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On Extended Tempered Fractional Calculus

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Abstract: In this paper we will introduce an extension of the Riemann-Liouville tempered fractional derivative (integral) replacing the exponential factor by the one parameter Mittag-Leffler function. We will calculate the Fourier Transform of this new tempered fractional operator.

Keywords: Fractional Calculus, Tempered Fractional Calculus, Mittag-Leffler function, Fourier Transform.

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1. Introduction and Preliminaries

It is known the importance of the fractional calculus in phenomena exemplification studied by diverse sciences (see [1–3, 5–10] and [12]). The tempered fractional calculus is a generalization of the fractional calculus, it is natural to find in its focused area of study problems that generalize the pre-existent ones, for example, tempered fractional diffusion equations, tempered fractional Browian motion, turbulence (for further details see [1, 4, 8] and [12]). In this paper we will introduce an extension of the Riemann-Liouville tempered fractional derivative (integral) replacing the exponential factor by the one parameter Mittag-Leffler function. We will calculate the Fourier Transform of this new tempered fractional operator. To do this we will start recalling some definitions and Lemmas.

Definition 1.1. *Let $f \in W^{1,2}[a,b]$, a Sobolev space, $\lambda \geq 0$. The left and the right Riemann-Liouville tempered fractional integral of order $\alpha > 0$ respectively, is defined as*

$$I_{a,x}^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) e^{-\lambda(x-s)} ds \tag{1}$$

$$I_{x,b}^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s) e^{-\lambda(s-x)} ds \tag{2}$$

where Γ presents the Gamma Euler, $I_{a,x}^{\alpha}$ denoted the left Riemann-Liouville factional integral

$$I_{a,x}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) ds \tag{3}$$

and $I_{x,b}^{\alpha}$ denotes the right Riemann-Liouville fractional integral

$$I_{x,b}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (s-x)^{\alpha-1} f(s) ds \tag{4}$$

Note that if $\lambda = 0$ the tempered fractional integral (1), (2) it reduces to the classical Riemann-Liouville fractional integral (3), (4) see(cf. [2])

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Definition 1.2. *Let $f \in W^{1,2}[a,b]$, a Sobolev space, $\lambda \geq 0$, $n-1 \leq \alpha < n$, $n \in N$. The left and the right Riemann-Liouville tempered fractional derivative of order $\alpha > 0$ respectively, is defined as*

$$\begin{aligned} D_{a,x}^{\alpha,\lambda} f(x) &= \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-s)^{n-\alpha-1} f(s) e^{-\lambda(x-s)} ds\right) \\ &= D^n \left(I_{a,x}^{n-\alpha,\lambda} f(x)\right) \end{aligned} \tag{5}$$

$$\begin{aligned} D_{x,b}^{\alpha,\lambda} f(x) &= \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_x^b (s-x)^{n-\alpha-1} f(s) e^{-\lambda(s-x)} ds\right) \\ &= D^n \left(I_{x,b}^{n-\alpha,\lambda} f(x)\right) \end{aligned} \tag{6}$$

where $D_{a,x}^{\alpha}$ denoted the left Riemann-Liouville fractional derivative

$$D_{a,x}^{\alpha} f(x) = \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_a^x (x-s)^{n-\alpha-1} f(s) ds\right) \tag{7}$$

and $D_{x,b}^{\alpha}$ denoted the right Riemann-Liouville fractional derivative

$$D_{x,b}^{\alpha} f(x) = \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_x^b (s-x)^{n-\alpha-1} f(s) ds\right) \tag{8}$$

Note that if $\lambda = 0$ the left and right tempered fractional derivative (5), (6) it reduces to the classical left and right Riemann-Liouville fractional derivative (7), (8) (see [2])

Lemma 1.3. *Let $f \in W^{1,2}[a,b]$, a Sobolev Space. The Fourier Transform of the left and right Riemann-Liouville tempered fractional derivative (or integral) is*

$$\mathcal{F}[I_{a,x}^{\alpha,\lambda} f(x)](w) = (\lambda + iw)^{-\alpha}\, \mathcal{F}[f(x)](w) \tag{9}$$

$$\mathcal{F}[I_{x,b}^{\alpha,\lambda} f(x)](w) = (\lambda - iw)^{-\alpha}\, \mathcal{F}[f(x)](w) \tag{10}$$

$$\mathcal{F}[D_{a,x}^{\alpha,\lambda} f(x)](w) = (iw)^n (\lambda + iw)^{\alpha-n}\, \mathcal{F}[f(x)](w) \tag{11}$$

$$\mathcal{F}[D_{x,b}^{\alpha,\lambda} f(x)](w) = (iw)^n (\lambda - iw)^{\alpha-n}\, \mathcal{F}[f(x)](w) \tag{12}$$

Note that if $\lambda = 0$ the Fourier transform of the left and right Riemann-Liouville tempered fractional derivative (or integral) (9), (10), (11), (12), it reduces to the Fourier transform of the left and right Riemann-Liouville fracional derivative (or integral). For further details see (see.[2, 9])

2. Main Result

Let us following generalized Mittag-Leffler function, is given by:

$$E_\alpha(-x^\alpha) = \sum_{k=0}^{\infty} \frac{(-x)^{\alpha k}}{\Gamma(\alpha k + 1)} \tag{13}$$

The Taylor series of $e^{-\lambda(x-s)}$ at point x is given by:

$$e^{-\lambda(x-s)} = \sum_{k=0}^{\infty} \frac{[-\lambda(x-s)]^k}{k!} \tag{14}$$

If we replace (14) in (1) we have that:

$$\begin{aligned} I_{a,x}^{\alpha,\lambda} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) \sum_{k\geq 0} \frac{[-\lambda(x-s)]^k}{k!} ds \\ &= \frac{1}{\Gamma(\alpha)} \sum_{k\geq 0} \frac{-\lambda^k}{k!} \int_a^x (x-s)^{\alpha-1} f(s)(x-s)^k ds \end{aligned}$$

If in the previous expression, we replace $k!$ with $\Gamma(\mu k+1)$ and $(x-s)^k$ with $(x-s)^{\mu k}$, we obtain:

$$I_{a,x}^{\alpha,\lambda} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) \sum_{k\geq 0} \frac{[-\lambda(x-s)^\mu]^k}{\Gamma(\mu k+1)} ds \tag{15}$$

Taking into account that:

$$\sum_{k\geq 0} \frac{[-\lambda(x-s)^\mu]^k}{\Gamma(\mu k+1)} = E_\mu[-\lambda(x-s)^\mu] \tag{16}$$

replacing (16) in (15), we obtain the following

Definition 2.1. *Let $f \in W^{1,2}[a,b]$, a Sobolev space, $\lambda \geq 0$, $\mu \in \mathbb{R}^+$. The left and the right extended Riemann-Liouville tempered fractional integral of order $\alpha > 0$ respectively is defined as*

$$I_{a,x}^{\alpha,\lambda,\mu} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-s)^{\alpha-1} f(s) E_\mu[-\lambda(x-s)^\mu] ds \tag{17}$$

$$I_{x,b}^{\alpha,\lambda,\mu} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (s-x)^{\alpha-1} f(s) E_\mu[-\lambda(s-x)^\mu] ds \tag{18}$$

Definition 2.2. *Let $f \in W^{1,2}[a,b]$, a Sobolev space, $\lambda \geq 0$, $\mu \in \mathbb{R}^+$, $n-1 \leq \alpha < n$, $n \in N$. The left and the right extended Riemann-Liouville tempered fractional derivative of order $\alpha > 0$ respectively is defined as*

$$\begin{aligned} D_{a,x}^{\alpha,\lambda,\mu} f(x) &= \frac{d^n}{dx^n} \left(\frac{1}{\Gamma(n-\alpha)} \int_a^x (x-s)^{n-\alpha-1} f(x) E_\mu[-\lambda(x-s)^\mu] ds \right) \\ &= D^n \left(I_{a,x}^{n-\alpha,\lambda,\mu} f(x) \right) \end{aligned} \tag{19}$$

$$\begin{aligned} D_{x,b}^{\alpha,\lambda,\mu} f(x) &= \frac{d^n}{dx^n} \left(\frac{1}{\Gamma(n-\alpha)} \int_x^b (s-x)^{n-\alpha-1} f(x) E_\mu[-\lambda(s-x)^\mu] ds \right) \\ &= D^n \left(I_{x,b}^{n-\alpha,\lambda,\mu} f(x) \right) \end{aligned} \tag{20}$$

Note that $I_{a,x}^{\alpha,\lambda,\mu} f(x) \to I_{a,x}^{\alpha,\lambda} f(x)$, $I_{x,b}^{\alpha,\lambda,\mu} f(x) \to I_{a,x}^{\alpha,\lambda} f(x)$ when $\mu \to 1$ and $D_{a,x}^{\alpha,\lambda,\mu} f(x) \to D_{a,x}^{\alpha,\lambda} f(x)$, $D_{x,b}^{\alpha,\lambda,\mu} f(x) \to D_{a,x}^{\alpha,\lambda} f(x)$ when $\mu \to 1$.

Lemma 2.3. *Let $f \in W^{1,2}[a,b]$, a Sobolev Space. The Fourier Transform of the left and right the extended Riemann-Liouville tempered fractional integral is*

$$\mathcal{F}[I_{a,x}^{\alpha,\lambda,\mu} f(x)](w) = \left(\lambda^{\frac{1}{\mu}} + iw\right)^{-\alpha} \mathcal{F}[f(x)](w) \tag{21}$$

$$\mathcal{F}[I_{x,b}^{\alpha,\lambda,\mu} f(x)](w) = \left(\lambda^{\frac{1}{\mu}} - iw\right)^{-\alpha} \mathcal{F}[f(x)](w) \tag{22}$$

Proof. Taking into account that $\mathcal{F}[f*g](w)=\mathcal{F}[f](w)\cdot\mathcal{F}[g](w)$, $\mathcal{L}\left\{x^{\alpha}\right\}(s)=\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$, $(\alpha)_n=\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}$ Symbol Pochhamer and Binomial Theorem, we obtain

$$\begin{aligned}
\mathcal{F}[I_{a,x}^{\alpha,\lambda,\mu}f(x)](w) &= \frac{1}{\Gamma(\alpha)}\mathcal{F}\left[f(x)*x^{\alpha-1}E_{\mu}(-\lambda x^{\mu})\right](w)\\
&= \frac{1}{\Gamma(\alpha)}\mathcal{F}[f(x)](w)\cdot\mathcal{F}\left[x^{\alpha-1}E_{\mu}(-\lambda x^{\mu})\right](w)\\
&= \frac{1}{\Gamma(\alpha)}\mathcal{F}[f(x)](w)\int_{\mathbb{R}}e^{-iwx}\left(x^{\alpha-1}E_{\mu}(-\lambda x^{\mu})\right)dx\\
&= \frac{1}{\Gamma(\alpha)}\mathcal{F}[f(x)](w)\int_{\mathbb{R}}e^{-iwx}x^{\alpha-1}\sum_{k\geq 0}\frac{(-\lambda x^{\mu})^k}{\Gamma(\mu k+1)}dx\\
&= \frac{1}{\Gamma(\alpha)}\mathcal{F}[f(x)](w)\sum_{k\geq 0}\frac{(-\lambda)^k}{\Gamma(\mu k+1)}\int_{\mathbb{R}}e^{-iwx}x^{\alpha+\mu k-1}dx
\end{aligned}$$

If we make a change of variable $s=iw$, we obtain

$$\begin{aligned}
\mathcal{F}[I_{a,x}^{\alpha,\lambda,\mu}f(x)](w) &= \frac{1}{\Gamma(\alpha)}\mathcal{F}[f(x)](w)\sum_{k\geq 0}\frac{(-\lambda)^k}{\Gamma(\mu k+1)}\int_{\mathbb{R}}e^{-sx}x^{\alpha+\mu k-1}dx\\
&= \frac{1}{\Gamma(\alpha)}\mathcal{F}[f(x)](w)\sum_{k\geq 0}\frac{(-\lambda)^k}{\Gamma(\mu k+1)}\mathcal{L}\left\{x^{\alpha+\mu k-1}\right\}(s)\\
&= \frac{1}{\Gamma(\alpha)}\mathcal{F}[f(x)](w)\sum_{k\geq 0}\frac{(-1)^k(\lambda)^k}{\Gamma(\mu k+1)}\frac{\Gamma(\alpha+\mu k)}{s^{\alpha+\mu k}}\\
&= \frac{1}{\Gamma(\alpha)}\mathcal{F}[f(x)](w)\sum_{k\geq 0}\frac{(-1)^k(\lambda)^k}{\Gamma(\mu k+1)}\Gamma(\alpha+\mu k)\cdot iw^{-\alpha-\mu k}\\
&= \mathcal{F}[f(x)](w)\sum_{k\geq 0}\frac{(-1^{\frac{1}{\mu}})^{\mu k}(\alpha)_{\mu k}}{\Gamma(\mu k+1)}(\lambda^{\frac{1}{\mu}})^{\mu k}\cdot iw^{-\alpha-\mu k}\\
&= \mathcal{F}[f(x)](w)\sum_{k\geq 0}\frac{(-1)^{\mu k}(-(-\alpha)_{\mu k}}{\Gamma(\mu k+1)}(\lambda^{\frac{1}{\mu}})^{\mu k}\cdot iw^{-\alpha-\mu k}\\
&= \mathcal{F}[f(x)](w)\sum_{k\geq 0}\binom{-\alpha}{\mu k}(\lambda^{\frac{1}{\mu}})^{\mu k}\cdot iw^{-\alpha-\mu k}\\
&= \left(\lambda^{\frac{1}{\mu}}+iw\right)^{-\alpha}\mathcal{F}[f(x)](w)
\end{aligned}$$

□

Analogously to the previous case, it may be proved that

$$\mathcal{F}[I_{x,b}^{\alpha,\lambda,\mu}f(x)](w)=\left(\lambda^{\frac{1}{\mu}}-iw\right)^{-\alpha}\mathcal{F}[f(x)](w)$$

Lemma 2.4. *Let $f\in W^{1,2}[a,b]$, a Sobolev Space. The Fourier Transform of the left and right extended Riemann-Liouville tempered fractional derivative is*

$$\mathcal{F}[D_{a,x}^{\alpha,\lambda,\mu}f(x)](w)=(-iw)^n\left(\lambda^{\frac{1}{\mu}}+iw\right)^{\alpha-n}\mathcal{F}[f(x)](w) \tag{23}$$

$$\mathcal{F}[D_{x,b}^{\alpha,\lambda,\mu}f(x)](w)=(-iw)^n\left(\lambda^{\frac{1}{\mu}}-iw\right)^{\alpha-n}\mathcal{F}[f(x)](w) \tag{24}$$

Proof. Taking into account the definition (19) and (20), we obtain

$$\begin{aligned}
\mathcal{F}[D_{a,x}^{\alpha,\lambda,\mu}f(x)](w) &= \mathcal{F}[D^n\left(I_{a,x}^{n-\alpha,\lambda,\mu}f(x)\right)](w)\\
&= (iw)^n\mathcal{F}[I_{a,x}^{n-\alpha,\lambda,\mu}f(x)](w)\\
&= (iw)^n\left(\lambda^{\frac{1}{\mu}}+iw\right)^{\alpha-n}\mathcal{F}[f(x)](w)
\end{aligned}$$

□

Analogous to the previous case, it is proved that

$$\mathcal{F}[D_{x,b}^{\alpha,\lambda,\mu} f(x)](w) = (iw)^n \left(\lambda^{\frac{1}{\mu}} - iw\right)^{\alpha-n} \mathcal{F}[f(x)](w)$$

Note that if $\mu = 1$ the Fourier transform of the left and right on the extended Riemann-Liouville tempered fractional derivative (or integral) it reduce to the Fourier transform of the left and right Riemann-Liouville tempered fractional derivative (or integral)

References

[1] Albaro Cartea and Diego del Castillo-Negrete, *Fractional diffusions model of option prices in marhets with jumps*, Physica A: Statistirel Mechanics and its Appllications, 374(2)(2016), 749-763.

[2] Changpin Li and Weihua Deng, *Remarks on Fractional Derivatives*, Applied Mathematics and Computation, 187(2)(2007), 777-784.

[3] Dumitru Baleanu, *Fractional Calculus, Model and numerical Methods*, Vol 3, World Scientific, (2012).

[4] Farzad Sabzihar, Marh M. Meerschaert and Jinghua Chen, *Tempered Fractional Calculus*, Journal of Computational Physics, (2014).

[5] I. M. Sokolov, A. V. Chechkin and J. Ktafler, *Fractional Diffusion Equation for a power-law-fruncated levy process*, Physica A: Statistirel Mechanics and its Appllications, 336(3)(2004), 245-251.

[6] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, (1999).

[7] M. H. Chen and W. H. Deng, *Discretized fractional subtantial calculus*, ESAIM: Math. Model., 49(2)(2015), 373-394.

[8] O. Maron and E. Momoniat, *A comparison of numerical solutions of fractional diffusion model in finance*, Nonl Anal: R.W.A, 10(6)(2009), 3435-3442.

[9] S. Mubeen and M. Habibullah, *k-fractional Integrals and Application*, Int. J. Contemp. Math. Science, 7(2)(2012).

[10] M. M. Meerschaert and F. Sabzihar, *Tempered fractional Browian motion*, Stat. Probab. Lett, 83(2013), 2269-2275.

[11] S. Sanko, A. Kilbas and O. Marichev, *Fractional integral and derivative: theory and applications*, Gorden and Breach, Yverdon, (1993).

[12] V. Pipiras and M. Taqqu, *Integration questions related to fractional Browian motion, probab*, Theory Relat. Field, 118(2000), 251-291.