)G_k^*(w(t))dt + h(x) = 0, \end{aligned}$$

which together with $S_k(0) \geq u_0, S_k(1) \geq u_1$ proves that S_k for $k \geq 1$ is an upper solution.

(ii) To prove that $s_k \leq S_k$ in I, we subtract Equation (19) from (18) to have

$$\begin{aligned} (s_k - S_k)'' \;&=\; \int_0^x K(x,t)(G_k(w(t)) - g_k(v(t)))dt + \int_0^1 K^*(x,t)(G_k^*(w(t)) - g_k^*(v(t)))dt \\ &\geq\; \int_0^x K(x,t)(G_k(w(t)) - g_k(w(t)))dt + \int_0^1 K^*(x,t)(G_k^*(w(t)) - g_k^*(w(t)))dt \\ &\geq\; 0. \end{aligned}$$

Applying the Maximum Principle on $Z'' = (s_k - S_k)'' \geq 0$ with $Z(0), Z(1) \leq 0$ implies that $s_k \leq S_k$ in I as desired. □

The following theorem discusses the uniform convergence of the sequences $\{s_k\}$ and $\{S_k\}$ that already constructed in the Theorem 2.5.

Theorem 2.6. *Consider the nonlinear Volterra-Fredholm integro-differential equation (1)-(2) with $F(u(t)), F^*(u(t))$ are decreasing and $K, K^* \geq 0 in D$. Let $\{s_k\}$ and $\{S_k\}$ be, respectively, the sequences of lower and upper solutions as constructed in Theorem 2.5. Then the sequences $\{s_k\}$ and $\{S_k\}$ converge uniformly to s^* and S^*, respectively, with $s^* \leq u \leq S^*$ where u is the exact solution of (1)-(2).*

Proof. The sequence $\{S_k\}$ is monotonically decreasing and bounded below by $s_0 = w$, therefore it is convergent to a continuous function S^*. Also, since the sequence $\{s_k\}$ is monotonically increasing and bounded above by $S_0 = v$, it is convergent to a continuous function s^*. Moreover, since $\{s_k\}$ and $\{S_k\}$ are sequences of continuous real-valued functions on the compact set $I := [0,1]$, Dini's theorem [5] implies that the convergence of those sequences is uniform. To show that $s^* \leq S^*$, recall part (iii) of Theorem 2.5 then take the limit of both sides, we arrive at

$$s^* = \lim_{k \longrightarrow \infty} s_k \leq u \leq \lim_{k \longrightarrow \infty} S_k = S^*,$$

as desired. □

Note that, it can be easily verified that s^* and S^* are, respectively, lower and upper solutions to (1)-(2) on I. A possible improvement in the method of constructing monotone sequences of lower and upper solutions to problem (1)-(2) on I can be done by reconsidering Eqs. (18) and (19) by calculating the new iterates, say s_k and S_k, using the previously computed ones s_{k-1} and S_{k-1}, respectively. The following theorem is a modified version of Theorem 2.5.

Theorem 2.7. *Consider the nonlinear Volterra-Fredholm integro-differential equation* (1)-(2) *with* $F(u(t)), F^*(u(t))$ *are decreasing and* $K, K^* \geq 0 in D$. *Let* $s_0 = w$ *and* $S_0 = v$ *be an initial ordered lower and upper solutions of* (1)-(2) *on* I. *Let* $g_k(y)$ *and* $G_k(y), k \geq 1$, *be decreasing functions in* $[w, v]$ *with*

$$g_1(y) \leq g_2(y) \leq \cdots \leq g_k(y) \cdots \leq F(y) \leq \cdots \leq G_k(y) \cdots \leq G_2(y) \leq G_1(y)$$
$$g_1^*(y) \leq g_2^*(y) \leq \cdots \leq g_k^*(y) \cdots \leq F^*(y) \leq \cdots \leq G_k^*(y) \cdots \leq G_2^*(y) \leq G^*1(y).$$

Let s_k *and* S_k *for* $k \geq 1$ *be, respectively, the solutions of*

$$s_k'' + \int_0^x K(x,t)g_k(S_{k-1}(t))dt + \int_0^1 K^*(x,t)g_k^*(S_{k-1}(t))dt + h(x) = 0, \tag{20}$$
$$s_k(0) = s_1(0) \leq u_0, \quad s_k(1) = s_1(1) \leq u_1, \quad \forall k \geq 1.$$
$$S_k'' + \int_0^x K(x,t)G_k(s_{k-1}(t))dt + \int_0^1 K^*(x,t)G_k^*(s_{k-1}(t))dt + h(x) = 0, \tag{21}$$
$$S_k(0) = S_1(0) \geq u_0, \quad S_k(1) = S_1(1) \geq u_1, \quad \forall k \geq 1.$$

Then we have

(i). $s_k \leq S_k$, *for* $k \geq 1$.

(ii). $\{s_k\}$ *and* $\{S_k\}$ *are, respectively, increasing sequence of lower solutions and decreasing sequence of upper solutions to* (1)-(2) *on* I.

Proof. (i) To prove that $s_k \leq S_k$ in I, we implement the method of mathematical induction. The case when $k = 1$ follows from Lemma 2.3. If we assume that $s_n \leq S_n$ in I for $k = n$ then we must prove that $s_{n+1} \leq S_{n+1}$. Subtracting Equation (22) from (21) at $k = n+1$, we have

$$(s_{n+1} - S_{n+1})'' = \int_0^x K(x,t)(G_{n+1}(s_n(t)) - g_{n+1}(S_n(t)))dt + \int_0^1 K^*(x,t)(G_{n+1}^*(s_n(t)) - g_{n+1}^*(S_n(t)))dt. \tag{22}$$

Applying the mean value theorem after using the property $g_{n+1}(S_n) \leq G_{n+1}(S_n), g_{n+1}^*(S_n) \leq G_{n+1}^*(S_n)$ in Equation (22), we obtain

$$(s_{n+1} - S_{n+1})'' \geq \int_0^x K(x,t)\frac{\partial G_{n+1}}{\partial y}(\lambda_1)(s_n(t) - S_n(t))dt + \int_0^1 K^*(x,t)\frac{\partial G_{n+1}^*}{\partial y}(\lambda_2)(s_n(t) - S_n(t))dt, \tag{23}$$

where $\lambda_1, \lambda_2 \in [s_n, S_n]$. Since G_{n+1}, G_{n+1}^* are decreasing with respect to y in $[w, v]$, $\frac{\partial G_{n+1}}{\partial y}(\lambda_1) < 0$ and $\frac{\partial G_{n+1}^*}{\partial y}(\lambda_2) < 0$. Therefore, the integrand $\frac{\partial G_{n+1}}{\partial y}(\lambda_1)(s_n(t) - S_n(t))$ and $\frac{\partial G_{n+1}^*}{\partial y}(\lambda_2)(s_n(t) - S_n(t))$ are positive on the interval $[0, x]$ and, hence, $(s_{n+1} - S_{n+1})'' \geq 0$ in I with $s_{n+1}(0) - S_{n+1}(0) \leq 0$ and $s_{n+1}(1) - S_{n+1}(1) \leq 0$. By the Maximum Principle we have $s_{n+1} - S_{n+1} \leq 0$ in I, and the result follows.

(ii) Note that, from Equations (21) and (22), the construction of s_k or S_k requires knowledge of both s_{k-1} and S_{k-1}. To prove (ii), we utilized the method of mathematical induction to show that

$$S_k - S_{k-1} \leq 0, \quad \forall k \geq 1. \tag{24}$$

and

$$s_k - s_{k-1} \geq 0, \quad \forall k \geq 1. \tag{25}$$

The case when $k = 1$ follows from Lemma 2.3 where we showed that $S_1 \leq v = S_0$ and $s_1 \geq w = s_0$. If we assume that the statements (24) and (25) hold for $k = n$ then we must prove that (24) and (25) are true for $k = n + 1$. Since the proof of (25) is similar to that of (24) we prove only (24). To this end, Equation (22) gives

$$\begin{aligned}(S_{n+1} - S_n)'' &= \int_0^x K(x,t)(G_n(s_{n-1}(t)) - G_{n+1}(s_n(t)))dt + \int_0^1 K^*(x,t)(G_n^*(s_{n-1}(t)) - G_{n+1}^*(s_n(t)))dt \\ &\geq \int_0^x K(x,t)(G_n(s_{n-1}(t)) - G_n(s_n(t)))dt + \int_0^1 K^*(x,t)(G_n^*(s_{n-1}(t)) - G_n^*(s_n(t)))dt \\ &\geq 0.\end{aligned}$$

Thus, we have $(S_{n+1} - S_n)'' \geq 0$ in I. Setting $Z = S_{n+1} - S_n$ gives $Z'' \geq 0$ in I with $Z(0) = 0, Z(1) = 0$. It follows from the Maximum Principle that $Z \leq 0$ in I as desired. Now, we show that S_k is an upper solution to (1)-(2) on I.

$$\begin{aligned}AS_k &:= S_k'' + \int_0^x K(x,t)F(S_k(t))dt + \int_0^1 K^*(x,t)F^*(S_k(t))dt + h(x) \\ &\leq S_k'' + \int_0^x K(x,t)G_k(S_k(t))dt + \int_0^1 K^*(x,t)G_k^*(S_k(t))dt + h(x).\end{aligned} \tag{26}$$

Using the result of part (i) in (26), we have

$$\begin{aligned}AS_k &\leq S_k'' + \int_0^x K(x,t)G_k(s_k(t))dt + \int_0^1 K^*(x,t)G_k^*(s_k(t))dt + h(x) = 0, \\ &\leq S_k'' + \int_0^x K(x,t)G_k(s_{k-1}(t))dt + \int_0^1 K^*(x,t)G_k^*(s_{k-1}(t))dt + h(x) = 0.\end{aligned} \tag{27}$$

Consequently, the result of (27) with the associate conditions $S_k(0) = v(0) \geq u_0, S_k(1) = v(1) \geq u_1$ proves that S_k for $k \geq 1$ is an upper solution. Similarly, we can prove that $s_k, k \geq 1$, is a lower solution. □

It should be noted that the sequences $\{s_k\}$ and $\{S_k\}$ constructed in Theorem 2.6 converge uniformly to s^* and S^*, respectively, with $s^* \leq u \leq S^*$ where u is the exact solution of (1)–(2). Moreover, s^* and S^* are lower and upper solutions to (1)–(2) respectively.

3. Numerical Results

In this section, we solve nonlinear Volterra-Fredholm integro-differential equations by using the monotone iterative technique to demonstrate the performance and efficiency of this technique.

Example 3.1. *Consider the Volterra-Fredholm integro-differential equation (1) with*

$$K(x,t) = (x-t)^2, K^*(x,t) = 0, F(u) = F^*(u) = e^{-u}$$

$$h(x) = \frac{e^{-x-\frac{1}{4}}}{8}\left(e^{x+\frac{1}{4}}(18-8x) + e^{x^2+\frac{1}{4}}(4x-2) + e^x\sqrt{\pi}(4(x-1)x-1) \times erfi\left(\frac{1}{2}-x\right) - e^x\sqrt{\pi}(4(x-1)x-1)erfi\left(\frac{1}{2}\right)\right),$$

with the boundary conditions

$$u(0) = 0, \quad u(1) = 0.$$

Here erfi(x) is the imaginary error function.

The exact solution is

$$u(x) = x - x^2.$$

Obviously, K is a positive on $I \times I$ and $F(u) = e^{-u}$ is a decreasing function. The functions $w(x) = 0$ and $v(x) = 1 - x^2$ form, respectively, an initial ordered lower and upper solutions of our problem on I. The decreasing functions $g_k(u)$ and $G_k(u), k \geq 1$, on $[w, v]$ are constructed using Taylor series expansion for $F(u) = e^{-u}$ about $u = 0$, we choose them to have the forms

$$g_k(u) = \sum_{n=0}^{2k} \frac{(-1)^n}{n!} u^n, \quad G_k(u) = \sum_{n=0}^{2k-1} \frac{(-1)^n}{n!} u^n, \quad k \geq 1.$$

Therefore, Theorem 2.7 applies. Solving Eqs.(21)-(22) for $k = 1$, we obtain

$$\begin{aligned} s_1(x) &= \frac{e^{-x-\frac{1}{4}}}{40320}(-210e^{x^2+\frac{1}{4}}(2x(4x^2 - 6x - 7) + 9) + e^{x+\frac{1}{4}}(4x(8723 - 4x) \\ &\quad \times (2x^5 - 420x + 2835)) - 105e^x\sqrt{\pi}(8x(x(2(x-2)x - 3) + 5) + 1)\text{erfi}(\frac{1}{2} - x) \\ &\quad +105e^x\sqrt{\pi}(2x(4x(2(x-2)x - 3) + 19) + 1)\text{erfi}(\frac{1}{2})), \\ S_1(x) &= \frac{e^{-x-\frac{1}{4}}}{1920}(-10e^{x^2-\frac{1}{4}}(2x(4x^2 - 6x - 7) + 9) + e^{x+\frac{1}{4}}(90 - 4x(8x^4 - 80x^2 + 540x - 423)) \\ &\quad -5e^x\sqrt{\pi}(8x(x(2(x-2)x - 3) + 5) + 1)\text{erfi}(\frac{1}{2} - x) \\ &\quad +5e^x\sqrt{\pi}(2x(4x(2(x-2)x - 3) + 19) + 1)\text{erfi}(\frac{1}{2})). \end{aligned}$$

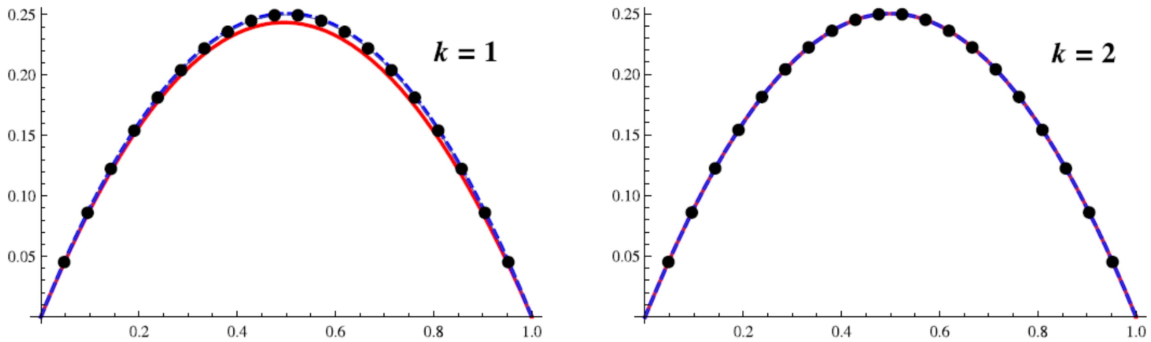

Figure 1. **The graphs of the exact and approximate solutions for $k = 1, 2$.**

Remark 3.2. *The results show that the error bounds for $k = 0, 1, 2$. It certainly appears that the convergence of the lower solution (s_k (dashed)) and upper solution (S_k (solid)) to the exact solution (u (dotted)) is very rapid and the error is almost negligible after only three iterations.*

4. Conclusions

In this paper, we have established an efficient algorithm based on a monotone iterative sequences for the solution of a class of nonlinear Volterra-Fredholm integro-differential equations. This algorithm involves a clear description of constructing two sequences of increasing lower solutions, $\{s_k\}$ and decreasing upper solutions, $\{S_k\}$, which are uniformly convergent. Numerical results has proved the efficiency of the proposed algorithm in terms of accuracy and rapid convergence of the proposed technique.

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