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# Stochastic Inventory System with Multi Optional Service and Finite Source

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**Abstract:** In this article, we consider a continuous review perishable inventory system with a finite number of homogeneous sources of demands. The maximum storage capacity is fixed as $S$. The operating policy is $(s, S)$ policy, that is, whenever the inventory level drops to $s$, an order for $Q(= S - s)$ items is placed. The ordered items are received after a random time which is distributed as exponential. The life time of each items is assumed to be exponential. All arriving customers demand first the essential service (regular service) and some of them may further demand one of other optional services: Type 1, Type 2, ..., and Type $N$ service. The service times of the essential service and of the Type $j$ $(j = 1, 2, \ldots, N)$ service are assumed to be exponentially distributed. The joint probability distribution of the number of customers in the waiting hall and the inventory level is obtained for the steady state case. We have derived the Laplace-Stieljes transforms of waiting time distribution of customers in the waiting hall. Some important system performance measures in the steady state are derived, and the long-run total expected cost rate is also derived.

**Keywords:** $(s, S)$ policy, Continuous review, Perishable commodity, Optional service, Markov process, Finite population.

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## 1. Introduction

Many authors assumed that after completion of the service immediately the customers leave the system. But in many real life situation, all the arriving customers first require an essential service and only some may require additional optional service immediately after completion of the first essential service by the same server. The concept of the additional optional service with queue has been studied by several researchers in the past. As a related work we refer [3–5]. In this paper, we consider a continuous review $(s, S)$ perishable inventory system with $N$ additional options for service and a finite populations is motivated by the service facility system with restricted customers for example military canteen providing service to soldiers or a company canteen serving the members of the specific working community in the company. Machine service problems within an industry is also a problem which motivated me to create the present stochastic model. The problem we consider is more relevant to the real life situation. For a detailed study in stochastic inventory models with finite source the reader is referred to [6–9, 11–16]. The same model with infinite population has discussed by Jeganathan [10]. This paper is presented as follows. In the next section, the mathematical model and the notations used in this paper are described. Analysis of the model and the steady state solutions of the model are obtained in section 3. In section 4, we have derived the Laplace-Stieltjes transform of waiting time distribution of customers in the waiting hall. Some key system performance measures are derived in section 5. In the last section, we have derived the total expected cost rate.

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## 2. Mathematical Model

Consider a continuous review perishable inventory system at a service facility with the maximum capacity of $S$ units and $N$ additional options for service. The demands are generated by a finite number of homogeneous sources $M$, $0 < M < \infty$ and the demand time points form a quasi-random distribution with parameter $\lambda(> 0)$. That is, the probability that any particular source generates a demand in any interval $(t, dt)$ is $\lambda dt + o(dt)$ as $dt \to 0$ if the source is idle at time $t$, and zero if the source is in the service facility at time $t$, independently of the behavior of any other sources. The waiting customers receive their service one by one and they demand single item. Furthermore, customers who arrive and find either server busy or inventory level is zero must wait in the waiting hall until the server is available with positive inventory level. $(s, S)$ ordering policy is adapted in this paper. The lead time is assumed to be distributed as negative exponential with parameter $\beta(> 0)$. The life time of the commodity is assumed to be distributed as negative exponential with parameter $\gamma(> 0)$. We have assumed that an item of inventory that makes it into the service process cannot perish while in service. There is a single server that provides the first essential service(regular service) as well as one of the $N$ optional services (Type 1, Type 2, ..., Type $N$) to each arriving customer. The items are issued to the demanding customers only after some random time due to some service on it. In this article the latter type of service referred to as first essential service(regular service). The first essential service of a customer is assumed to be exponentially distributed with parameter $\mu_0$. As soon as the first essential service of a customer is completed, then with probability $r_j$ the customer may ask for Type $j$ service (i.e immediately customer requests additional service on their item), in which case his Type $j$ service will immediately commence, or with probability $r_0$ he may opt to leave the system, in which case if both the inventory level and waiting hall size are positive, the customer will be taken for first essential service immediately by the server. Otherwise (i.e., either inventory level is zero or customer level is zero or both), server becomes idle. The service time of the $j$th optional service is assumed to be exponential with parameter $\mu_{\alpha_j}$, where $j = 1, 2, \ldots, N$ and $\sum_{j=0}^{N} r_j = 1$. The following notions are used in this paper.

$$
\begin{aligned}
\pi &: \text{a column vector of appropriate dimension containing all ones} \\
\delta_{ij} &: \begin{cases} 1 & \text{if} j = i \\ 0 & \text{otherwise} \end{cases} \\
k \in V_i^j &: k = i, i+1, \ldots j \\
H(x) &: \text{Heaviside function}
\end{aligned}
$$

## 3. Analysis

Let $L(t)$, $Y(t)$ and $X(t)$ respectively, denote the inventory level, the server status and the number of customers in the waiting hall at time $t$. Further, server status $Y(t)$ be defined as follows:

$$
Y(t) : \begin{cases}
-1, & \text{if the server is idle at time t,} \\
0, & \text{if the server is busy with essential service at time t,} \\
1, & \text{if the server is busy with Type 1 service at time t,} \\
2, & \text{if the server is busy with Type 2 service at time t,} \\
3, & \text{if the server is busy with Type 3 service at time t,} \\
\vdots & \vdots \\
N-1, & \text{if the server is busy with Type N-1 service at time t,} \\
N, & \text{if the server is busy with Type N service at time t,}
\end{cases}
$$

From the assumptions made on the input and output processes, it can be shown that the stochastic process $I(t) = \{(L(t), Y(t), X(t)), t \geq 0\}$ is a continuous time Markov chain with state space given by $E = E_1 \cup E_2 \cup E_3 \cup E_4$, where

$$\begin{aligned}
E_1 &: \{(0, -1, i_3) \mid i_3 = 0, 1, 2, \ldots, M, \} \\
E_2 &: \{(i_1, -1, 0) \mid i_1 = 1, 2, \ldots, S, \} \\
E_3 &: \{(i_1, 0, i_3) \mid i_1 = 1, 2, \ldots, S,\ i_3 = 1, 2, \ldots, M, \} \\
E_4 &: \{(i_1, i_2, i_3) \mid i_1 = 0, 1, 2, \ldots, S,\ i_2 = 1, 2, \ldots, N,\ i_3 = 1, 2, \ldots, M, \}
\end{aligned}$$

Then the state space can be ordered as $(\lll 0 \ggg, \lll 1 \ggg, \ldots, \lll S \ggg)$ and infinitesimal generator $\Theta$ can be conveniently written in a block partitioned matrix with entries

$$\Theta = \begin{array}{c} S \\ S-1 \\ \vdots \\ s+1 \\ s \\ s-1 \\ \vdots \\ 1 \\ 0 \end{array} \left( \begin{array}{cccccccccc}
A_S & B_S & & & & & & & & \\
 & A_{S-1} & B_{S-1} & & & & & & & \\
 & & \cdots & & & & & & & \\
 & & & \cdots & A_{s+1} & B_{s+1} & & & & \\
C & & & & & A_s & B_s & & & \\
 & C & & & & & A_{s-1} & & & \\
 & & \cdots & & & & \cdots & & & \\
 & & C & & & & & \cdots & A_1 & B_1 \\
 & & & C_1 & & & & & & A_0
\end{array} \right)$$

It may be noted that $A_{i_1}$, $B_{i_1}$, $i_1 = 1, 2, \ldots, S$, $A_0$, $C_1$ and $C$ are square matrices of order $(N+1)M+1$.

- The sub-matrix $C_1$ and $C$ are denotes the transitions from (i) to (i + Q) $(i = 0, 1, \ldots, s)$;
- The sub-matrix $B_i$ $(i = 1, \ldots, S)$ denotes the transitions from (i) to (i-1);
- The sub-matrix $A_i$ $(i = 0, 1, \ldots, S)$ denotes the transitions from (i) to (i);

## 3.1. Steady State Analysis

It can be seen from the structure of $\Theta$ that the homogeneous Markov process $\{(L(t), Y(t), X(t)) : t \geq 0\}$ on the finite space $E$ is irreducible, aperiodic and persistent non-null. Hence the limiting distribution

$$\phi^{(i_1, i_2, i_3)} = \lim_{t \to \infty} Pr[L(t) = i_1, Y(t) = i_2, X(t) = i_3 | L(0), Y(0), X(0)] \text{ exists.}$$

Let $\mathbf{\Phi}$ denote the steady state probability vector of the generator $\Theta$. The computation of steady state probability vector $\mathbf{\Phi} = (\mathbf{\Phi^{(0)}}, \mathbf{\Phi^{(1)}}, \ldots, \mathbf{\Phi^{(S)}})$, by solving the following set of equations,

$$\begin{aligned}
\mathbf{\Phi^{i_1}} B_{i_1} + \mathbf{\Phi^{i_1 - 1}} A_{i_1 - 1} &= \mathbf{0}, & i_1 &= 1, 2, \ldots, Q, \\
\mathbf{\Phi^{i_1}} B_{i_1} + \mathbf{\Phi^{i_1 - 1}} A_{i_1 - 1} + \mathbf{\Phi^{(i_1 - 1 - Q)}} C_1 &= \mathbf{0}, & i_1 &= Q + 1, \\
\mathbf{\Phi^{i_1}} B_{i_1} + \mathbf{\Phi^{i_1 - 1}} A_{i_1 - 1} + \mathbf{\Phi^{(i_1 - 1 - Q)}} C &= \mathbf{0}, & i_1 &= Q + 2, Q + 3, \ldots, S, \\
\mathbf{\Phi^{S}} A_S + \mathbf{\Phi^{s}} C &= \mathbf{0}.
\end{aligned}$$

subject to conditions $\mathbf{\Phi}\Theta = \mathbf{0}$ and $\sum\sum\sum_{(i_1, i_2, i_3)} \phi^{(i_1, i_2, i_3)} = 1$. This is done by the following algorithm.

**Step 1.** Solve the following system of equations to find the value of $\mathbf{\Phi^{Q}}$

$$\mathbf{\Phi^{Q}}\left[\left\{(-1)^{Q}\sum_{j=0}^{s-1}\left[\left(\overset{s+1-j}{\underset{k=Q}{\Omega}}B_kA_{k-1}^{-1}\right)CA_{S-j}^{-1}\left(\overset{Q+2}{\underset{l=S-j}{\Omega}}B_lA_{l-1}^{-1}\right)\right]\right\}B_{Q+1}+A_Q+\left\{(-1)^{Q}\overset{1}{\underset{j=Q}{\Omega}}B_jA_{j-1}^{-1}\right\}C\right]=0,$$

and

$$\mathbf{\Phi^{Q}}\left[\sum_{i_1=0}^{Q-1}\left((-1)^{Q-i_1}\overset{i_1+1}{\underset{j=Q}{\Omega}}B_jA_{j-1}^{-1}\right)+I+\sum_{i_1=Q+1}^{S}\left((-1)^{2Q-i_1+1}\sum_{j=0}^{S-i_1}\left[\left(\overset{s+1-j}{\underset{k=Q}{\Omega}}B_kA_{k-1}^{-1}\right)CA_{S-j}^{-1}\left(\overset{i_1+1}{\underset{l=S-j}{\Omega}}B_lA_{l-1}^{-1}\right)\right]\right)\right]\boldsymbol{\pi} \;=\; 1.$$

**Step 2.** Compute the values of

$$\begin{aligned}\mathbf{\Omega_{i_1}} &= (-1)^{Q-i_1}\mathbf{\Phi^{Q}}\overset{i_1+1}{\underset{j=Q}{\Omega}}B_jA_{j-1}^{-1}, && i_1=Q-1,Q-2,\dots,0\\ &= (-1)^{2Q-i_1+1}\mathbf{\Phi^{Q}}\sum_{j=0}^{S-i_1}\left[\left(\overset{s+1-j}{\underset{k=Q}{\Omega}}B_kA_{k-1}^{-1}\right)CA_{S-j}^{-1}\left(\overset{i_1+1}{\underset{l=S-j}{\Omega}}B_lA_{l-1}^{-1}\right)\right], && i_1=S,S-1,\dots,Q+1\\ &= I, && i_1=Q\end{aligned}$$

**Step 3.** Using $\mathbf{\Phi^{(Q)}}$ and $\Omega_{i_1}, i_1=0,1,\dots,S$, calculate the value of $\mathbf{\Phi^{(i_1)}}, i_1=0,1,\dots,S$. That is,

$$\mathbf{\Phi^{(i_1)}} \;=\; \mathbf{\Phi^{(Q)}}\Omega_{i_1}, \quad i_1=0,1,\dots,S.$$

## 4. Waiting Time of the Customers

In this section, our aim is to derive the waiting time for the customer. We deal with the arriving (tagged) customer waiting time, defined as the time between the arrival epoch of a customer till the instant at which the customer request is satisfied. We will represent this continuous random variable as $W$. The objective is to describe the probability that a customer has to wait, the distribution of the waiting time and $n^{th}$ order moments. Note that $W$ is zero when the arriving customer finds positive stock and the server is free. Consequently, the probability that the customer does not have to wait is given by

$$P\{W=0\} \;=\; \sum_{i_1=1}^{S}\phi^{(i_1,-1,0)}$$

In order to get the distribution of $W$, we will define some auxiliary variables. Let us consider the Markov process at an arbitrary time $t$ and suppose that it is at state $(i_1,i_2,i_3), i_3>0$. We tag any of those waiting customer and $W_{(i_1,i_2,i_3)}$ denotes the time until the selected customer gets the desired item. Let $W^*(y)=E[e^{-yW}]$ and $W^*_{(i_1,i_2,i_3)}(y)=E[e^{-yW_{(i_1,i_2,i_3)}}]$ be the corresponding Laplace-Stieltjes transforms for unconditional and conditional waiting time.
Let $W^*(y)=\left(W^*_{(0)}(y),W^*_{(1)}(y),\dots,W^*_{(S)}(y)\right)$, where

$$\begin{aligned}W^*_{(0)}(y) &= \left(W^*_{(0,-1)}(y),W^*_{(0,1)}(y),W^*_{(0,2)}(y),\dots,W^*_{(0,N)}(y)\right);\\ W^*_{(i_1)}(y) &= \left(W^*_{(i_1,-1)}(y),W^*_{(i_1,0)}(y),W^*_{(i_1,1)}(y),W^*_{(i_1,2)}(y)\dots,W^*_{(i_1,N)}(y)\right);\ 1\le i_1\le S;\\ W^*_{(0,-1)}(y) &= \left(W^*_{(0,-1,1)}(y),W^*_{(0,-1,2)}(y),\dots,W^*_{(0,-1,M)}(y)\right);\\ W^*_{(i_1,0)}(y) &= \left(W^*_{(i_1,0,1)}(y),W^*_{(i_1,0,2)}(y),\dots,W^*_{(i_1,0,M)}(y)\right);1\le i_1\le S;\\ W^*_{(i_1,i_2)}(y) &= \left(W^*_{(i_1,i_2,1)}(y),W^*_{(i_1,i_2,2)}(y),\dots,W^*_{(i_1,i_2,M)}(y)\right);\ 0\le i_1\le S;\ 1\le i_2\le N;\end{aligned}$$

The conditional waiting time of the primary customer is given in the following theorem.

**Theorem 4.1.** *The Laplace-Stieltjes transforms* $\{W^*_{(i_1,i_2,i_3)}(y), (i_1,i_2,i_3) \in E^*(= E_1 \cup E_3 \cup E_4 \cup \{**\}), i_3 > 0\}$ *satisfy the following system*

$$\Theta_1(y)W^*(y) = z, \tag{1}$$

*where* $\Theta_1(y) = \Theta_1 - yI_{(S+1)(N+1)M}$*, the matrix* $\Theta_1$ *is obtained from* $\Theta$ *by removing the following states* $(0,-1,0) \cup \{(i_1,-1,0), 1 \leq i_1 \leq S\}$ *form the state space and we denote* $\{**\}$ *as an absorbing state. The absorption occurs, when the customer gets his requested item.*

*Proof.* We use a first step analysis to get the conditional waiting time is
For $i_1 = 0, \quad i_2 = -1, \quad 1 \leq i_3 \leq M$

$$w_1 W^*_{(i_1,i_2,i_3)}(y) - (M-i_3)\lambda\bar{\delta}_{i_3M}W^*_{(i_1,i_2,i_3+1)}(y) - \beta\delta_{i_30}W^*_{(i_1+Q,i_2,i_3)}(y) - \beta\bar{\delta}_{i_30}W^*_{(i_1+Q,i_2,i_3)}(y) = 0 \tag{2}$$

where $w_1 = y + (M-i_3)\lambda\bar{\delta}_{i_3M} + \beta\delta_{i_30} + \beta\bar{\delta}_{i_30}$.
For $1 \leq i_1 \leq S, \quad i_2 = 0, \quad 1 \leq i_3 \leq M,$

$$\begin{aligned} w_2 W^*_{(i_1,i_2,i_3)}(y) &- (M-i_3)\lambda\bar{\delta}_{i_3M}W^*_{(i_1,i_2,i_3+1)}(y) - \beta H(s-i_1)W^*_{(i_1+Q,0,i_3)}(y) - (i_1-1)\gamma\bar{\delta}_{i_11}W^*_{(i_1-1,i_2,i_3)}(y) \\ &- r_0\mu_0\bar{\delta}_{i_11}\bar{\delta}_{i_31}W^*_{(i_1-1,i_2,i_3-1)}(y) - r_0\mu_0\delta_{i_11}W^*_{(i_1-1,-1,i_3-1)}(y) - r_0\mu_0\delta_{i_31}W^*_{(i_1-1,-1,i_3-1)}(y) \\ &- r_1\mu_0W^*_{(i_1,1,i_3)}(y) - r_2\mu_0W^*_{(i_1,2,i_3)}(y) - r_3\mu_0W^*_{(i_1,3,i_3)}(y)\ldots - r_N\mu_0W^*_{(i_1,N,i_3)}(y) = 0 \end{aligned} \tag{3}$$

where

$$w_2 = y + (M-i_3)\lambda\bar{\delta}_{i_3M} + \beta H(s-i_1) + (i_1-1)\gamma\bar{\delta}_{i_11} + r_0\mu_0\bar{\delta}_{i_11}\bar{\delta}_{i_31} + r_0\mu_0\delta_{i_11} + r_0\mu_0\delta_{i_31} + r_1\mu_0 + r_2\mu_0 + \ldots + r_N\mu_0$$

For $0 \leq i_1 \leq S, \quad 1 \leq i_2 \leq N, \quad 1 \leq i_3 \leq M,$

$$\begin{aligned} w_3 W^*_{(i_1,i_2,i_3)}(y) &- (M-i_3)\lambda\bar{\delta}_{i_3M}W^*_{(i_1,i_2,i_3+1)}(y) - i_1\gamma\bar{\delta}_{i_10}W^*_{(i_1-1,i_2,i_3)}(y) - \beta H(s-i_1)W^*_{(i_1+Q,i_2,i_3)}(y) \\ &- \mu_{i_2}\bar{\delta}_{i_10}W^*_{(i_1,-1,i_3-1)}(y) - \mu_{i_2}\delta_{i_11}\delta_{i_31}W^*_{(i_1,-1,i_3-1)}(y) - \mu_{i_2}\delta_{i_10}\bar{\delta}_{i_31}W^*_{(i_1,0,i_3-1)}(y) = 0 \end{aligned} \tag{4}$$

where $w_3 = y + (M-i_3)\lambda\bar{\delta}_{i_3M} + i_1\gamma\bar{\delta}_{i_10} + \beta H(s-i_1) + \mu_{i_2}\delta_{i_10} + \mu_{i_2}\delta_{i_11}\delta_{i_31} + \mu_{i_2}\delta_{i_10}\bar{\delta}_{i_31}$. By expressing equations (2)-(4) in matrix form we obtain the expression (1). □

**Theorem 4.2.** *The* $n^{th}$ *moments of the conditional waiting time is*

$$\Theta_1(y)\frac{d^{(n+1)}}{dy^{(n+1)}}W^*(y) - (n+1)\frac{d^{(n)}}{dy^{(n)}}W^*(y) = \mathbf{0} \tag{5}$$

*and* $\frac{d^{(n+1)}}{dy^{(n+1)}}W^*_{(i_1,i_2,i_3)}(y)|_{y=0} = E\left[W^{(n+1)}_{(i_1,i_2,i_3)}\right], (i_1,i_2,i_3) \in E^*$.

*Proof.* Differentiating $(n+1)$ times equation (1) and evaluating at $y = 0$, we get the following equations
For $i_1 = 0, \quad i_2 = -1, \quad 1 \leq i_3 \leq M$

$$w_4E\left[W^{(n+1)}_{(i_1,i_2,i_3)}\right] - (M-i_3)\lambda\bar{\delta}_{i_3M}E\left[W^{(n+1)}_{(i_1,i_2,i_3+1)}\right] - \beta\delta_{i_30}E\left[W^{(n+1)}_{(i_1+Q,i_2,i_3)}\right] - \beta\bar{\delta}_{i_30}E\left[W^{(n+1)}_{(i_1+Q,i_2,i_3)}\right] = (n+1)E\left[W^{(n)}_{(i_1,i_2,i_3)}\right]$$

where $w_4 = (M - i_3)\lambda\bar{\delta}_{i_3M} + \beta\delta_{i_30} + \beta\bar{\delta}_{i_30}$.

For $1 \leq i_1 \leq S, \quad i_2 = 0, \quad 1 \leq i_3 \leq M,$

$$\begin{aligned} w_5E\left[W^{(n+1)}_{(i_1,i_2,i_3)}\right] - (M-i_3)\lambda\bar{\delta}_{i_3M}E\left[W^{(n+1)}_{(i_1,i_2,i_3+1)}\right] - \beta H(s-i_1)E\left[W^{(n+1)}_{(i_1+Q,0,i_3)}\right] \\ -(i_1-1)\gamma\bar{\delta}_{i_11}E\left[W^{(n+1)}_{(i_1-1,i_2,i_3)}\right] - r_0\mu_0\bar{\delta}_{i_11}\bar{\delta}_{i_31}E\left[W^{(n+1)}_{(i_1-1,i_2,i_3-1)}\right] \\ -r_0\mu_0\delta_{i_11}E\left[W^{(n+1)}_{(i_1-1,-1,i_3-1)}\right] - r_0\mu_0\delta_{i_31}E\left[W^{(n+1)}_{(i_1-1,-1,i_3-1)}\right] - r_1\mu_0E\left[W^{(n+1)}_{(i_1,1,i_3)}\right] \\ -r_2\mu_0E\left[W^{(n+1)}_{(i_1,2,i_3)}\right] - r_3\mu_0E\left[W^{(n+1)}_{(i_1,3,i_3)}\right]\ldots - r_N\mu_0E\left[W^{(n+1)}_{(i_1,N,i_3)}\right] = (n+1)E\left[W^{(n)}_{(i_1,i_2,i_3)}\right] \end{aligned} \tag{6}$$

where

$$w_5 = (M-i_3)\lambda\bar{\delta}_{i_3M} + \beta H(s-i_1) + (i_1-1)\gamma\bar{\delta}_{i_11} + r_0\mu_0\bar{\delta}_{i_11}\bar{\delta}_{i_31} + r_0\mu_0\delta_{i_11} + r_0\mu_0\delta_{i_31} + r_1\mu_0 + r_2\mu_0 + \ldots + r_N\mu_0$$

For $0 \leq i_1 \leq S, \quad 1 \leq i_2 \leq N, \quad 1 \leq i_3 \leq M,$

$$\begin{aligned} w_6E\left[W^{(n+1)}_{(i_1,i_2,i_3)}\right] - (M-i_3)\lambda\bar{\delta}_{i_3M}E\left[W^{(n+1)}_{(i_1,i_2,i_3+1)}\right] - i_1\gamma\bar{\delta}_{i_10}E\left[W^{(n+1)}_{(i_1-1,i_2,i_3)}\right] \\ -\beta H(s-i_1)E\left[W^{(n+1)}_{(i_1+Q,i_2,i_3)}\right] - \mu_{i_2}\bar{\delta}_{i_10}E\left[W^{(n+1)}_{(i_1,-1,i_3-1)}\right] - \mu_{i_2}\delta_{i_11}\delta_{i_31}E\left[W^{(n+1)}_{(i_1,-1,i_3-1)}\right] \\ -\mu_{i_2}\delta_{i_10}\bar{\delta}_{i_31}E\left[W^{(n+1)}_{(i_1,0,i_3-1)}\right] = (n+1)E\left[W^{(n)}_{(i_1,i_2,i_3)}\right] \end{aligned} \tag{7}$$

where

$$w_6 \;=\; (M-i_3)\lambda\bar{\delta}_{i_3M} + i_1\gamma\bar{\delta}_{i_10} + \beta H(s-i_1) + \mu_{i_2}\delta_{i_10} + \mu_{i_2}\delta_{i_11}\delta_{i_31} + \mu_{i_2}\delta_{i_10}\bar{\delta}_{i_31}$$

Equations (6)-(8) are used to determine the unknowns $E\left[W^{(n+1)}_{(i_1,i_2,i_3)}\right], (i_1,i_2,i_3) \in E^*$ in terms of the moments of one order less. Noticing that $E\left[W^{(n)}_{(i_1,i_2,i_3)}\right] = 1$, for $n = 0$. We can obtain the moments up to a desired order in a recursive way. □

Next, we find the unconditional waiting time of the customer

**Theorem 4.3.** *The Laplace-Stieltjes transform of the unconditional waiting time is given by*

$$W^*(y) = \sum_{i_3=0}^{M-1}\phi^{(0,-1,i_3)}W^*_{(0,-1,i_3+1)}(y) + \sum_{i_1=1}^{S}\sum_{i_3=0}^{M-1}\phi^{(i_1,0,i_3)}W^*_{(i_1,0,i_3+1)}(y) + \sum_{i_1=0}^{S}\sum_{i_2=1}^{N}\sum_{i_3=1}^{M-1}\phi^{(i_1,i_2,i_3)}W^*_{(i_1,i_2,i_3+1)}(y) \tag{8}$$

*Proof.* By using PASTA property, the unconditional Laplace-Stieltjes transform of the waiting time with respect to all possible initial states $(i_1,i_2,i_3)$ and the corresponding states $(i_1,i_2,i_3-1)$ before an arrive we have

$$W^*(y) = \mathbf{\Phi}^{(i_1)}W^*_{(i_1)}(y), 0 \leq i_1 \leq S. \tag{9}$$

where $\mathbf{\Phi}$ is a steady-state probability vector. The expression (10) we get $W^*(y)$ for a given $y$. By using the Euler and Post-Widder algorithms in Abate and Whitt [1] for the nuerical inversion of $W^*(y)$, we get the waiting time distribution of $W$. □

In Corollary 1, we given the $n^{th}$ moments of the unconditional waiting time of the customer.

**Corollary 4.4.** *The $n^{th}$ moments of the unconditional waiting time is* $E[W^{(n)}] = \mathbf{\Phi}^{(i_1)}\frac{d^{(n)}}{dy^{(n)}}W^*_{(i_1)}(y)|_{y=0}$ $0 \leq i_1 \leq S$ *(i.e.)*

$$\begin{aligned} E[W^{(n)}] = \delta_{0n} + \left[\sum_{1_3=1}^{M-1}\phi^{(0,-1,i_3)}E\left[W^{(n)}_{(0,-1,i_3+1)}\right] + \sum_{i_1=1}^{S}\sum_{i_3=1}^{M-1}\phi^{(i_1,0,i_3)}E\left[W^{(n)}_{(i_1,0,i_3+1)}\right] \right. \\ \left. + \sum_{i_1=0}^{S}\sum_{i_2=1}^{N}\sum_{i_3=1}^{M-1}\phi^{(i_1,i_2,i_3)}E\left[W^{(n)}_{(i_1,i_2,i_3+1)}\right]\right](1-\delta_{0n}) \end{aligned}$$

Which provides the $n^{th}$ moments of the unconditional waiting time in terms of conditional moments of the same order.

## 5. System performance measures

In this section, we derive some measures of system performance in the steady state. Using this, we calculate the total expected cost rate.

(1). **Expected Inventory Level**

$$\eta_I = \sum_{i_1=1}^{S} i_1 \Phi^{(i_1)} \mathbf{e}$$

(2). **Expected Reorder Rate**

$$\eta_R = (s+1)\gamma\phi^{(s+1,-1,0)} + \sum_{i_2=1}^{N}\sum_{i_3=1}^{M}(s+1)\gamma\phi^{(s+1,i_2,i_3)} + \sum_{i_3=1}^{M}(r_0\mu_0 + s\gamma)\phi^{(s+1,0,i_3)}.$$

(3). **Expected Perishable Rate**

$$\eta_P = \sum_{i_1=1}^{S} i_1\gamma\phi^{(i_1,-1,0)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{M}(i_1-1)\gamma\phi^{(i_1,0,i_3)} + \sum_{i_1=1}^{S}\sum_{i_2=1}^{N}\sum_{i_3=1}^{M} i_1\gamma\phi^{(i_1,i_2,i_3)}$$

(4). **Expected Number of Customers in the Waiting Hall**

$$\Gamma = \sum_{i_1=1}^{S}\sum_{i_3=1}^{M} i_3\phi^{(i_1,0,i_3)} + \sum_{i_1=0}^{S}\sum_{i_2=1}^{N}\sum_{i_3=1}^{M} i_3\phi^{(i_1,i_2,i_3)}$$

(5). **Probability that Server is Busy with Essential Service**

$$\eta_{SB} = \sum_{i_1=1}^{S}\sum_{i_3=1}^{M}\phi^{(i_1,0,i_3)}$$

(6). **Probability that Server is Idle**

$$\eta_{SI} = \sum_{i_1=0}^{S}\phi^{(i_1,-1,0)}$$

(7). **Probability that Server is Busy with Optional Service**

$$\eta_{SO} = \sum_{i_1=0}^{S}\sum_{i_2=1}^{N}\sum_{i_3=1}^{M}\phi^{(i_1,i_2,i_3)}$$

(8). **Expected total cost:** The expected total cost per unit time (expected total cost rate) in the steady state for this model is defined to be

$$TC(S,s,N,M) = c_h\eta_I + c_s\eta_R + c_p\eta_P + c_w\eta_W,$$

$c_h$ : The inventory carrying cost per unit item per unit time,

$c_s$ : Setup cost per order,

$c_p$ : Perishable cost per unit item per unit time,

$c_w$ : Waiting time cost of a customer per unit time,

Substituting the values of $\eta$'s, we get

$$TC(S,s,N,M) = c_s\left[(s+1)\gamma\phi^{(s+1,-1,0)} + \sum_{i_3=1}^{M}\left((r_0\mu_0 + s\gamma)\phi^{(s+1,0,i_3)} + \sum_{i_2=1}^{N}\sum_{i_3=1}^{M}(s+1)\gamma\phi^{(s+1,i_2,i_3)}\right)\right] +$$
$$c_h\left[\sum_{i_1=1}^{S} i_1\phi^{(i_1)}\mathbf{e}\right] + c_p\sum_{i_1=1}^{S}\left[i_1\gamma\phi^{(i_1,0,0)} + \sum_{i_1=1}^{M}(i_1-1)\gamma\phi^{(i_1,0,i_3)} + \sum_{i_2=1}^{N}\sum_{i_3=1}^{M} i_1\gamma\phi^{(i_1,i_2,i_3)}\right] + c_w\left[\frac{\Gamma}{\eta_{AR}}\right]$$

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