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Products and Factors of Bresets

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Abstract: The aim of this paper is a. to generalize the conjunctive and co-normal products of digraphs to an arbitrary family of digraphs or sets with one binary relation each b. to introduce sectional product, for an arbitrary family of digraphs and study their properties and c. to introduce and study properties of 1. conjunctive factor 2. disjunctive factor and 3. sectional factor, for a set with a binary relation whose underlying set is a Cartesian product of a family of sets, thus introducing and elementarily studying factorization theory of digraphs. Further, several relations between these products and factors are studied along with algorithms to compute both products and factors.

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1. Introduction

Binary relations are one of the earliest relations known to both mankind and Mathematicians. Basic relations like reflexive, symmetric, transitive, irreflexive, anti-symmetric, cyclic, etcetera are all binary relations which play an important role in the studies of several order-structures like semi ordered set, well ordered set, totally ordered set, partial ordered set and higher-order structures like, join (meet) semi lattice, (distributive, modular, deMorgan) lattice, (join,meet) and still higher order structures like complete lattice, infinite (join, meet) distributive lattice, completely distributive complete lattice etcetera. Notice that the under lying object for all of them is a set with a binary relation. For more information in the studies of these higher ordered objects, one can refer to Szasz [25], Birkhoff [3] etcetera.

Another important class of objects, on a set with a binary relation, is the graph which is nothing but a finite set (of nodes) together with a binary relation (of edges). The Theory of Graphs is well known for its applications both in Hardware and Software of Computer Science. In Hardware, it is used in the feasibility, design and analysis of Circuits. For more details in this direction, one can refer to Charles Desoer and Ernest Kuh [5], Krishnaiyan Thulasiraman [26] and Narasingh Deo [19]. In Software, there are several applications for both notions and algorithms of Graph Theory. To name a few, shortest path and minimal spanning tree are chiefly studied in communication/transportation networks; various types of connectedness, cycles etcetera are used in Digital Image Processing/Detection; BFS (Breadth First Search), DFS (Depth First Search) are extensively used in Searching. For more information in this direction one can refer to Kenneth H Rosen [20], J.P. Trembly and R. Manohar [27] and Joe L. Mott, Abraham Kandel and Theodore P. Baker [13]. Interestingly, there are even computer languages like HINT (an extension of LISP), GRASPE (another extension of LISP), GEA (Graphic Extended ALGOL,

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an extension of ALGOL), GIRL (Graph Information Retrieval Language), GTPL (Graph Theoretic Processing Language) etcetera and also packages like GASP (Graph Algorithms Software Package), SPANTREE (To find a spanning tree in the given graph) to exclusively process Graph Theoretic ideas/algorithms.

Fuzzy binary relations were extensively studied in Murthy-Ravi [16]. In fact, in Murthy-Ravi [16], for any L-fuzzy set X, they construct a crisp set in such a way that there is a Galois connection between the set of all L-fuzzy reflexive (irreflexive, symmetric, antisymmetric, transitive) relations on the former and certain reflexive (irreflexive, symmetric, antisymmetric, transitive) relations on the later. The above Galois connection is also shown to have extended to between L-fuzzy equivalence relations, L-fuzzy I-ary relations and L-fuzzy partitions on the former and certain equivalence relations, I-ary relations and partitions respectively on the later. Further, they also construct a Galois connection between the set of all L-fuzzy partitions and the set of all L-fuzzy equivalence relations on the L-fuzzy set X. For each of these Galois Connections, both the onward and return order preserving maps are characterized in terms of the complete lattice L being certain chains.

However, a set with a binary relation, as an object by itself is *not* exclusively studied and primarily this aspect is taken up in Murthy-Sujatha [17, 18]. There a set with a binary relation is called a breset (Binarily RElated SET) and studies of this object and morphisms between such objects are made.

Notice that a set with an associative binary operation is called a semi group and is extensively studied. For example, see Clifford and Preston [6], Lidl and Pilz [11].

Now coming back to bresets, since a digraph is a finite set with a (finite) binary relation and since the notion of a breset has *no* restriction of finiteness either on the number of elements of the underlying set or on the size of the binary relation, a breset can be regarded as an (in) finite digraph and hence a generalization of digraph. Of course, (in) finite (di) graphs were studied in conjunction with groups and/or vector spaces. To see some work in this direction, one can refer to, Finucane [8], Seifter [22], Soardi and Woess [24], Bondy and Hemminger [4] and Andrea [2] etcetera.

In this paper, we extend such notions of digraphs as conjunctive product (also known as tensor or categorical product), disjunctive product (also known as co-normal product) to *arbitrary* families of bresets and study them. Also we introduce and study such notions as factors, radicals for bresets and prove such results as: for a family of bresets $(\mathcal{A}_i)_{i\in I}$, $\mathcal{A}_i$ is reflexive for all $i \in I$ if and only if $\Pi^c_{i\in I}\mathcal{A}_i$ is reflexive whenever each $\mathcal{A}_i$ is non empty for all $i \in I$, $\mathcal{A}_i$ is symmetric for all $i \in I$ if and only if $\Pi^c_{i\in I}\mathcal{A}_i$ is symmetric whenever each $\beta_{\mathcal{A}_i}$ is non empty for all $i \in I$, $\Pi^c_{i\in I}\mathcal{A}_i$ is transitive for all $i \in I$ and $\mathcal{A}_i$ is symmetric for all $i \in I$ implies $\mathcal{A}_i$ is transitive for all $i \in I$ whenever each $\beta_{\mathcal{A}_i}$ is non empty for all $i \in I$, $\mathcal{A}_i$ is transitive for all $i \in I$ implies $\Pi^c_{i\in I}\mathcal{A}_i$ is transitive for all $i \in I$, $\Pi^c_{i\in I}\mathcal{A}_i$ is transitive for all $i \in I$ and $\mathcal{A}_i$ is reflexive for all $i \in I$ implies $\mathcal{A}_i$ is transitive for all $i \in I$ whenever each $\mathcal{A}_i$ is non empty, $\Pi^c_{i\in I}\mathcal{A}_i$ is anti-symmetric for all $i \in I$ and $\mathcal{A}_i$ is reflexive for all $i \in I$ implies $\mathcal{A}_i$ is anti-symmetric whenever each $\mathcal{A}_i$ is non empty for all $i \in I$ and $\mathcal{A}_i$ is anti-symmetric for all $i \in I$ if and only if $\Pi^c_{i\in I}\mathcal{A}_i$ is anti-symmetric for all $i \in I$. etcetera under some simple different conditions and (lattice) algebraic properties of (inverse) images of substructures of bresets under (function and relation) morphisms between bresets etcetera all of which are more of purely mathematical interest. It is for these reasons, in fact, that we preferred the word breset over the word (in) finite digraph.

The other major reason for bringing the underlying set of a digraph into mathematical consideration and calling it then a breset is that, in an unpublished manuscript, Murthy [15] observed that (a) factorization of a (finite or not) (di) graph is a very important tool since the notion of (di) graph is extensively used in Computer Science in several algorithms and (b) any factorization of a (di) graph most naturally involves a factorization of the underlying nodes set and so developed the notion of breset and elementary theory of factorization of bresets.

Since some of the new notions introduced and studied here, in the finite set up, are also relevant for computer scientists as well and since $(a_1, a_2) \leftrightarrow a_1a_2$ defines one-one correspondence between the product set $A_1 \times A_2$ and the string set A_1A_2, where $A_1 \times A_2 = \{(a_1, a_2)|a_i \in A_i, i = 1, 2, ..\}$ and $A_1A_2 = \{a_1a_2|a_i \in A_i, i = 1, 2, ..\}$, which is easily extendable to n-tuples and n-strings(strings of length n), in general, we do *not* distinguish between n-tuples and n-strings and in fact, we prefer to use n-strings in stead of n-tuples especially in our examples and counter examples.

In this paper first we study products of bresets and next study factors of bresets whose underlying set is a Cartesian product of sets. Sections 3 and 4 introduce and study some properties of J-conjunctive product, J-disjunctive and j_0-(sectional) product for a family $(\mathcal{A}_i)_{i\in I}$ of bresets where J is a subset of the index set I, generalizing the notions of conjunctive and disjunctive products for directed graphs. Notice that (1) when the index set is empty, the Cartesian product of empty family of sets is the singleton of empty mapping and (2) when the index set is non-empty and some A_i is empty, the Cartesian product of I-family of sets is the empty set. Thus the product in both these cases is uninteresting. Hence, we *always* assume that the index set I is non empty and *each* A_i is non-empty for $i \in I$. However, in situations where some ambiguity is possible or an emphasis looks better, we explicitly state them. Further, we also make a free use of the Axiom of Choice and/or its equivalent the existence of the Choice Function without an explicit mention of the same. Section 5 introduces and studies the notions of the K-conjunctive factor, K-disjunctive factor and k_0-sectional factor for a breset whose underlying set is a Cartesian product of a family $(A_i)_{i\in I}$ of sets and section 6 studies properties of the above. Section 7 studies several relations between these products and factors. Lastly, in Section 8, we study algorithms to compute products introduced in Section 3 and factors introduced in Section 5. We do not distinguish between n-tuples and n-strings and in fact, we prefer to use n-strings in stead of n-tuples especially in our examples and counter examples. Notice that whenever a notion that we used and/or introduced for bresets is already known in a (finite, infinite) (di) graph theory and we are aware, in almost all cases, we made an explicit mention of the same and explain the relation between them. In what follows, we recall the basic notions and results of bresets from Murthy-Sujatha [18].

2. Bresets, Substructures and Some Properties

In this section the notions of breset, lower sub breset or simply l-sub breset, upper sub breset or simply u-sub breset and sub breset are recalled from Murthy-Sujatha [18].

Definitions and Statements 2.1. *(a). A breset is any ordered pair (A, R), where A is called the* underlying set *or shortly the* u-set *of (A, R) and R is a binary relation on A.*

Let us recall that a binary relation R on a set A is any subset of $A \times A$.

(b). For any pair of bresets (A, R), (B, S), (A, R) is equal to (B, S), denoted by $(A, R) = (B, S)$, if and only if $A = B$ and $R = S$.

(c). A breset (A, R) is empty breset or simply empty, denoted by Φ, iff the underlying set $A = \phi$ and the binary relation $R = \phi$.

(d). Clearly, a breset (A, R) is empty if and only if u-set $A = \phi$ or equivalently a breset (A, R) is non empty iff u-set A is non empty. So, (i) in a non empty breset (A, R), it can so happen that the u-set A is nonempty but the binary relation R on A is empty and (ii) there can be several bresets with the same u-set A.

Since a breset (A, R) is uniquely determined by both its u-set A and the binary relation R on the set A and not by any one of them, here onwards for notation convenience, we denote the breset (A, R) by $\mathcal{A}$ and the binary relation R by $\beta_{\mathcal{A}}$. Further, through out this and other chapters on bresets, the script letters and the suffixed script letters always stand for the bresets. In other words, the $\mathcal{P}$ stands for the ordered pair $(P, \beta_{\mathcal{P}})$, etcetera.

(e). Let $\mathcal{A}$, $\mathcal{B}$ be a pair of bresets.

(i). $\mathcal{A}$ is said to be a sub system *of $\mathcal{B}$ iff $A \subseteq B$.*

(ii). $\mathcal{A}$ is said to be a lower sub breset *or simply l-sub breset of $\mathcal{B}$ iff $A \subseteq B$ and $\beta_{\mathcal{A}} \subseteq \beta_{\mathcal{B}} \cap (A \times A)$.*

(iii). For any breset $\mathcal{X}$, the set of all l-sub bresets of $\mathcal{X}$ is denoted by $\mathcal{S}_l(\mathcal{X})$.

(iv). $\mathcal{A}$ is said to be an upper sub breset *or simply u-sub breset of $\mathcal{B}$ iff $A \subseteq B$ and $\beta_{\mathcal{A}} \supseteq \beta_{\mathcal{B}} \cap (A \times A)$.*

(v). For any breset $\mathcal{X}$, the set of all upper sub bresets of $\mathcal{X}$ is denoted by $\mathcal{S}_u(\mathcal{X})$.

(vi). $\mathcal{A}$ is said to be a sub breset *of $\mathcal{B}$ iff $A \subseteq B$ and $\beta_{\mathcal{A}} = \beta_{\mathcal{B}} \cap (A \times A)$.*

Clearly, when the underlying set A of the breset $\mathcal{A}$ is finite, the notion of l-sub breset of a breset is equivalent to the notion of sub digraph of a digraph and the notion of sub breset of a breset is equivalent to the notion of the induced sub digraph of a digraph.

(f). For any breset $\mathcal{X}$

(i). Being l-sub breset is a binary relation on $\mathcal{S}_l(\mathcal{X})$ making it a breset and further, a poset.

(ii). Being u-sub breset is a binary relation on $\mathcal{S}_u(\mathcal{X})$ making it a breset and further, a poset.

(iii). Being sub breset is a binary relation on $\mathcal{S}(\mathcal{X})$ making it a breset and further, a poset.

Clearly, (1). Every sub breset is a lower (upper) sub breset.

(2). An l-sub breset need not *be a sub breset.*

(3). A u-sub breset need not *a sub breset.*

(4). For any breset $\mathcal{B}$ and for any subset A of B, the subset $\beta_{\mathcal{A}} = \beta_{\mathcal{B}} \cap (A \times A)$ of $B \times B$, is such that $\mathcal{A}$ is always a sub breset of $\mathcal{B}$, called the induced sub breset.

We now recall the notions of union and intersection for bresets which essentially generalize the existing notions of union (cf. P28, Jorgen-Gregory [10]) and intersection (cf. P195, Jorgen-Gregory [10]) for digraphs and next we use these notions to investigate order structures on collections of substructures of bresets.

(g). For any family of bresets $(\mathcal{A}_i)_{i\in I}$, $\mathcal{A}$, where $A = \cap_{i\in I} A_i$, $\beta_{\mathcal{A}} = \cap_{i\in I}\beta_{\mathcal{A}_i}$, is a breset.

(h). For any family of bresets $(\mathcal{A}_i)_{i\in I}$, the breset $\mathcal{A}$ defined as in (g) above is called the intersection of bresets $(\mathcal{A}_i)_{i\in I}$ and is denoted by $\cap_{i\in I}\mathcal{A}_i$.

In other words, for bresets $(\mathcal{A}_i)_{i\in I}$, (i) $\cap_{i\in I}\mathcal{A}_i = (\cap_{i\in I} A_i, \cap_{i\in I}\beta_{\mathcal{A}_i})$ (ii) $\beta_{\cap_{i\in I}\mathcal{A}_i} = \cap_{i\in I}\beta_{\mathcal{A}_i}$. Notice that the notion of intersection for both graphs and digraphs is available as follows: Refer page 177, Jorgen-Gregory [10] for finite intersection of digraphs with the same vertex set, refer page 3, Diestel [7] for intersection of graphs.

(i). For any family of bresets $(\mathcal{A}_i)_{i\in I}$, $\mathcal{A}$ where $A = \cup_{i\in I} A_i$, $\beta_{\mathcal{A}} = \cup_{i\in I}\beta_{\mathcal{A}_i}$, is a breset.

(j). For any family of bresets $(\mathcal{A}_i)_{i\in I}$, the breset $\mathcal{A}$ defined as in (i) above is called the union of bresets $(\mathcal{A}_i)_{i\in I}$ and is denoted by $\cup_{i\in I}\mathcal{A}_i$.

In other words, for bresets $(\mathcal{A}_i)_{i\in I}$, (i) $\cup_{i\in I}\mathcal{A}_i = (\cup_{i\in I} A_i, \cup_{i\in I}\beta_{\mathcal{A}_i})$ (ii) $\beta_{\cup_{i\in I}\mathcal{A}_i} = \cup_{i\in I}\beta_{\mathcal{A}_i}$. Notice that, (i) the notion of union is already available for (pseudo) (di) graphs etcetera and it is not unique. For example, refer page 10, Jorgen-Gregory [10] for union of pseudo digraphs, refer page 3. Diestel [7] for union of graphs (ii) the union (di) graphs can become a pseudo (di) graph and (iii) although when the underlying set is finite, the notions of digraph and breset are exactly the same, when one takes their union for a pair of digraphs, one may end up getting a pseudo digraph in which more than two edges can exist between a pair of nodes, which will not happen in our union because the set union does not allow multiple entries for a same set element (as in a multi set).

Although we could define arbitrary union and arbitrary intersection for even class-indexed families of bresets in the same way as we defined above, we do not take such collection of bresets for fear of unions of such collections of bresets can easily become a non-breset and in fact could make a pair of classes (cf. any book on Category Theory, for example, Herrlich-Strecker [9]). However, an application of the union and the intersection of a set-indexed family of bresets to (lower, upper) sub bresets of a given breset will be useful as can be seen in the following results.

(k). For any breset $\mathcal{X}$ and for any set indexed family of l-sub bresets $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{X}$, the set $A = \cap_{i\in I} A_i$ together with the binary relation $\beta_{\mathcal{A}} = \cap_{i\in I}\beta_{\mathcal{A}_i}$ is a subsystem $\mathcal{A}$ of $\mathcal{X}$ such that

(i). $\mathcal{A} = \cap_{i\in I}\mathcal{A}_i$ is an l-sub breset of $\mathcal{X}$

(ii). $\mathcal{A}$ is an l-sub breset of $\mathcal{A}_i$ for all $i \in I$

(iii). whenever $\mathcal{B}$ is an l-sub breset of $\mathcal{A}_i$ for all $i \in I$, $\mathcal{B}$ is an l-sub breset of $\mathcal{A}$.

(l). For any breset $\mathcal{X}$ and for any family of l-sub bresets $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{X}$, the l-sub breset $\mathcal{A}$ defined as in (k) is called the intersection of l-sub bresets $(\mathcal{A}_i)_{i\in I}$ and is denoted by $\cap_{i\in I}\mathcal{A}_i$.

(m). For any breset $\mathcal{X}$ and for any set indexed family of l-sub bresets $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{X}$, the set $A = \cup_{i\in I} A_i$ together with the binary relation $\beta_{\mathcal{A}} = \cup_{i\in I}\beta_{\mathcal{A}_i}$ is a subsystem $\mathcal{A}$ of $\mathcal{X}$ such that

(i). $\mathcal{A} = \cup_{i\in I}\mathcal{A}_i$ is an l-sub breset of $\mathcal{X}$

(ii). each $\mathcal{A}_i$ is an l-sub breset of $\mathcal{A}$

(iii). whenever $\mathcal{A}_i$ is an l-sub breset of $\mathcal{B}$ for all $i \in I$, $\mathcal{A}$ is an l-sub breset of $\mathcal{B}$.

(n). For any breset $\mathcal{X}$ and for any family of l-sub bresets $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{X}$, the l-sub breset $\mathcal{A}$ defined as in (m) is called the union of l-sub bresets $(\mathcal{A}_i)_{i\in I}$ and is denoted by $\cup_{i\in I}\mathcal{A}_i$.

(o). The set of all l-sub bresets of a breset is a complete lattice.

(p). For any breset $\mathcal{X}$ and for any family of u-sub bresets $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{X}$, the set $A = \cap_{i\in I} A_i$ together with the binary relation $\beta_{\mathcal{A}} = \cap_{i\in I}\beta_{\mathcal{A}_i}$ is a subsystem $\mathcal{A}$ of $\mathcal{X}$ such that

(i). $\mathcal{A} = \cap_{i\in I}\mathcal{A}_i$ is a u-sub breset of $\mathcal{X}$ but not necessarily of $\mathcal{A}_i$

(ii). whenever $\mathcal{B}$ is a u-sub breset of $\mathcal{A}_i$ for all $i \in I$, then $\mathcal{B}$ is a u-sub breset of $\mathcal{A}$.

An analogous of result with l-sub breset replaced by u-sub breset in (m) is no longer true. More precisely, for any breset $\mathcal{X}$ and for any family of u-sub bresets $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{X}$, the set $A = \cup_{i\in I} A_i$ together with the binary relation $\beta_{\mathcal{A}} = \cup_{i\in I}\beta_{\mathcal{A}_i}$ is a breset. However

(1). $\mathcal{A} = \cup_{i\in I}\mathcal{A}_i$ is not necessarily a u-sub breset of $\mathcal{X}$.

(2). $\mathcal{A}_i$ is not necessarily a u-sub breset of $\mathcal{A} = \cup_{i\in I}\mathcal{A}_i$.

(3). whenever $\mathcal{B}$ is a u-sub breset of $\mathcal{X}$ such that $\mathcal{A}_i$ is a u-sub breset of $\mathcal{B}$ then $\mathcal{A}$ is not necessarily a u-sub breset of $\mathcal{B}$.

(q). For any breset $\mathcal{X}$ and for any family of sub bresets $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{X}$, the set $A = \cap_{i\in I} A_i$ together with the binary relation $\beta_{\mathcal{A}} = \cap_{i\in I}\beta_{\mathcal{A}_i}$ is a subsystem of $\mathcal{X}$ such that

(i). $\cap_{i\in I}\mathcal{A}_i$ is a sub breset of $\mathcal{X}$

(ii). $\cap_{i\in I}\mathcal{A}_i$ is a sub breset of $\mathcal{A}_i$ for all $i \in I$

(iii). whenever $\mathcal{B}$ is a sub breset of $\mathcal{A}_i$ for all $i \in I$, $\mathcal{B}$ is a sub breset of $\mathcal{A}$.

(r). For any breset $\mathcal{X}$, the set $\mathcal{S}(\mathcal{X})$ of all sub bresets of $\mathcal{X}$ is an intersection complete semi lattice.

An analogous result with l-sub breset replaced by sub bresets in (m), is no longer true.

More precisely, for any breset $\mathcal{X}$ and for any family of sub bresets $(\mathcal{A}_i)_{i\in I}$ of $\mathcal{X}$, the set $A = \cup_{i\in I} A_i$ together with the binary relation $\beta_{\mathcal{A}} = \cup_{i\in I}\beta_{\mathcal{A}_i}$ is a breset. However,

(1). $\mathcal{A} = \cup_{i\in I}\mathcal{A}_i$ is not necessarily a sub breset of $\mathcal{X}$.

(2). $\mathcal{A}_i$ is not necessarily a sub breset of the breset $\mathcal{A} = \cup_{i\in I}\mathcal{A}_i$.

(3). If $\mathcal{B}$ is a sub breset of $\mathcal{X}$ such that $\mathcal{A}_i$ is a sub breset of $\mathcal{B}$, then $\mathcal{A}$ is not necessarily a sub breset of $\mathcal{B}$.

(s). For any set indexed family of bresets $(\mathcal{A}_i)_{i\in I}$, $\mathcal{A}_{i_0}$ is a sub breset of the breset $\cup_{i\in I}\mathcal{A}_i$ if and only if $\mathcal{A}_{i_0}$ is a u-sub breset of $\mathcal{A}_i$ for all $i \in I$.

(t). For any set indexed family of bresets $(\mathcal{A}_i)_{i\in I}$, $\mathcal{A}_{i_0}$ is a sub breset of the breset $\cap_{i\in I}\mathcal{A}_i$ if and only if $\mathcal{A}_{i_0}$ is an l-sub breset of $\mathcal{A}_i$ for all $i \in I$.

(u). For any set indexed family of bresets $(\mathcal{A}_i)_{i\in I}$, $\mathcal{A} = \cup_{i\in i}\mathcal{A}_i$ *is a breset such that*

(i). for each $i \in I$, $\mathcal{A}_i$ *is an l-sub breset of* $\mathcal{A}$ *and (ii) if* $\mathcal{B}$ *is any breset such that* $\mathcal{A}_i$ *is an l-sub breset of* $\mathcal{B}$ *for each* $i \in I$ *then* $\mathcal{A}$ *is an l-sub breset of* $\mathcal{B}$.

(v). For any set indexed family of bresets $(\mathcal{A}_i)_{i\in I}$, $\mathcal{A} = \cap_{i\in i}\mathcal{A}_i$ *is a breset such that*

(i). for each $i \in I$, $\mathcal{A}$ *is an l-sub breset of* $\mathcal{A}_i$ *and (ii) if* $\mathcal{B}$ *is any breset such that* $\mathcal{B}$ *is an l-sub breset of* $\mathcal{A}_i$ *for each* $i \in I$ *then* $\mathcal{B}$ *is an l-sub breset of* $\mathcal{A}$.

3. Products of Bresets

In this section, the notions of J-conjunctive product, J- disjunctive product and j_0-sectional product are introduced for a family of bresets and examples are mentioned for the same.

Definitions and Statements 3.1. *Let* $(\mathcal{A}_i)_{i\in I}$ *be a family of bresets and* J *be a subset of the index set* I. *Then*

(1). The J*-conjunctive product of the bresets* $(\mathcal{A}_i)_{i\in I}$, *denoted by* $\Pi^{J,c}_{i\in I}\mathcal{A}_i$, *is defined by the breset* $\mathcal{A}$, *where* $A = \Pi_{i\in I}A_i$ *and* $\beta_{\mathcal{A}} \subseteq A \times A$ *also denoted by* $\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i}$, *is defined by: for all* $f, g \in A$, $(f, g) \in \beta_{\mathcal{A}}$ *iff* $(fj, gj) \in \beta_{\mathcal{A}_j}$ *for all* $j \in J$.

In other words, (a). $\Pi^{J,c}_{i\in I}\mathcal{A}_i = (\Pi_{i\in I}A_i, \Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i})$

(b). $\beta_{\Pi^{J,c}_{i\in I}\mathcal{A}_i} = \Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i}$

(c). for all $f, g \in A$, $(f, g) \in \Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i}$ *iff for all* $j \in J, (fj, gj) \in \beta_{\mathcal{A}_j}$.

In particular, the I*-conjunctive product is simply called the conjunctive product of* $(\mathcal{A}_i)_{i\in I}$ *and is denoted by* $\Pi^{c}_{i\in I}\mathcal{A}_i$. *Further, whenever* $I = \{1, 2\}$, *the conjunctive product* $\Pi^{c}_{i\in I}\mathcal{A}_i$ *is denoted by* $\mathcal{A}_1 \times^c \mathcal{A}_2$ *and* $\beta_{\Pi^{c}_{i\in I}\mathcal{A}_i}$ *is denoted by* $\beta_{\mathcal{A}_1} \times^c \beta_{\mathcal{A}_2}$.

Clearly, (a). $\Pi^{J,c}_{i\in I}\mathcal{A}_i = \Phi$ *breset whenever* $A_i = \phi$ *for some* $i \in I$

(b). $\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i} = \phi$ *whenever* $\beta_{\mathcal{A}_j} = \phi$ *for some* $j \in J$ *or* $A_i = \phi$ *for some* $i \in I$

(c). $\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i} = A \times A$ *whenever* $J = \phi$, *where* $A = \Pi_{i\in I}A_i$. *However, the converse is not necessarily true as shown in Example 3.2 below.*

(2). The J*-disjunctive product of the bresets* $(\mathcal{A}_i)_{i\in I}$, *denoted by* $\Pi^{J,d}_{i\in I}\mathcal{A}_i$, *is defined by the breset* $\mathcal{A}$, *where* $A = \Pi_{i\in I}A_i$ *and* $\beta_{\mathcal{A}} \subseteq A \times A$ *also denoted by* $\Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i}$, *is defined by: for all* $f, g \in A$, $(f, g) \in \beta_{\mathcal{A}}$ *iff* $(fj, gj) \in \beta_{\mathcal{A}_j}$ *for some* $j \in J$.

In other words, (a). $\Pi^{J,d}_{i\in I}\mathcal{A}_i = (\Pi_{i\in I}A_i, \Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i})$

(b). $\beta_{\Pi^{J,d}_{i\in I}\mathcal{A}_i} = \Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i}$

(c). for all $f, g \in A$, $(f, g) \in \Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i}$ *iff* $(fj, gj) \in \beta_{\mathcal{A}_j}$ *for some* $j \in J$.

In particular, the I*-disjunctive product is simply called the disjunctive product of* $(\mathcal{A}_i)_{i\in I}$ *and is denoted by* $\Pi^{d}_{i\in I}\mathcal{A}_i$. *Further, whenever* $I = \{1, 2\}$, *the disjunctive product* $\Pi^{d}_{i\in I}\mathcal{A}_i$ *is denoted by* $\mathcal{A}_1 \times^d \mathcal{A}_2$ *and* $\beta_{\Pi^{d}_{i\in I}\mathcal{A}_i}$ *is denoted by* $\beta_{\mathcal{A}_1} \times^d \beta_{\mathcal{A}_2}$.

Clearly, (a). $\Pi^{J,d}_{i\in I}\mathcal{A}_i = \Phi$ *breset whenever* $A_i = \phi$ *for some* $i \in I$

(b). $\Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i} = \phi$ *whenever* $\beta_{\mathcal{A}_j} = \phi$ *for all* $j \in J$ *or* $A_i = \phi$ *for some* $i \in I$

(c). $\Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i} = \phi$ *whenever* $J = \phi$. *However, the converse is not necessarily true as shown in Example 3.3 below.*

(3). The j_0*-(sectional) product of the bresets* $(\mathcal{A}_i)_{i\in I}$, *denoted by* $\Pi^{I,j_0}_{i\in I}\mathcal{A}_i$, *is defined by the breset* $\mathcal{A}$, *where* $A = \Pi_{i\in I}A_i$ *and* $\beta_{\mathcal{A}} \subseteq A \times A$ *also denoted by* $\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i}$ *is defined by: for all* $f, g \in A$, $(f, g) \in \beta_{\mathcal{A}}$ *iff* $(fj_0, gj_0) \in \beta_{\mathcal{A}_{j_0}}$.

In other words, (a). $\Pi^{I,j_0}_{i\in I}\mathcal{A}_i = (\Pi_{i\in I}A_i, \Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i})$

(b). $\beta_{\Pi^{I,j_0}_{i\in I}\mathcal{A}_i} = \Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i}$

(c). for all $f, g \in A$, $(f, g) \in \Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i}$ *iff* $(fj_0, gj_0) \in \beta_{\mathcal{A}_{j_0}}$.

Further, whenever $I = \{1, 2\}$ *and* $j_0 = 1$, *the* j_0*-sectional product* $\Pi^{j_0}_{i\in I}\mathcal{A}_i$ *is denoted by* $\mathcal{A}_1 \times^1 \mathcal{A}_2$ *and* $\beta_{\Pi^{I,j_0}_{i\in I}\mathcal{A}_i}$ *is denoted by* $\beta_{\mathcal{A}_1} \times^1 \beta_{\mathcal{A}_2}$.

Clearly, (a). $\Pi^{I,j_0}_{i\in I}\mathcal{A}_i = \Phi$ *breset whenever* $A_i = \phi$ *for some* $i \in I$

(b). $\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i} = \phi$ *whenever* $\beta_{\mathcal{A}_{j_0}} = \phi$ *or* $A_{j_0} = \phi$.

Observe that when $J = \{j_0\}$, all the three products, namely, J-conjunctive product, J-disjunctive product and j_0-sectional product are the same.

Example 3.2. *Let* $I = J = \{1,2\}$, $A_i = \{a_i\}$, $\beta_{\mathcal{A}_i} = \{a_i a_i\}$ *where* $i = 1,2$. *Then* $A = A_1 \times A_2$, $A \times A = \{a_1a_2a_1a_2\}$, $\Pi^{c}_{i\in I}\beta_{\mathcal{A}_i} = \{a_1a_2a_1a_2\} = A \times A$ *but* $J \neq \phi$.

Example 3.3. *Let* $I = J = \{1,2\}$, $A_i = \{a_i\}$, $\beta_{\mathcal{A}_i} = \phi$ *where* $i = 1,2$. *Then* $A = A_1 \times A_2$, $A \times A = \{a_1a_2a_1a_2\}$, $\Pi^{d}_{i\in I}\beta_{\mathcal{A}_i} = \phi$ *and* $J \neq \phi$.

For more examples of the above notions, please see Examples 4.2, 5.2, 6.2, 6.5, 7.5-7.9, 7.13-7.17.

Clearly, (1). $\Pi^{J,c}_{i\in I}\mathcal{A}_i = \Phi$ breset if and only if $\mathcal{A}_{j_0} = \Phi$ breset for some $j_0 \in J$.

(2). $\beta_{\Pi^{J,c}_{i\in I}\mathcal{A}_i} = \phi$ if and only if there exists $j_0 \in J$ such that $\beta_{\mathcal{A}_{j_0}} = \phi$; when $A_i \neq \phi$ for all $i \in I$ and $J \neq \phi$.

(3). $\Pi^{J,d}_{i\in I}\mathcal{A}_i = \Phi$ if and only if $\mathcal{A}_{i_0} = \Phi$ for some $i_0 \in I$.

(4). $\beta_{\Pi^{J,d}_{i\in I}\mathcal{A}_i} = \phi$ if and only if for all $j \in J$, $\beta_{\mathcal{A}_j} = \phi$; whenever $A_i \neq \phi$ for all $i \in I$ and $J \neq \phi$.

(5). $\Pi^{I,j_0}_{i\in I}\mathcal{A}_i = \Phi$ breset if and only if $\mathcal{A}_{i_0} = \Phi$ breset for some $i_0 \in I$.

(6). $\beta_{\Pi^{I,j_0}_{i\in I}\mathcal{A}_i} = \phi$ if and only if $\beta_{\mathcal{A}_{j_0}} = \phi$; whenever each $A_i \neq \phi$ for $i \in I$.

The following Example shows that in all (4),(2) and (6) above, each A_i is non-empty, is necessary.

Example 3.4. *Let* $I = \{1,2\}$, $J = \{2\}$, $j_0 = \{2\}$, $A_1 = \beta_{\mathcal{A}_1} = \phi$, $A_2 = \{b\}$, $\beta_{\mathcal{A}_2} = \{bb\}$ *and* $\mathcal{C} = \mathcal{A}_1 \times^d \mathcal{A}_2$. *Then* $C = A_1 \times A_2 = \phi = \beta_{\mathcal{C}}$, *so that* $\beta_{\mathcal{A}_1\times^c\mathcal{A}_2} = \beta_{\mathcal{A}_1\times^d\mathcal{A}_2} = \beta_{\mathcal{A}_1\times^2\mathcal{A}_2} = \phi$ *but* $\beta_{\mathcal{A}_2} \neq \phi$.

4. Properties of Products

In this section first we study some natural relations between the conjunctive product, disjunctive product, sectional product of a family of bresets and later we show that, the conjunctive product is in fact the breset intersection of sectional products and the disjunctive product is in fact the breset union of sectional products.

Theorem 4.1. *For any family of bresets* $(\mathcal{A}_i)_{i\in I}$, *for any* $\phi \neq J \subseteq I$ *and for any* $j_0 \in J$, *the following are true:*

(1). $\Pi^{J,c}_{i\in I}\mathcal{A}_i$ *is an l-sub breset of* $\Pi^{I,j_0}_{i\in I}\mathcal{A}_i$

(2). $\Pi^{I,j_0}_{i\in I}\mathcal{A}_i$ *is an l-sub breset of* $\Pi^{J,d}_{i\in I}\mathcal{A}_i$.

Proof. (1): From the definitions of l-sub breset, J-conjunctive product and j_0-product, it is enough to show that $\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i} \subseteq \Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i}$. Let $\alpha \in \Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i}$. $\alpha = (f,g)$, $f,g \in A = \Pi_{i\in I}A_i$, $(fi,gi) \in \beta_{\mathcal{A}_i}$ for all $i \in I$, which implies $\alpha = (f,g)$, $f,g \in \Pi_{i\in I}A_i$, $(fj_0,gj_0) \in \beta_{\mathcal{A}_{j_0}}$ in particular, which in turn implies $\alpha = (f,g) \in \Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i}$. Therefore, $\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i} \subseteq \Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i}$ or $\Pi^{J,c}_{i\in I}\mathcal{A}_i$ is an l-sub breset of $\Pi^{I,j_0}_{i\in I}\mathcal{A}_i$.

(2): From the definitions of l-sub breset, J-disjunctive product and j_0-product, it is enough to show that $\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i} \subseteq \Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i}$. Let $\alpha \in \Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i}$. $\alpha = (f,g)$, $f,g \in A = \Pi_{i\in I}A_i$, $(fj_0,gj_0) \in \beta_{\mathcal{A}_{j_0}}$, which implies $\alpha = (f,g)$, $f,g \in \Pi_{i\in I}A_i$, $(fj_0,gj_0) \in \beta_{\mathcal{A}_{j_0}}$ for some $j_0 \in I$ which in turn implies $\alpha = (f,g) \in \Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i}$. Therefore, $\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i} \subseteq \Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i}$ or $\Pi^{I,j_0}_{i\in I}\mathcal{A}_i$ is an l-sub breset of $\Pi^{J,d}_{i\in I}\mathcal{A}_i$. $\square$

Strict inequalities can hold in previous Theorem 4.1 above as shown in the following Example:

Example 4.2. *Let* $I = J = \{1,2\}$, $A_1 = \{p,q\}$, $A_2 = \{a,b,c\}$, $\beta_{\mathcal{A}_1} = \{pq\}$ *and* $\beta_{\mathcal{A}_2} = \{ab,ac,bc\}$. *Then* $A_1 \times A_2 = \{pa,pb,pc,qa,qb,qc\}$, $(A_1\times A_2)^2 = \{papa,papb,papc,paqp,paqb,paqc,pbpa,pbpb,pbpc,pbqp,pbqb,pbqc,pcpa,pcpb,pcpc,pcqp,$ $pcqb,pcqc,qapa,qapb,qapc,qaqa,qaqb,qaqc,qbpa,qbpb,qbpc,qbqp,qbqb,qbqc,qcpa,qcpb,qcpc,qcqa,qcqb,qcqc\}$,

$\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i} = \{paqb,\ paqc,\ pbqc\};$

$\Pi^{I,1}_{i\in I}\beta_{\mathcal{A}_i} = \{paqa, paqb,\ paqc, pbqa,\ pbqb, pbqc,\ pcqa, pcqb,\ pcqc\};$

$\Pi^{I,2}_{i\in I}\beta_{\mathcal{A}_i} = \{papb, papc, paqb, paqc, pbpc, pbqc, qapb, qapc, qaqb, qaqc, qbpc, qbqc\};$

$\Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i} = \{paqa,\ paqb, paqc, pbqa, pbqb, pbqc, pcqa, pcqb,\ pcqc, papb, papc,\ pbpc, qapb, qapc,\ qaqb, qaqc, qbpc, qbqc\}.$

Clearly, strict inequalities in (1) and (2) of 4.1 above are holding.

Corollary 4.3. *For any family of bresets $(\mathcal{A}_i)_{i\in I}$, for any $\phi \neq J \subseteq I$, the following are true:*

(1). $\Pi^{J,c}_{i\in I}\mathcal{A}_i = \cap_{j_0\in J}(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$

(2). $\cup_{j_0\in J}(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i) = \Pi^{J,d}_{i\in I}\mathcal{A}_i.$

Proof. (1): From the definitions of equality, intersection, J-conjunctive product and j_0-product of bresets, the u-sets on both sides are the same and so, it is enough to show that $\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i} = \cap_{j_0\in J}(\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i})$.

From the proof of 4.1(1), it follows that $\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i} \subseteq \cap_{j_0\in J}(\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i})$.

If $\cap_{j_0\in J}(\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i})$ is empty, then any way $\cap_{j_0\in J}(\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i}) \subseteq \Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i}$.

Let $\alpha \in \cap_{j_0\in J}(\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i})$. Since $\cap_{j_0\in J}(\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i}) \subseteq \Pi_{i\in I}A_i \times \Pi_{i\in I}A_i$, $\alpha = (f,g)$, where $f,g \in \Pi_{i\in I}A_i$. Now $\alpha \in \Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i}$ for all $j_0 \in J$ implies $(fj_0, gj_0) \in \beta_{\mathcal{A}_{j_0}}$ for all $j_0 \in J$ which implies $(f,g) \in \Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i}$ or $\cap_{j_0\in J}(\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i}) \subseteq \Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i}$.

(2): From the definitions of equality, union, J-disjunctive product and j_0-product of bresets, the u-sets on both sides are the same and so, it is enough to show that $\cup_{j_0\in J}(\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i}) = \Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i}$.

From the proof of 4.1(2), it follows that $\cup_{j_0\in J}(\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i}) \subseteq \Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i}$.

If $\Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i}$ is empty, then any way $\Pi^{d}_{i\in I}\beta_{\mathcal{A}_i} \subseteq \cup_{j_0\in J}(\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i})$.

Let $\alpha \in \Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i}$. Since $\Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i} \subseteq \Pi_{i\in I}A_i \times \Pi_{i\in I}A_i$, $\alpha = (f,g)$, where $f,g \in \Pi_{i\in I}A_i$. Now $\alpha \in \Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i}$ implies there exists $j_0 \in J$, $(fj_0, gj_0) \in \beta_{\mathcal{A}_{j_0}}$ which implies $(f,g) \in \cup_{j_0\in J}(\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i})$ or $\Pi^{J,d}_{i\in I}\beta_{\mathcal{A}_i} \subseteq \cup_{j_0\in J}(\Pi^{I,j_0}_{i\in I}\beta_{\mathcal{A}_i})$. □

5. Factors of Bresets

In this section, the notions of K-conjunctive factor, K- disjunctive factor and k_0-sectional factor are introduced for a breset whose underlying set is a Cartesian product of a family of sets and examples are mentioned for the same.

Definitions and Statements 5.1. *Let $\mathcal{A}$ be a breset such that the u-set A of $\mathcal{A}$ is the Cartesian product $A = \Pi_{i\in I}A_i$ of sets $(A_i)_{i\in I}$ and $K \subseteq I$. Then*

(1). The K-conjunctive factor of the breset $\mathcal{A}$, denoted by $\wedge_K\mathcal{A}$, is defined by the breset $\mathcal{B}$ where $B = \cap_{i\in I}A_i$ and $\beta_{\mathcal{B}} \subseteq B\times B$, also denoted by $\beta_{\wedge_K\mathcal{A}}$, is defined by: for all $a,b \in B$, $(a,b) \in \beta_{\mathcal{B}}$ iff there exists $(f,g) \in \beta_{\mathcal{A}}$ such that $(fk, gk) = (a,b)$ for all $k \in K$.

In other words, (a). $\wedge_K\mathcal{A} = (\cap_{i\in I}A_i, \beta_{\wedge_K\mathcal{A}})$

(b). for all $a,b \in \cap_{i\in I}A_i$, $(a,b) \in \beta_{\wedge_K\mathcal{A}}$ if and only if there exists $(f,g) \in \beta_{\mathcal{A}}$ such that $(fk, gk) = (a,b)$ for all $k \in K$.

Clearly, $\beta_{\wedge_K\mathcal{A}} = \cap_{i\in I}A_i \times \cap_{i\in I}A_i$ whenever $\beta_{\mathcal{A}} = \phi$ or $K = \phi$. However, the converse is not necessarily true as shown in example later.

The I-conjunctive factor of the breset $\mathcal{A}$ is simply called the conjunctive factor of $\mathcal{A}$ and is denoted by $\wedge\mathcal{A}$.

(2). The K-disjunctive factor of the breset $\mathcal{A}$, denoted by $\vee_K\mathcal{A}$, is defined by the breset $\mathcal{B}$ where $B = \cup_{i\in I}A_i$ and $\beta_{\mathcal{B}} \subseteq B\times B$, also denoted by $\beta_{\vee_K\mathcal{A}}$, is defined by: for $a,b \in B$, $(a,b) \in \beta_{\mathcal{B}}$ iff there exists $(f,g) \in \beta_{\mathcal{A}}$ such that $(fk_0, gk_0) = (a,b)$ for some $k_0 \in K$.

In other words, (a). $\vee_K\mathcal{A} = (\cup_{i\in I}A_i, \beta_{\vee_K\mathcal{A}})$

(b). for all $a,b \in \cup_{i\in I}A_i$, $(a,b) \in \beta_{\vee_K\mathcal{A}}$ if and only if there exists $(f,g) \in \beta_{\mathcal{A}}$ such that $(fk_0, gk_0) = (a,b)$ for some $k_0 \in K$.

Clearly, $\beta_{\vee_K \mathcal{A}} = \Phi$ *whenever* $\beta_{\mathcal{A}} = \phi$ *or* $K = \phi$. *However, the converse is not necessarily true as shown in Example 5.2 below.*

The I-disjunctive factor of the breset $\mathcal{A}$ *is simply called the disjunctive factor of* $\mathcal{A}$ *and is denoted by* $\vee\mathcal{A}$.

(3). The k_0*-(sectional) factor of the breset* $\mathcal{A}$*, denoted by* $(\mathcal{A})_{k_0}$*, is defined by the breset* $\mathcal{B}$ *where* $B = A_{k_0}$ *and* $\beta_{\mathcal{B}} \subseteq B \times B$*, also denoted by* $\beta_{(\mathcal{A})_{k_0}}$*, is defined by: for all* $a, b \in B$*,* $(a,b) \in \beta_{\mathcal{B}}$ *iff there exists* $(f,g) \in \beta_{\mathcal{A}}$ *such that* $(fk_0, gk_0) = (a,b)$*.*

In other words, (a). $(\mathcal{A})_{k_0} = (A_{k_0}, \beta_{(\mathcal{A})_{k_0}})$

(b). for all $a, b \in A_{k_0}$*,* $(a,b) \in \beta_{(\mathcal{A})_{k_0}}$ *if and only if there exists* $(f,g) \in \beta_{\mathcal{A}}$ *such that* $(fk_0, gk_0) = (a,b)$*.*

Clearly, whenever the set $A_{k_0} = \phi$*,* $(\mathcal{A})_{k_0} = \Phi$*, but not conversely as will be seen in an example later.*

Example 5.2. *Let* $I = K = \{1,2\}$*,* $A_1 = \{b,a\}$*,* $A_2 = \{c,a\}$*,* $\beta_{\mathcal{A}} = \{aaaa\}$ *and* $\mathcal{B} = \wedge_K \mathcal{A}$*. Then* $B = A_1 \cap A_2 = \{a\}$*,* $\beta_{\mathcal{B}} = \beta_{\wedge_K \mathcal{A}} = \{aa\} = B \times B = \cap_{i\in I} A_i \times \cap_{i \in I} A_i$ *but* $\beta_{\mathcal{A}} \neq \phi$ *and* $J = \{1,2\} \neq \phi$*.*

For more examples of the above notions, please see Examples 5.2, 6.2, 6.5, 7.5-7.9, 7.13-7.17.

6. Properties of Factors

Lemma 6.1. *For any breset* $\mathcal{A}$ *whose u-set* A *is the Cartesian product* $\Pi_{i\in I} A_i$ *of a family of sets* $(A_i)_{i\in I}$*, and for any* $\phi \neq K \subseteq I$*, the following are true:*

(1). $\cup_{k\in K} (\mathcal{A})_k = \vee_K \mathcal{A}$

(2). $\wedge_K \mathcal{A}$ *is an l-sub breset of* $\cap_{k\in K} (\mathcal{A})_k$ *where* $(\mathcal{A})_k$ *is the k-factor of the breset* $\mathcal{A}$ *.*

Proof. (1): From the definitions of (i) equality of bresets (ii) union of bresets and (iii) K-disjunctive factor of a breset, it is enough if we show that the binary relations on both sides are the same. In other words, it is enough to show that $\cup_{i\in K} \beta_{(\mathcal{A})_i} = \beta_{\vee_K \mathcal{A}}$.

(a) Now, $(a,b) \in \cup_{k\in K} \beta_{(\mathcal{A})_k}$ implies there exists $k_0 \in K$ such that $(a,b) \in \beta_{(\mathcal{A})_{k_0}}$ which implies there exists $(f,g) \in \beta_{\mathcal{A}}$ such that $fk_0 = a$, $gk_0 = b$ and $k_0 \in K$ which in turn implies $(a,b) \in \beta_{\vee_K \mathcal{A}}$. Therefore, $\cup_{k\in K} \beta_{(\mathcal{A})_k} \subseteq \beta_{\vee_K \mathcal{A}}$.

(b) $(a,b) \in \beta_{\vee_K \mathcal{A}}$ implies there exists $k_0 \in K$, $(f,g) \in \beta_{\mathcal{A}}$ such that $fk_0 = a$ and $gk_0 = b$ which in turn implies $(a,b) \in \cup_{k\in K} \beta_{(\mathcal{A})_k}$. Therefore, $\beta_{\vee_K \mathcal{A}} \subseteq \cup_{k\in K} \beta_{(\mathcal{A})_k}$.

(2): From the definitions of (i) l-sub breset (ii) intersection of bresets and (iii) K-conjunctive factor of a breset, it is enough if we show that $\beta_{\wedge_K \mathcal{A}} \subseteq \beta_{\cap_{k\in K} (\mathcal{A})_k} = \cap_{k\in K} \beta_{(\mathcal{A})_k}$. $(a,b) \in \beta_{\wedge_K \mathcal{A}}$ implies there exists $(f,g) \in \beta_{\mathcal{A}}$ such that $fk = a$ and $gk = b$ for all $k \in K$ which in turn implies $(a,b) \in \beta_{(\mathcal{A})_k}$ for all $k \in K$ or $(a,b) \in \cap_{k\in K} \beta_{(\mathcal{A})_k}$. Therefore, $\beta_{\wedge_K \mathcal{A}} \subseteq \cap_{k\in K} \beta_{(\mathcal{A})_k}$. □

A strict inequality can hold in Lemma 6.1(2) above as shown in the following Example:

Example 6.2. *Let* $I = K = \{1,2\}$*,* $A_1 = \{a,b\}$*,* $A_2 = \{b,c\}$*. Then* $A_1 \cap A_2 = \{b\}$*,* $A = A_1 \times A_2 = \{ab, ac, bb, bc\}$ *and* $A \times A = \{abab, abac, abbb, abbc, acab, acac, acbb, acbc, bbab, bbac, bbbb, bbbc, bcab, bcac, bcbb, bcbc\}$*. Let* $\beta_{\mathcal{A}} \subseteq A \times A$ *be defined by* $\beta_{\mathcal{A}} = \{bcbc, abab\}$*. Then* $\beta_{\wedge \mathcal{A}} = \phi$*,* $\beta_{(\mathcal{A})_1} = \{bb, aa\}$*,* $\beta_{(\mathcal{A})_2} = \{cc, bb\}$ *and* $\beta_{(\mathcal{A})_1} \cap \beta_{(\mathcal{A})_2} = \{bb\} \supset \beta_{\wedge \mathcal{A}} = \phi$*.*

Note: In the above example if $\beta_{\mathcal{A}} \subseteq A \times A$ is given by $\beta_{\mathcal{A}} = \{bbbb\}$, then $\beta_{\wedge \mathcal{A}} = \{bb\} = \beta_{(\mathcal{A})_1} \cap \beta_{(\mathcal{A})_2} = \{bb\} \cap \{bb\} = \{bb\}$. So equality can hold some times in Lemma 6.1(2) above.

Corollary 6.3. *For any breset* $\mathcal{A}$ *whose u-set* A *is the Cartesian product* $\Pi_{i\in I} A_i$ *of a family of sets* $(A_i)_{i\in I}$*, and for any* $\phi \neq K \subseteq I$*, the following are true:*

(a). the K-conjunctive factor $\wedge_K \mathcal{A}$ *is an l-sub breset of k-factor* $(\mathcal{A})_k$ *for all* $k \in K$

(b). the k-factor $(\mathcal{A})_k$ *is an l-sub breset of the K-disjunctive factor* $\vee_K \mathcal{A}$ *for all* $k \in K$*.*

Proof. (a) It follows from 6.1(2). (b) It follows from 6.1(1). □

Theorem 6.4. *For any pair of bresets $\mathcal{A}$, $\mathcal{B}$ whose u-sets are the Cartesian product $\Pi_{i\in I}C_i$ of sets $(C_i)_{i\in I}$ such that $\mathcal{A}$ is an l-sub breset of $\mathcal{B}$ and for any $\phi \neq K \subseteq I$, the following are true:*

(a). the K-conjunctive factor $\wedge_K\mathcal{A}$ is an l-sub breset of the k-conjunctive factor $\wedge_K\mathcal{B}$

(b). the K-disjunctive factor $\vee_K\mathcal{A}$ is an l-sub breset of K-disjunctive factor $\vee_K\mathcal{B}$

(c). k_0- factor $(\mathcal{A})_{k_0}$ is an l-sub breset of k_0- factor $(\mathcal{B})_{k_0}$ for all $k_0 \in K$.

Proof. (a): From the definitions of l-sub breset and K-conjunctive factor of a breset, it is enough to show that $\beta_{\wedge_K\mathcal{A}} \subseteq \beta_{\wedge_K\mathcal{B}}$. Let $\alpha \in \beta_{\wedge_K\mathcal{A}}$. Then $\alpha = (a, b)$, where $a, b \in C = \cap_{i\in I}C_i$ and there exist $(f, g) \in \beta_{\mathcal{A}}$ such that $(fk, gk) = (a, b)$ for all $k \in K$. Since $\beta_{\mathcal{A}} \subseteq \beta_{\mathcal{B}}$, $(f, g) \in \beta_{\mathcal{B}}$ is such that $(fk, gk) = (a, b)$ for all $k \in K$, $a, b \in C$; which of course implies $(a, b) = \alpha \in \beta_{\wedge_K\mathcal{B}}$ or $\beta_{\wedge_K\mathcal{A}} \subseteq \beta_{\wedge_K\mathcal{B}}$.

(b): From the definitions of l-sub breset and K-disjunctive factor of a breset, it is enough to show that $\beta_{\vee_K\mathcal{A}} \subseteq \beta_{\vee_K\mathcal{B}}$. Let $\alpha \in \beta_{\vee_K\mathcal{A}}$. Then $\alpha = (a, b)$, where $a, b \in C = \cup_{i\in I}C_i$ and there exist $(f, g) \in \beta_{\mathcal{A}}$ such that $(fk, gk) = (a, b)$ for some $k \in K$. Since $\beta_{\mathcal{A}} \subseteq \beta_{\mathcal{B}}$, $(f, g) \in \beta_{\mathcal{B}}$ is such that $(fk, gk) = (a, b)$ for some $k \in K$, $a, b \in C$; which of course implies $(a, b) = \alpha = \beta_{\vee_K\mathcal{B}}$ or $\beta_{\vee_K\mathcal{A}} \subseteq \beta_{\vee_K\mathcal{B}}$.

(c): From the definitions of l-sub breset and k_0-factor of a breset, it is enough to show that $\beta_{(\mathcal{A})_{k_0}} \subseteq \beta_{(\mathcal{B})_{k_0}}$. Let $\alpha \in \beta_{(\mathcal{A})_{k_0}}$. Then $\alpha = (a, b)$, where $a, b \in C = C_{k_0}$ and there exist $(f, g) \in \beta_{\mathcal{A}}$ such that $(fk_0, gk_0) = (a, b)$. Since $\beta_{\mathcal{A}} \subseteq \beta_{\mathcal{B}}$, $(f, g) \in \beta_{\mathcal{B}}$ is such that $(fk_0, gk_0) = (a, b)$, $a, b \in C$; which implies $(a, b) = \alpha \in \beta_{(\mathcal{B})_{k_0}}$ or $\beta_{(\mathcal{A})_{k_0}} \subseteq \beta_{(\mathcal{B})_{k_0}}$. □

A strict inequality can hold in all (a), (b) and (c) of Theorem 6.4 above as shown in the following Example:

Example 6.5. *Let $I = \{1, 2, 3\}$, $\phi \neq K = \{1, 2\}$, $C_1 = \{a, b\}$, $C_2 = \{a, q\}$, $C_3 = \{a, n\}$, $C_1 \times C_2 = \{aa, aq, ba, bq\}$, $C_1 \times C_2 \times C_3 = \{aaa, aan, aqa, aqn, baa, ban, bqa, bqn\}$, $\beta_{\mathcal{A}}, \beta_{\mathcal{B}} \subseteq (C_1 \times C_2 \times C_3)^2 = \{aaaaaa, aaaaan, aaaaqa, aaaaqn, aaabaa, aaaban, aaabqa, aaabqn, aanaaa, aanaan, aanaqa, aanaqn, aanbaa, aanban, aanbqa, aanbqn, aqaaaa, aqaaan, aqaaqa, aqaaqn, aqabaa, aqaban, aqabqa, aqabqn, aqnaaa, aqnaan, aqnaqa, aqnaqn, aqnbaa, aqnban, aqnbqa, aqnbqn, baaaaa, baaaan, baaaqa, baaaqn, baabaa, baaban, baabqa, baabqn, banaaa, banaan, banaqa, banaqn, banbaa, banban, banbqa, banbqn, bqaaaa, bqaaan, bqaaqa, bqaaqn, bqabaa, bqaban, bqabqa, bqabqn, bqnaaa, bqnaan, bqnaqa, bqnaqn, bqnbaa, bqnban, bqnbqa, bqnbqn\}$, $\beta_{\mathcal{A}} = \{aaaban, bqabqa\}$, $\beta_{\mathcal{B}} = \{aaaaaa, aaaban, bqabqa, banaqn\}$. Then $\beta_{\mathcal{A}} \subseteq \beta_{\mathcal{B}}$, $\cap_{i\in I}C_i = \{a\}$, $\beta_{\wedge_K\mathcal{A}} = \phi$, $\beta_{\wedge_K\mathcal{B}} = \{aa\}$, $\beta_{\vee_K\mathcal{A}} = \{ab, aa, an, bb, qq\}$, $\beta_{\vee_K\mathcal{B}} = \{aa, ab, an, bb, qq, ba, aq, nn\}$, $\beta_{(\mathcal{A})_1} = \{ab, bb\}$ and $\beta_{(\mathcal{B})_1} = \{aa, ab, bb, ba\}$. Clearly, strict inequalities in all of (a), (b), and (c) of 6.4 above are holding.*

7. Properties of Products Versus Factors

In this section, first we study a. for a family of bresets, several relations between various products of the family of bresets and various factors of these products and b. for a breset whose underlying set is Cartesian product of a family of sets, several relations between various factors of the breset and various products of these factors. Next, we justify and establish the poset diagram shown in the end of this section after Corollary 7.14 whose nodes are the various products of a fixed family of bresets and various factors of these products, and whose upward edge between any two nodes is the relation, *is an l-sub breset of.*

Lemma 7.1. *For any family of bresets $(\mathcal{A}_i)_{i\in I}$, the following are true:*

(a)(i) $\wedge_K(\Pi^{J,c}_{i\in I}\mathcal{A}_i)$ is an l-sub breset of $\wedge_K(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$ (ii) $\wedge_K(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$ is an l-sub breset of $\wedge_K(\Pi^{J,d}_{i\in I}\mathcal{A}_i)$

(b)(i) $(\Pi^{J,c}_{i\in I}\mathcal{A}_i)_{k_0}$ is an l-sub breset of $(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)_{k_0}$ (ii) $(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)_{k_0}$ is an l-sub breset of $(\Pi^{J,d}_{i\in I}\mathcal{A}_i)_{k_0}$

(c)(i) $\vee_K(\Pi^{J,c}_{i\in I}\mathcal{A}_i)$ *is an l-sub breset of* $\vee_K(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$ *(ii)* $\vee_K(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$ *is an l-sub breset of* $\vee_K(\Pi^{J,d}_{i\in I}\mathcal{A}_i)$

(d)(i) $\wedge_K(\Pi^{J,c}_{i\in I}\mathcal{A}_i)$ *is an l-sub breset of* $(\Pi^{J,c}_{i\in I}\mathcal{A}_i)_{k_0}$ *(ii)* $(\Pi^{J,c}_{i\in I}\mathcal{A}_i)_{k_0}$ *is an l-sub breset of* $\vee_K(\Pi^{J,c}_{i\in I}\mathcal{A}_i)$

(e)(i) $\wedge_K(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$ *is an l-sub breset of* $(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)_{k_0}$ *(ii)* $(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)_{k_0}$ *is an l-sub breset of* $\vee_K(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$

(f)(i) $\wedge_K(\Pi^{J,d}_{i\in I}\mathcal{A}_i)$ *is an l-sub breset of* $(\Pi^{J,d}_{i\in I}\mathcal{A}_i)_{k_0}$ *(ii)* $(\Pi^{J,d}_{i\in I}\mathcal{A}_i)_{k_0}$ *is an l-sub breset of* $\vee_K(\Pi^{J,d}_{i\in I}\mathcal{A}_i)$.

Proof. From 4.1, 6.4 and 6.3, we have, (i)(a) $\Pi^{J,c}_{i\in I}\mathcal{A}_i$ is an l-sub breset of $\Pi^{I,j_0}_{i\in I}\mathcal{A}_i$ (b) $\Pi^{I,j_0}_{i\in I}\mathcal{A}_i$ is an l-sub breset of $\Pi^{J,d}_{i\in I}\mathcal{A}_i$ (ii) (a) $\wedge_K\mathcal{A}$ is an l-sub breset of $\wedge_K\mathcal{B}$ (b) $\vee_K\mathcal{A}$ is an l-sub breset of $\vee_K\mathcal{B}$ (c) $(\mathcal{A})_{k_0}$ is an l-sub breset of $(\mathcal{B})_{k_0}$ where $\mathcal{A}$, $\mathcal{B}$ are bresets with the same u-set $A = \Pi_{i\in I}A_i$ such that $\mathcal{A}$ is an l-sub breset of $\mathcal{B}$.

(iii) (a) $\wedge_K\mathcal{A}$ is an l-sub breset of $(\mathcal{A})_{k_0}$ and (b) $(\mathcal{A})_{k_0}$ is an l-sub breset of $\vee_K\mathcal{A}$, where $\mathcal{A}$ is a breset with the u-set $A = \Pi_{i\in I}A_i$.

(a): (i) It follows from (i)(a) and (ii)(a). (ii) It follows from (i)(b) and (ii)(a).

(b): (i) It follows from (i)(a) and (ii)(c). (ii) It follows from (i)(b) and (ii)(c).

(c): (i) It follows from (i)(a) and (ii)(b). (ii) It follows from (i)(b) and (ii)(b).

(d)(i), (e)(i) and (f)(i) follow from (iii)(a). (d)(ii), (e)(ii) and (f)(ii) follow from (iii)(b). □

Corollary 7.2. *For any breset* $\mathcal{A}$*, the following are true:*

(a)(i) $\wedge_K(\Pi^{J,c}_{i\in I}(\mathcal{A})_i)$ *is an l-sub breset of* $\wedge_K(\Pi^{I,j_0}_{i\in I}(\mathcal{A})_i)$ *(ii)* $\wedge_K(\Pi^{I,j_0}_{i\in I}(\mathcal{A})_i)$ *is an l-sub breset of* $\wedge_K(\Pi^{J,d}_{i\in I}(\mathcal{A})_i)$

(b)(i) $(\Pi^{J,c}_{i\in I}(\mathcal{A})_i)_{k_0}$ *is an l-sub breset of* $(\Pi^{I,j_0}_{i\in I}(\mathcal{A})_i)_{k_0}$ *(ii)* $(\Pi^{I,j_0}_{i\in I}(\mathcal{A})_i)_{k_0}$ *is an l-sub breset of* $(\Pi^{J,d}_{i\in I}(\mathcal{A})_i)_{k_0}$

(c)(i) $\vee_K(\Pi^{J,c}_{i\in I}(\mathcal{A})_i)$ *is an l-sub breset of* $\vee_K(\Pi^{I,j_0}_{i\in I}(\mathcal{A})_i)$ *(ii)* $\vee_K(\Pi^{I,j_0}_{i\in I}(\mathcal{A})_i)$ *is an l-sub breset of* $\vee_K(\Pi^{J,d}_{i\in I}(\mathcal{A})_i)$

(d)(i) $\wedge_K(\Pi^{J,c}_{i\in I}(\mathcal{A})_i)$ *is an l-sub breset of* $(\Pi^{J,c}_{i\in I}(\mathcal{A})_i)_{k_0}$ *(ii)* $(\Pi^{J,c}_{i\in I}(\mathcal{A})_i)_{k_0}$ *is an l-sub breset of* $\vee_K(\Pi^{J,c}_{i\in I}(\mathcal{A})_i)$

(e)(i) $\wedge_K(\Pi^{I,j_0}_{i\in I}(\mathcal{A})_i)$ *is an l-sub breset of* $(\Pi^{I,j_0}_{i\in I}(\mathcal{A})_i)_{k_0}$ *(ii)* $(\Pi^{I,j_0}_{i\in I}(\mathcal{A})_i)_{k_0}$ *is an l-sub breset of* $\vee_K(\Pi^{I,j_0}_{i\in I}(\mathcal{A})_i)$

(f)(i) $\wedge_K(\Pi^{J,d}_{i\in I}(\mathcal{A})_i)$ *is an l-sub breset of* $(\Pi^{J,d}_{i\in I}(\mathcal{A})_i)_{k_0}$ *(ii)* $(\Pi^{J,d}_{i\in I}(\mathcal{A})_i)_{k_0}$ *is an l-sub breset of* $\vee_K(\Pi^{J,d}_{i\in I}(\mathcal{A})_i)$.

Proof. It follows from 7.1. □

Theorem 7.3. *For any index set* I*, for any pair of subsets* J, K *of* I *and for any family of bresets* $(\mathcal{A}_i)_{i\in I}$*, the following are true:*

(1). always $\cap_{i\in I}\mathcal{A}_i$ *is an l-sub breset of* $\wedge_K(\Pi^{J,c}_{i\in I}\mathcal{A}_i)$*; however equality holds whenever* $I = J = K$

(2). $(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)_{k_0} = \mathcal{A}_{k_0}$*, the* k_0*-th breset of the family of bresets* $(\mathcal{A}_i)_{i\in I}$*, whenever each u-set* A_i *of the breset* $\mathcal{A}_i$ *is non empty and* $k_0 = j_0$ *and* $J = I$

(3). $\vee_K(\Pi^{J,c}_{i\in I}\mathcal{A}_i) = \cup_{i\in I}\mathcal{A}_i$ *whenever each binary relation* $\beta_{\mathcal{A}_i}$ *of the breset* $\mathcal{A}_i$ *is non empty and* $I = J = K$

(4). $(\Pi^{J,c}_{i\in I}\mathcal{A}_i)_{k_0} = \mathcal{A}_{k_0}$*, the* k_0*-th breset of the family of bresets* $(\mathcal{A}_i)_{i\in I}$*, whenever each binary relation* $\beta_{\mathcal{A}_i}$ *of the breset* $\mathcal{A}_i$ *is non empty and* $J = I$.

Proof. (1): Let $\mathcal{A} = \Pi^{J,c}_{i\in i}\mathcal{A}_i$. First observe that, from 3.1(1), 5.1(1) and 2.1(h), we get that the u-set of both sides is $\cap_{i\in I}A_i$ and if this is empty we have nothing to prove as both sides equal to the empty breset Φ. So let it be non empty. By 2.1(h), it is enough to show that $\cap_{i\in I}\beta_{\mathcal{A}_i} \subseteq \beta_{\wedge_K\mathcal{A}}$.

(i) if $\cap_{i\in I}\beta_{\mathcal{A}_i}$ is empty, we have nothing to prove. If $\alpha \in \cap_{i\in I}\beta_{\mathcal{A}_i}$, $\alpha \in \beta_{\mathcal{A}_i} \subseteq A_i \times A_i$, for all $i \in I$. Consequently, there exists $a, b \in \cap_{i\in I}A_i$ such that $\alpha = (a, b)$. Then one can define $f, g \in \Pi_{i\in I}A_i$ such that for all $i \in I$, $(fi, gi) = (a, b) = \alpha \in \beta_{\mathcal{A}_i}$, so that $(f, g) \in \beta_{\Pi^{J,c}_{i\in I}\mathcal{A}_i} = \beta_{\mathcal{A}}$ for any $\phi \neq J \subseteq I$ and $\alpha = (a, b) \in \beta_{\wedge_K\mathcal{A}}$ for any $\phi \neq K \subseteq I$ or $\cap_{i\in I}\beta_{\mathcal{A}_i} \subseteq \beta_{\wedge_K\mathcal{A}}$.

(ii) Let $I = J = K$. If $\beta_{\wedge\mathcal{A}}$ is empty then of course, $\beta_{\wedge\mathcal{A}} \subseteq \cap_{i\in I}\beta_{\mathcal{A}_i}$. If $\alpha \in \beta_{\wedge\mathcal{A}}$, then there exists $(f, g) \in \beta_{\mathcal{A}}$ such that $(fi, gi) = (a, b) = \alpha$ for all $i \in I$ and $a, b \in \cap_{i\in I}A_i$. But $(f, g) \in \beta_{\mathcal{A}} = \beta_{\Pi^c_{i\in I}\mathcal{A}_i}$ implies, by the definition of conjunctive product, $\alpha = (a, b) = (fi, gi) \in \beta_{\mathcal{A}_i}$ for all $i \in I$ or $\alpha \in \cap_{i\in I}\beta_{\mathcal{A}_i}$, so that $\beta_{\wedge\mathcal{A}} \subseteq \cap_{i\in I}\beta_{\mathcal{A}_i}$.

(2): First observe that the u-sets of bresets on both sides are same that is A_{k_0} which is non empty by assumption. Hence it is enough to show that the binary relations of bresets on both sides are the same.

Let $\mathcal{A} = \Pi^{I,j_0}_{i\in I}\mathcal{A}_i$. Then we have to show that $\beta_{(\mathcal{A})_{k_0}} = \beta_{\mathcal{A}_{k_0}}$.

(i) Let $\alpha \in \beta_{(\mathcal{A})_{k_0}}$. Then by the definition of k_0-factor of $\mathcal{A}$ (5.1(3)), $\alpha = (a,b)$ for some $a,b \in A_{k_0}$ and there exists $f,g \in \beta_{\mathcal{A}}$ such that $(fk_0, gk_0) = (a,b)$. But since $\mathcal{A} = \Pi^{j_0}_{i\in I}\mathcal{A}_i$, $k_0 = j_0$ and $(f,g) \in \beta_{\mathcal{A}}$ by 3.1(3), $\alpha = (a,b) = (fj_0, gj_0) = (fk_0, gk_0) \in \beta_{\mathcal{A}_{k_0}}$, so that $\beta_{(\mathcal{A})_{k_0}} \subseteq \beta_{\mathcal{A}_{k_0}}$.

(ii) Let $\alpha \in \beta_{\mathcal{A}_{k_0}} \subseteq A_{k_0} \times A_{k_0}$. Then $\alpha = (a,b)$ for some $a,b \in A_{k_0}$. Define $f,g : I \to \cup_{i\in I}A_i$ such that $fk_0 = a$ and $gk_0 = b$ and $fi, gi \in A_i$ for all $i \neq k_0$. Such f,g exists in $A = \Pi_{i\in I}A_i$ because $A_i \neq \phi$ for all $i \in I$. Then $(f,g) \in \Pi^{j_0}_{i\in I}\beta_{\mathcal{A}_i}$ because $k_0 = j_0$ and $(fj_0, gj_0) = (a,b) \in \beta_{\mathcal{A}_{j_0}} = \beta_{\mathcal{A}_{k_0}}$.

Now $\alpha = (a,b) \in \beta_{(\mathcal{A})_{k_0}}$ because there exists $(f,g) \in \beta_{\mathcal{A}} = \Pi^{j_0}_{i\in I}\beta_{\mathcal{A}_i}$ such that $(fj_0, gj_0) = (a,b)$, so that $\beta_{\mathcal{A}_{k_0}} \subseteq \beta_{(\mathcal{A})_{j_0}} = \beta_{(\mathcal{A})_{k_0}}$.

(3): Let $\mathcal{A} = \Pi^{J,c}_{i\in I}\mathcal{A}_i$. First observe that, from 3.1(1), 5.1(2) and 2.1(j), we get that the u-set of bresets on both sides is $\cup_{i\in I}A_i$.

So it is enough to show that the binary relations of bresets on both sides are the same.

By 2.1(j), it is enough to show that $\beta_{\vee\mathcal{A}} = \cup_{i\in I}\beta_{\mathcal{A}_i}$.

(i) Let $\alpha \in \beta_{\vee\mathcal{A}}$. Then $\alpha = (a,b)$ where there exists $(f,g) \in \beta_{\mathcal{A}}$, there exists $k_0 \in I$ such that $\alpha = (a,b) = (fk_0, gk_0)$. But then $(f,g) \in \beta_{\mathcal{A}}$ implies $(f_i, g_i) \in \beta_{\mathcal{A}_i}$ for all $i \in I$. In particular $\alpha = (a,b) = (fk_0, gk_0) \in \beta_{\mathcal{A}_{k_0}} \subseteq \cup_{i\in I}\beta_{\mathcal{A}_i}$. So that $\beta_{\vee\mathcal{A}} \subseteq \cup_{i\in I}\beta_{\mathcal{A}_i}$.

(ii) Let $\alpha \in \cup_{i\in I}\beta_{\mathcal{A}_i}$. There exists $k_0 \in I$ such that $\alpha \in \beta_{\mathcal{A}_{k_0}}$. Since $\beta_{\mathcal{A}_{k_0}} \subseteq A_{k_0} \times A_{k_0}$, $\alpha = (a,b)$ for some $a,b \in A_{i_0}$.

Since each $\beta_{\mathcal{A}_i} \neq \phi$ for all $i \in I$, there exists $\alpha_i = (a_i, b_i) \in \beta_{\mathcal{A}_i}$ for all $i \in I - \{k_0\}$.

Let $fi = a_i$ for all $i \neq k_0$ and $gi = b_i$ for all $i \neq k_0$ and $fi_0 = a$, $gi_0 = b$. Then $(f,g) \in \Pi^{c}_{i\in I}\beta_{\mathcal{A}_i}$ such that $\alpha = (fk_0, gk_0) = (a,b) \in \beta_{\mathcal{A}_{k_0}}$ implying that $\alpha \in \beta_{\vee\mathcal{A}}$, so that $\cup_{i\in I}\beta_{\mathcal{A}_i} \subseteq \beta_{\vee\mathcal{A}}$.

(4): Let $\mathcal{A} = \Pi^{J,c}_{i\in I}\mathcal{A}_i$. First observe that, from 3.1(1), 5.1(3) we get that the u-set of bresets on both sides is A_{k_0}. So by 2.1(b), it suffices to show that the binary relations of the bresets on both sides are the same, namely $\beta_{(\mathcal{A})_{k_0}} = \beta_{\mathcal{A}_{k_0}}$.

(i) If $\beta_{(\mathcal{A})_{k_0}}$ is empty, then $\beta_{(\mathcal{A})_{k_0}} \subseteq \beta_{\mathcal{A}_{k_0}}$. Let $\alpha \in \beta_{(\mathcal{A})_{k_0}}$. Then by 5.1(3), $\alpha = (a,b)$ where $a,b \in A_{k_0}$, there exists $(f,g) \in \beta_{\mathcal{A}}$ such that $(fk_0, gk_0) = (a,b) = \alpha$. Since $(f,g) \in \beta_{\mathcal{A}}$ by 3.1(1), $\alpha = (a,b) = (fk_0, gk_0) \in \beta_{\mathcal{A}_{k_0}}$, so that $\beta_{(\mathcal{A})_{k_0}} \subseteq \beta_{\mathcal{A}_{k_0}}$.

(ii) If $\beta_{\mathcal{A}_{k_0}}$ is empty, then $\beta_{\mathcal{A}_{k_0}} \subseteq \beta_{(\mathcal{A})_{k_0}}$. Let $\alpha \in \beta_{\mathcal{A}_{k_0}} \subseteq A_{k_0} \times A_{k_0}$. Then $\alpha = (a,b)$ for some $a,b \in A_{k_0}$. Now define $f,g : I \to \cup_{i\in I}A_i$ such that $(fk_0, gk_0) = (a,b) = \alpha$ and $(fi, gi) = r_i \in \beta_{\mathcal{A}_i}$ for $i \neq k_0$. Notice that such $(fi, gi) = r_i$ exists because $\beta_{\mathcal{A}_i}$ is non empty for all $i \in I$. Clearly, $(f,g) \in \Pi^{c}_{i\in I}\beta_{\mathcal{A}_i}$ by its construction and $\alpha = (fj_0, gj_0) \in \beta_{(\mathcal{A})_{k_0}}$, so that $\beta_{\mathcal{A}_{k_0}} \subseteq \beta_{(\mathcal{A})_{k_0}}$. □

Corollary 7.4. *For any breset $\mathcal{A}$, the following are true:*

(1). $\wedge(\Pi^{c}_{i\in I}(\mathcal{A})_i) = \cap_{i\in I}(\mathcal{A})_i$

(2). $(\Pi^{I,j_0}_{i\in I}(\mathcal{A})_i)_{k_0} = (\mathcal{A})_{k_0}$*, the k_0 factor of the breset $\mathcal{A}$, whenever the u-set A of the breset $\mathcal{A}$ is non empty and $k_0 = j_0$*

(3). $\vee(\Pi^{c}_{i\in I}(\mathcal{A})_i) = \cup_{i\in I}(\mathcal{A})_i$ *whenever the binary relation $\beta_{\mathcal{A}}$ of the breset $\mathcal{A}$ is non empty*

(4). $(\Pi^{c}_{i\in I}(\mathcal{A})_i)_{k_0} = (\mathcal{A})_{k_0}$*, the k_0 factor of the breset $\mathcal{A}$, whenever the binary relation $\beta_{\mathcal{A}}$ of the breset $\mathcal{A}$ is non empty.*

Proof. It follows from 7.3 above. □

The following Example shows that each u-set A_i being non empty is necessary in statement (2) of the Theorem 7.3 above:

Example 7.5. *Let $A_1 = \phi$, $A_2 = \{a,b\}$, $\beta_{A_1} = \phi$ and $\beta_{A_2} = \{ab\}$. Then $A_1 \times A_2 = \phi$, $(A_1 \times A_2)^2 = \phi$, $\Pi^2_{i\in I}\beta_{\mathcal{A}_i} = \phi$ implies $(\Pi^2_{i\in I}\beta_{\mathcal{A}_i})_2 = \phi \neq \{ab\} = \beta_{\mathcal{A}_2}$.*

The following Example shows that $k_0 = j_0$ is necessary in statement (2) of the Theorem 7.3 above:

Example 7.6. *Let* $A_1 = \{p, q\}$, $A_2 = \{a, b, c\}$, $\beta_{A_1} = \{pq\}$ *and* $\beta_{A_2} = \{ab, ac, bc\}$. *Then* $A_1 \times A_2 = \{pa, pb, pc, qa, qb, qc\}$, $(A_1 \times A_2)^2 = \{papa, papb, papc, paqp, paqb, paqc, pbpa, pbpb, pbpc, pbqp, pbqb, pbqc, pcpa, pcpb, pcpc, pcqp, pcqb, pcqc, qapa, qapb, qapc, qaqa, qaqb, qaqc, qbpa, qbpb, qbpc, qbqp, qbqb, qbqc, qcpa, qcpb, qcpc, qcqa, qcqb, qcqc\}$; $\Pi^1_{i\in I}\beta_{\mathcal{A}_i} = \{paqa, paqb, paqc, pbqa, pbqb, pbqc, pcqa, pcqb, pcqc\}$; $\Pi^2_{i\in I}\beta_{\mathcal{A}_i} = \{papb, papc, paqb, paqc, pbpc, pbqc, qapb, qapc, qaqb, qaqc, qbpc, qbqc\}$; $(\Pi^1_{i\in I}\beta_{\mathcal{A}_i})_1 = \{pq\} = \beta_{\mathcal{A}_1}$; $(\Pi^2_{i\in I}\beta_{\mathcal{A}_i})_2 = \{ab, ac, bc\} = \beta_{\mathcal{A}_2}$; $(\Pi^1_{i\in I}\beta_{\mathcal{A}_i})_2 = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\} \neq \{\beta_{\mathcal{A}_1}\}$ *or* $\{\beta_{\mathcal{A}_2}\}$ *and* $(\Pi^2_{i\in I}\beta_{\mathcal{A}_i})_1 = \{pp, pq, qp, qq\} \neq \{\beta_{\mathcal{A}_1}\}$ *or* $\{\beta_{\mathcal{A}_2}\}$.

The following Example shows that each binary relation $\beta_{\mathcal{A}_i}$ being non empty is necessary in statement (3) of the Theorem 7.3 above:

Example 7.7. *Let* $A_1 = \{p, q\}$, $A_2 = \{a, b, c\}$, $\beta_{A_1} = \phi$ *and* $\beta_{A_2} = \{ab, ac, bc\}$. *Then* $A_1 \times A_2 = \{pa, pb, pc, qa, qb, qc\}$, $(A_1 \times A_2)^2 = \{papa, papb, papc, paqp, paqb, paqc, pbpa, pbpb, pbpc, pbqp, pbqb, pbqc, pcpa, pcpb, pcpc, pcqp, pcqb, pcqc, qapa, qapb, qapc, qaqa, qaqb, qaqc, qbpa, qbpb, qbpc, qbqp, qbqb, qbqc, qcpa, qcpb, qcpc, qcqa, qcqb, qcqc\}$; $\Pi^c_{i\in I}\beta_{\mathcal{A}_i} = \phi$ *implies* $\vee(\Pi^c_{i\in I}\beta_{\mathcal{A}_i}) = \phi \neq \{ab, ac, bc\} = \cup_{i\in I}\beta_{\mathcal{A}_i}$.

The following Example shows that each binary relation $\beta_{\mathcal{A}_i}$ being non empty is necessary in statement (4) of the Theorem 7.3 above:

Example 7.8. *Let* $A_1 = \{p, q\}$, $A_2 = \{a, b, c\}$, $\beta_{A_1} = \phi$ *and* $\beta_{A_2} = \{ab, ac, bc\}$. *Then* $A_1 \times A_2 = \{pa, pb, pc, qa, qb, qc\}$, $(A_1 \times A_2)^2 = \{papa, papb, papc, paqp, paqb, paqc, pbpa, pbpb, pbpc, pbqp, pbqb, pbqc, pcpa, pcpb, pcpc, pcqp, pcqb, pcqc, qapa, qapb, qapc, qaqa, qaqb, qaqc, qbpa, qbpb, qbpc, qbqp, qbqb, qbqc, qcpa, qcpb, qcpc, qcqa, qcqb, qcqc\}$; $\Pi^c_{i\in I}\beta_{\mathcal{A}_i} = \phi$ *implies* $(\Pi^c_{i\in I}\beta_{\mathcal{A}_i})_2 = \phi \neq \{ab, ac, bc\} = \beta_{\mathcal{A}_2}$.

The following Example shows that $I = J = K$ is necessary for equality to hold in the statements (1), (3) and (4) of the Theorem 7.3 above:

Example 7.9. *Let* $I = \{1, 2, 3\}$, $J = \{1, 2\}$, $K = \{2, 3\}$, $A_1 = \{a, b\}$, $A_2 = \{a, p, q\}$, $A_3 = \{a, s, t\}$, $\beta_{A_1} = \{aa, ab\}$, $\beta_{A_2} = \{aa, ap\}$ *and* $\beta_{A_3} = \{st\}$. *Then* $\cap_{i\in I}A_i = \{a\}$, $\cap_{i\in I}\beta_{A_i} = \phi$, $A_1 \times A_2 = \{aa, ap, aq, ba, bp, bq\}$, $A_1 \times A_2 \times A_3 = \{aaa, aas, aat, apa, aps, apt, aqa, aqs, aqt, baa, bas, bat, bpa, bps, bpt, bqa, bqs, bqt\}$, $A \times A = (A_1 \times A_2 \times A_3)^2 = \{aaaaaa, aaaaas, aaaaat, aaaapa, aaaaps, aaaapt, aaaaqa, aaaaqs, aaaaqt, aaabaa, aaabas, aaabat, aaabpa, aaabps, aaabpt, aaabqa, aaabqs, aaabqt, aasaaa, aasaas, aasaat, aasapa, aasaps, aasapt, aasaqa, aasaqs, aasaqt, aasbaa, aasbas, aasbat, aasbpa, aasbps, aasbpt, aasbqa, aasbqs, aasbqt, aataaa, aataas, aataat, aatapa, aataps, aatapt, aataqa, aataqs, aataqt, aatbaa, aatbas, aatbat, aatbpa, aatbps, aatbpt, aatbqa, aatbqs, aatbqt, ..\}$, $\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i} = \{aaaaaa, aaaaas, aaaaat, aaaapa, aaaaps, aaaapt, aaabaa, aaabas, aaabat, aaabpa, aaabps, aaabpt, aasaaa, aasaas, aasaat, aasapa, aasaps, aasapt, aasbaa, aasbas, aasbat, aasbpa, aasbps, aasbpt, aataaa, aataas, aataat, aatapa, aataps, aatapt, aatbaa, aatbas, aatbat, aatbpa, aatbps, aatbpt\}$, $\wedge_K(\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i}) = \{aa\}$, $\cap_{i\in I}\beta_{\mathcal{A}_i} = \phi$, $\vee_K(\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i}) = \{aa, as, at, ap, sa, ss, st, ta, ts, tt\}$, $\cup_{i\in I}\beta_{\mathcal{A}_i} = \{aa, ab, ap, st\}$, $(\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i})_3 = \{aa, as, at, sa, ss, st, ta, ts, tt\}$, $\beta_{\mathcal{A}_3} = \{st\}$. *Therefore,*

(1) $\wedge_K(\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i}) = \{aa\} \supset \phi = \cap_{i\in I}\beta_{\mathcal{A}_i}$

(2) $\vee_K(\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i}) = \{aa, as, at, ap, sa, ss, st, ta, ts, tt\} \neq \{aa, ab, ap, st\} = \cup_{i\in I}\beta_{\mathcal{A}_i}$

(3) $\beta_{\mathcal{A}_3} = \{st\} \subset \{aa, as, at, sa, ss, st, ta, ts, tt\} = (\Pi^{J,c}_{i\in I}\beta_{\mathcal{A}_i})_3$.

Lemma 7.10. *For a family of bresets* $(\mathcal{A}_i)_{i\in I}$, *the following are true:*

(1). $\vee_K(\Pi^{J,d}_{i\in I}\mathcal{A}_i) = \cup_{i\in I}(\Pi^{J,d}_{i\in I}\mathcal{A}_i)_i$

(2). $\vee_K(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i) = \cup_{i\in I}(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)_i$

(3). $\vee_K(\Pi^{J,c}_{i\in I}\mathcal{A}_i) = \cup_{i\in I}(\Pi^{J,c}_{i\in I}\mathcal{A}_i)_i$.

Proof. It follows from Lemma 6.1(1). □

Corollary 7.11. *For any breset $\mathcal{A}$, the following are true:*

(1). $\vee_K(\Pi^{J,d}_{i\in I}(\mathcal{A})_i) = \cup_{i\in I}(\Pi^{J,d}_{i\in I}(\mathcal{A})_i)_i$

(2). $\vee_K(\Pi^{I,j_0}_{i\in I}(\mathcal{A})_i) = \cup_{i\in I}(\Pi^{I,j_0}_{i\in I}(\mathcal{A})_i)_i$

(3). $\vee_K(\Pi^{J,c}_{i\in I}(\mathcal{A})_i) = \cup_{i\in I}(\Pi^{c}_{i\in I}(\mathcal{A})_i)_i$.

Proof. It follows from Lemma 6.1(1). □

Lemma 7.12. *For any breset $\mathcal{A}$ whose u-set A is the cartesian product of a family of sets $(A_i)_{i\in I}$, the following are true:*

(1). $\mathcal{A}$ *is an l-sub breset of* $\Pi^{J,c}_{k\in I}(\mathcal{A})_k$

(2). $\Pi^{J,c}_{k\in I}(\mathcal{A})_k$ *is an l-sub breset of* $\Pi^{I,j_0}_{k\in I}(\mathcal{A})_k$

(3). $\Pi^{I,j_0}_{k\in I}(\mathcal{A})_k$ *is an l-sub breset of* $\Pi^{J,d}_{k\in I}(\mathcal{A})_k$.

Proof. (1): Let $\mathcal{B} = \Pi^{J,c}_{k\in I}(\mathcal{A})_k$. Then by definitions 5.1(3) and 3.1(1) it follows that the u-sets of bresets on both sides are the same, namely $\Pi_{k\in I}A_k$. So by 2.1(e)(ii), it suffices to show that $\beta_{\mathcal{A}} \subseteq \Pi^{J,c}_{k\in I}\beta_{(\mathcal{A})_k}$. Let $\alpha \in \beta_{\mathcal{A}} \subseteq (\Pi_{k\in I}A_i)^2$. Then $\alpha = (f, g)$ where $f, g \in \Pi_{k\in I}A_i$. Since $(f, g) \in \beta_{\mathcal{A}}$, $(fk, gk) \in \beta_{(\mathcal{A})_k}$ for all $k \in J$, so that $\alpha = (f, g) \in \Pi^{J,c}_{k\in I}\beta_{(\mathcal{A})_k}$ and $\beta_{\mathcal{A}} \subseteq \Pi^{J,c}_{k\in I}\beta_{(\mathcal{A})_k}$.

(2): Let $\mathcal{B} = \Pi^{J,c}_{i\in I}(\mathcal{A})_k$ and $\mathcal{C} = \Pi^{I,j_0}_{k\in I}(\mathcal{A})_k$. Then by 5.1(3), 3.1(1) and 3.1(3), it follows that the u-sets of bresets on both sides are the same, namely $\Pi_{k\in I}A_k$. So by 2.1(e)(ii), it suffices to show that $\beta_{\mathcal{B}} = \Pi^{J,c}_{k\in I}\beta_{(\mathcal{A})_k} \subseteq \beta_{\mathcal{C}} = \Pi^{I,j_0}_{k\in I}\beta_{(\mathcal{A})_k}$. But using 4.1(1) for the family of bresets $((\mathcal{A})_k)_{k\in I}$, we get that $\beta_{\mathcal{B}} \subseteq \beta_{\mathcal{C}}$.

(3): Let $\mathcal{B} = \Pi^{I,j_0}_{k\in I}(\mathcal{A})_k$ and $\mathcal{C} = \Pi^{J,d}_{k\in I}(\mathcal{A})_k$. Then by 5.1(3), 3.1(2) and 3.1(3), it follows that the u-sets of bresets on both sides are the same, namely $\Pi_{k\in I}A_k$. So by 2.1(e)(ii), it suffices to show that $\beta_{\mathcal{B}} = \Pi^{I,j_0}_{k\in I}\beta_{(\mathcal{A})_k} \subseteq \beta_{\mathcal{C}} = \Pi^{J,d}_{k\in I}\beta_{(\mathcal{A})_k}$. But using 4.1(2) for the family of bresets $((\mathcal{A})_k)_{k\in I}$, we get that $\beta_{\mathcal{B}} \subseteq \beta_{\mathcal{C}}$. □

A strict inequality can hold in all (1),(2) and (3) of Lemma 7.12 above as shown in the following Examples:

Example 7.13. $A_1 = \{a, b, c, d\}$, $A_2 = \{p, q, r, s, t\}$, $a = q, b = r, c = s$. *Then* $A_1 \cap A_2 = \{a(= q), b(= r), c(= s)\}$, $A = A_1 \times A_2 = \{ap, aq, ar, as, at, bp, bq, br, bs, bt, cp, cq, cr, cs, ct, dp, dq, dr, ds, dt, \}$. $\beta_{\mathcal{A}} \subseteq (A_1 \times A_2)^2$, $\beta_{\mathcal{A}} = \{aqbr, brcs, apbt\}$, *then* $\beta_{(\mathcal{A})_1} = \{ab, bc\}$, $\beta_{(\mathcal{A})_2} = \{qr, rs, pt\}$, $\Pi^{c}_{i\in I}\beta_{(\mathcal{A})_i} = \{aqbr, arbs, apbt, bqcr, brcs, bpct\}$. *Clearly* $\beta_{\mathcal{A}} \subset \Pi^{c}_{i\in I}\beta_{(\mathcal{A})_i}$.

Therefore a strict inequality can hold in 7.12(1) above.

$\Pi^{1}_{i\in I}\beta_{(\mathcal{A})_i} = \{aqbr,\ arbs,\ apbt,\ bqcr,\ brcs,\ bpct,\ aqbt, ..\}$, *clearly* $\Pi^{c}_{i\in I}\beta_{(\mathcal{A})_i} \subset \Pi^{1}_{i\in I}\beta_{(\mathcal{A})_i}$.

Therefore a strict inequality can hold in 7.12(2) above.

$\Pi^{1}_{i\in I}\beta_{(\mathcal{A})_i} = \{apbp, apbq, apbr, apbs, apbt, aqbp, aqbq, aqbr, aqbs, aqbt, arbp, arbq, arbr, arbs, arbt, asbp, asbq, asbr, asbs, asbt, atbp, atbq, atbr, atbs, atbt, bpcp, bpcq, bpcr, bpcs, bpct, bqcp, bqcq, bqcr, bqcs, bqct, brcp, brcq, brcr, brcs, brct, bscp, bscq, bscr, bscs, bsct, btcp, btcq, btcr, btcs, btct\}$.

$\Pi^{d}_{i\in I}\beta_{(\mathcal{A})_i} = \{apbp, apbq, apbr, apbs, apbt, aqbp, aqbq, aqbr, aqbs, aqbt, arbp, arbq, arbr, arbs, arbt, asbp, asbq, asbr, asbs, asbt, atbp, atbq, atbr, atbs, atbt, bpcp, bpcq, bpcr, bpcs, bpct, bqcp, bqcq, bqcr, bqcs, bqct, brcp, brcq, brcr, brcs, brct, bscp, bscq, bscr, bscs, bsct, btcp, btcq, btcr, btcs, btct, apct, ..\}$.

$\Pi^{1}_{i\in I}\beta_{(\mathcal{A})_i} \subset \Pi^{d}_{i\in I}\beta_{(\mathcal{A})_i}$. *Therefore a strict inequality can hold in 7.12(3) above.*

Corollary 7.14. *For any family of bresets $(\mathcal{A}_i)_{i\in I}$, the following are true:*

(1). $\wedge(\Pi^{c}_{i\in I}\mathcal{A}_i)$ *is an l-sub breset of* $\mathcal{A}_i$

(2). $\cap_{i\in I}\mathcal{A}_i$ *is an l-sub breset of* $(\Pi^{c}_{i\in I}\mathcal{A}_i)_{i_0}$

(3). $(\Pi^{c}_{i\in I}\mathcal{A}_i)_{i_0}$ *is an l-sub breset of* $\cup_{i\in I}\mathcal{A}_i$

(4). $\mathcal{A}_i$ *is an l-sub breset of* $\vee(\Pi^{c}_{i\in I}\mathcal{A}_i)$

(5). $\cap_{i\in I}\mathcal{A}_i$ *is an l-sub breset of* $\wedge(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$

(6). $\cap_{i\in I}\mathcal{A}_i$ *is an l-sub breset of* $(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)_{i_0}$

(7). $\cup_{i\in I}\mathcal{A}_i$ is an l-sub breset of $\vee(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$ *(8). $\cap_{i\in I}\mathcal{A}_i$ is an l-sub breset of $\wedge(\Pi^{d}_{i\in I}\mathcal{A}_i)$*

(9). $\mathcal{A}_i$ is an l-sub breset of $(\Pi^{d}_{i\in I}\mathcal{A}_i)_{i_0}$ *(10). $\cup_{i\in I}\mathcal{A}_i$ is an l-sub breset of $\vee(\Pi^{d}_{i\in I}\mathcal{A}_i)$.*

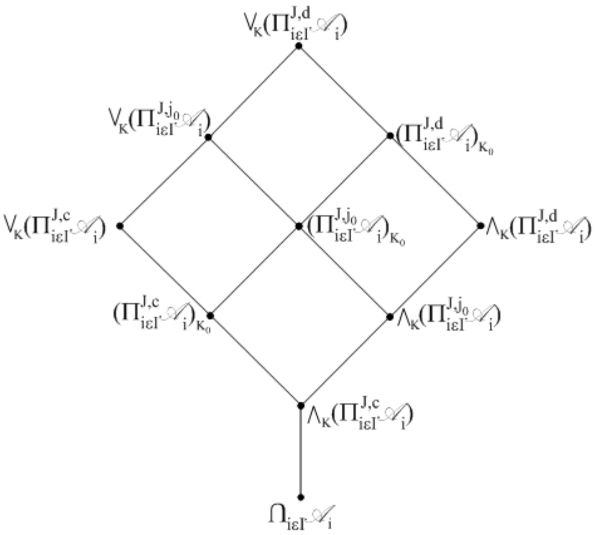

Figure 1. **Products-Factors-Relations for a family of bresets $(\mathcal{A}_i)_{i\in I}$ and for any $\phi \neq J \subseteq I$, $j_0 \in J$, $\phi \neq K \subseteq I$ and $k_0 \in K$**

Proof. (1): From 7.3(1), $\wedge(\Pi^c_{i\in I}\mathcal{A}_i) = \cap_{i\in I}\mathcal{A}_i$, it follows that $\wedge(\Pi^c_{i\in I}\mathcal{A}_i)$ is an l-sub breset of $\mathcal{A}_i$.

(2): From 7.3(4), $(\Pi^c_{i\in I}\mathcal{A}_i)_{i_0} = \mathcal{A}_{i_0}$, whenever each $\beta_{\mathcal{A}_i}$ is non empty, it follows that $\cap_{i\in I}\mathcal{A}_i$ is an l-sub breset of $(\Pi^c_{i\in I}\mathcal{A}_i)_{i_0}$.

(3): From 7.3(4), $(\Pi^c_{i\in I}\mathcal{A}_i)_{i_0} = \mathcal{A}_{i_0}$, whenever each $\beta_{\mathcal{A}_i}$ is non empty, it follows that $(\Pi^c_{i\in I}\mathcal{A}_i)_{i_0}$ is an l-sub breset of $\cup_{i\in I}\mathcal{A}_i$.

(4): From 7.3(3), $\vee(\Pi^c_{i\in I}\mathcal{A}_i) = \cup_{i\in I}\mathcal{A}_i$, whenever each $\beta_{\mathcal{A}_i}$ is non empty, it follows that $\mathcal{A}_i$ is an l-sub breset of $\vee(\Pi^c_{i\in I}\mathcal{A}_i)$.

(5): From 7.3(1), $\wedge(\Pi^c_{i\in I}\mathcal{A}_i) = \cap_{i\in I}\mathcal{A}_i$ and 7.1(a)(i), $\wedge(\Pi^c_{i\in I}\mathcal{A}_i) \subseteq \wedge(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$, it follows that $\cap_{i\in I}\mathcal{A}_i$ is an l-sub breset of $\wedge(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$.

(6): From 7.3(1), $\wedge(\Pi^c_{i\in I}\mathcal{A}_i) = \cap_{i\in I}\mathcal{A}_i$, 7.1(a)(i), $\wedge(\Pi^c_{i\in I}\mathcal{A}_i) \subseteq \wedge(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$ and 7.1(e)(i), $\wedge(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i) \subseteq (\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)_{i_0}$, it follows that $\cap_{i\in I}\mathcal{A}_i$ is an l-sub breset of $(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)_{i_0}$.

(7): From 7.3(3), $\vee(\Pi^c_{i\in I}\mathcal{A}_i) = \cup_{i\in I}\mathcal{A}_i$, whenever each $\beta_{\mathcal{A}_i}$ is non empty and 7.1(c)(i), $\vee(\Pi^c_{i\in I}\mathcal{A}_i) \subseteq \vee(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$, it follows that $\cup_{i\in I}\mathcal{A}_i$ is an l-sub breset of $\vee(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$.

(8): From 7.3(1), $\cap_{i\in I}\mathcal{A}_i = \wedge(\Pi^c_{i\in I}\mathcal{A}_i)$, 7.1(a)(i), $\wedge(\Pi^c_{i\in I}\mathcal{A}_i) \subseteq \wedge(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)$ and 7.1(a)(ii), $\wedge(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i) \subseteq \wedge(\Pi^d_{i\in I}\mathcal{A}_i)$, it follows that $\cap_{i\in I}\mathcal{A}_i$ is an l-sub breset of $\wedge(\Pi^d_{i\in I}\mathcal{A}_i)$.

(9): From 7.1(b)(ii), $(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)_{i_0} \subseteq (\Pi^d_{i\in I}\mathcal{A}_i)_{i_0}$ and 7.3(2), $(\Pi^{I,j_0}_{i\in I}\mathcal{A}_i)_{i_0} = \mathcal{A}_{i_0}$, whenever each $\mathcal{A}_i$ is non empty, it follows that $\mathcal{A}_{i_0}$ is an l-sub breset of $(\Pi^d_{i\in I}\mathcal{A}_i)_{i_0}$.

(10): Let $\Pi_{i\in I}\mathcal{A}_i = \mathcal{A}$. From 2.1(e)(ii), 2.1(j), 3.1(2), 5.1(2), it is enough to show that, $\cup_{i\in I}\beta_{\mathcal{A}_i} \subseteq \beta_{\vee\mathcal{A}}$.

Let $\alpha \in \cup_{i\in I}\beta_{\mathcal{A}_i}$. Then $\alpha = (a, b)$ for some $a, b \in A_{i_0}$ and for some $i_0 \in I$. Since $A_i \neq \phi$ for all $i \in I$, there exists $f, g \in \Pi_{i\in I}A_i$ such that $(fi_0, gi_0) = (a, b)$ and $fi, gi \in A_i$ for all $i \in I$.

Since there exists $i_0 \in I$ such that $(fi_0, gi_0) = (a, b) \in \beta_{\mathcal{A}_{i_0}}$ and $(f, g) \in \beta_{\mathcal{A}}$.

Since there exists $(f, g) \in \beta_{\vee\mathcal{A}}$ such that $(fi_0, gi_0) = (a, b) = \alpha$ for some $i_0 \in I$, $\alpha \in \beta_{\vee\mathcal{A}}$, implying that $\cup_{i\in I}\mathcal{A}_i$ is an l-sub breset of $\vee(\Pi^d_{i\in I}\mathcal{A}_i)$. □

A strict containment can hold in (1),(2),(3) and (4) of 7.14 as shown in the following Example.

Example 7.15. *Let* $A_1 = \{p,q\}$, $A_2 = \{p,b,c\}$, $\beta_{\mathcal{A}_1} = \{pq\}$, $\beta_{\mathcal{A}_2} = \{pb,pc,bc\}$ *and* $\mathcal{A} = \mathcal{A}_1 \times^c \mathcal{A}_2$. *Then* $A_1 \times A_2 = \{pp,pb,pc,qp,qb,qc\}$, $(A_1 \times A_2)^2 = \{pppp,pppb,pppc,ppqp,ppqb,ppqc,pbpp,pbpb,pbpc,pbqp,pbqb,$ $pbqc,pcpp,pcpb,pcpc,pcqp,pcqb,pcqc,qppp,qppb,qppc,qpqp,qpqb,qpqc,qbpp,qbpb,qbpc,qbqp,qbqb,qbqc,qcpp,qcpb,qcpc,qcqp,$ $qcqb,qcqc\}$.

$\beta_{\mathcal{A}} = \{ppqb,ppqc,pbqc\}$ *which implies* $\beta_{\wedge\mathcal{A}} = \phi \subset \beta_{\mathcal{A}_1} = \{pq\}$. *Thus a strict containment can hold in 7.14(1).*

$\beta_{(\mathcal{A})_1} = \{pq\}$, $\beta_{(\mathcal{A})_2} = \{ab,ac,bc\}$. *Therefore,* $\beta_{\mathcal{A}_1} \cap \beta_{\mathcal{A}_2} \subset \beta_{(\mathcal{A})_1}$. *Thus a strict containment can hold in 7.14(2).*

And $\beta_{(\mathcal{A})_1} = \{pq\} \subset \beta_{\mathcal{A}_1} \cup \beta_{\mathcal{A}_2} = \{pq,ab,ac,bc\}$. *Thus a strict containment can hold in 7.14(3).*

$\beta_{\vee\mathcal{A}} = \{pq,ab,ac,bc\} \supset \{pq\} = \beta_{\mathcal{A}_1}$. *Thus a strict containment can hold in 7.14(4).*

A strict containment can hold in (5), (6), (7) and (8) of 7.14 as shown in the following Example.

Example 7.16. $A_1 = \{a,b\}$, $A_2 = \{b,c\}$, $A_1 \cap A_2 = \{b\}$, $\beta_{\mathcal{A}_1} = \{ab,bb\}$, $\beta_{\mathcal{A}_2}$=$\{bc\}$ *and* $\mathcal{A}$=$\mathcal{A}_1 \times^1 \mathcal{A}_2$. *Then* $A = A_1 \times A_2$ $= \{ab,ac,bb,bc\}$, $(A_1 \times A_2)^2 = \{abab,abac,abbb,abbc,acab,acac,acbb,acbc,bbab,bbac,bbbb,bbbc,bcab,bcac,bcbb,bcbc\}$.

$\beta_{\mathcal{A}} = \{bbbb,bbbc,bcbb,bcbc,abbb,abbc,acbb,acbc\}$, $\beta_{\wedge\mathcal{A}} = \{bb\}$ *and* $\beta_{\mathcal{A}_1} \cap \beta_{\mathcal{A}_2} = \phi$, *so that* $\beta_{\wedge\mathcal{A}} \supset \phi = \cap_{i\in I}\beta_{\mathcal{A}_i}$. *Thus a strict containment can hold in 7.14(5).*

$\beta_{(\mathcal{A})_1} = \{bb,ab\} \supset \phi = \beta_{\mathcal{A}_1} \cap \beta_{\mathcal{A}_2}$. *Thus a strict containment can hold in 7.14(6).*

$\beta_{\vee\mathcal{A}} = \{pq,aa,ab,ac,ba,bb,bc,ca\} \supset \phi = \beta_{\mathcal{A}_1} \cap \beta_{\mathcal{A}_2}$. *Thus a strict containment can hold in 7.14(7).*

$\Pi^d_{i\in I}\beta_{\mathcal{A}_i} = \{bbbb,bbbc,bcbb,bcbc,abbb,abbc,acbb,acbc,abac,bbac\}$, $\wedge(\Pi^d_{i\in I}\beta_{\mathcal{A}_i}) = \{bb\} \supset \phi = \beta_{\mathcal{A}_1} \cap \beta_{\mathcal{A}_2}$. *Thus a strict containment can hold in 7.14(8).*

A strict containment can hold in (9) and (10) of 7.14 as shown in the following Example.

Example 7.17. *Let* $A_1 = \{p,q\}$, $A_2 = \{a,b,c\}$, $\beta_{\mathcal{A}_1} = \{pq\}$ *and* $\beta_{\mathcal{A}_2} = \{ab,ac,bc\}$. *Then* $\cup_{i\in i}\beta_{\mathcal{A}_i} = \{ab,ac,bc,pq\}$, $P \times Q = \{pa,pb,pc,qa,qb,qc\}$, $(P \times Q)^2 = \{papa,papb,papc,paqa,paqb,paqc,pbpa,pbpb,pbpc,pbqa,pbqb,pbqc,pcpa,pcpb,$ $pcpc,pcqa,pcqb,pcqc,qapa,qapb,qapc,qaqa,qaqb,qaqc,qbpa,qbpb,qbpc,qbqa,qbqb,qbqc,qcpa,qcpb,qcpc,qcqa,qcqb,qcqc\}$,

$\Pi^d_{i\in I}\beta_{\mathcal{A}_i} = \{paqa,paqb,paqc,pbqa,pbqb,pbqc,pcqa,pcqb,pcqc,\ papb,papc,pbpc,qapb,qapc,\ qaqb,qaqc,qbpc,pbqc\}$.

$(\Pi^d_{i\in I}\beta_{\mathcal{A}_i})_1 = \{pp,pq,qp,qq\} \supset \beta_{\mathcal{A}_1} = \{pq\}$. *Thus a strict containment can hold in 7.14(9).* $\vee(\Pi^d_{i\in I}\beta_{\mathcal{A}_i}) =$ $\{pp,pq,qp,qq,aa,ab,ac,ba,bb,bc,ca,cb,cc\} \supset \cup_{i\in I}\beta_{\mathcal{A}_i} = \{pq,ab,ac,bc\}$. *Thus a strict containment can hold in 7.14(10).*

8. Algorithms to Compute Products and Factors

In the above section, we introduced and studied some properties of J-conjunctive, J-disjunctive and j_0-sectional products for a family $(\mathcal{A}_i)_{i\in I}$ of bresets where J is a subset of the index set I and $i_0 \in I$, generalizing the notions of conjunctive and disjunctive products for directed graphs. Later we introduced and studied the notions of the conjunctive factor, disjunctive factor and sectional factor for a breset whose underlying set A is a cartesian product of a family $(A_i)_{i\in I}$ of sets. Lastly, we studied several relations between various products and factors. Now in this section we study algorithms to compute conjunctive product, disjunctive product, conjunctive factor, disjunctive factor and i_0-sectional factor.

Algorithm 1: To compute J-conjunctive product:

Input: Well ordered set I such that $\exists\ i+\ \forall i \in I$, $(\mathcal{A}_i)_{i\in I}$ where $\beta_{\mathcal{A}_i} \subseteq A_i \times A_i$, for all $i \in I$, $J \subseteq I$.

Output: The J-conjunctive product $\Pi^c_{j\in J}\mathcal{A}_j = \mathcal{A}$, where $A = \Pi_{j\in J}A_j$, $R = \beta_{\mathcal{A}}$.

Procedure:

/* First compute $A = \Pi_{i\in I}A_i$ */

Set $S = A_0$ and $A = \phi$, where 0 is the least element of I

for each $i \in I \setminus \{0\}$

for each $b \in A_i$

for each $a \in S$

$c = \text{concat}(a, b)$

$A = A \cup \{c\}$

continue

continue

$S = A$

continue

/* To compute $R = \beta_{\mathcal{A}}$ */

Set $R = \phi$

for each $f \in A$

10 for each $g \in A$

for each $j \in J$

if $\{(fj, gj) \notin \beta_{\mathcal{A}_j}\}$ then { go to 10 }

continue

$R = R \cup \{(f, g)\}$

continue

continue

end

Algorithm 2: To compute J-disjunctive product:

Input: Well ordered set I such that $\exists\ i+\ \forall i \in I$, $(\mathcal{A}_i)_{i \in I}$ where $\beta_{\mathcal{A}_i} \subseteq A_i \times A_i$, for all $i \in I$ and $J \subseteq I$.

Output: The J-disjunctive product $\Pi^d_{j \in J} \mathcal{A}_j = \mathcal{A}$, where $A = \Pi_{j \in J} A_j$, $R = \beta_{\mathcal{A}}$.

Procedure:

/*First compute $A = \Pi_{i \in I} A_i$ */

Set $S = A_0$ and $A = \phi$, where 0 is the least element in I

for each $i \in I \setminus \{0\}$

for each $b \in A_i$

for each $a \in S$

$c = \text{concat}(a, b)$

$A = A \cup \{c\}$

continue

continue

$S = A$

continue

/* To compute $R = \beta_{\mathcal{A}}$ */

Set $R = \phi$

for each $f \in A$

10 for each $g \in A$

for each $j \in J$

if $\{(fj, gj) \in \beta_{\mathcal{A}_j}\}$ then $\{R = R \cup \{(f, g)\}$ and go to 10 $\}$

continue

continue

continue

end

Algorithm 3: To compute J-conjunctive factor

Input: Well ordered set I such that $\exists\ i+ \forall i \in I$, $(A_i)_{i\in I}$, $S \subseteq \Pi_{i\in I} A_i \times \Pi_{i\in I} A_i$.

Output: The J-conjunctive factor $\mathcal{B}$, where $B = \cap_{i\in I} A_i$, $\beta_{\mathcal{B}} = \wedge S$.

Procedure:

/* First compute $B = \cap_{i\in I} A_i$ */

Set $B = \phi$

10 For each $a \in A_0$, where 0 is the least element in I

For each $i \in I \setminus \{0\}$

if $\{a \notin A_i\}$ then $\{$go to 10 $\}$

Continue

$B = B \cup \{a\}$

Continue

/* compute $T = \wedge S$ */

Set $T = \phi$

If $\{B = \cap_{i\in I} A_i = \phi\}$ then $\{$exit $\}$

For each $(f, g) \in S$

For each $a \in B$

20 For each $b \in B$

For each $i \in I$

if $\{fi \neq a \vee gi \neq b\}$ then $\{$go to 20 $\}$

Continue

$T = T \cup \{(a, b)\}$

Continue

Continue

Continue

end

Algorithm 4: To compute J-disjunctive factor

Input: Well ordered set I such that $\exists\ i+ \forall i \in I$, $(A_i)_{i\in I}$, $S \subseteq \Pi_{i\in I} A_i \times \Pi_{i\in I} A_i$.

Output: The J-disjunctive factor $\mathcal{B}$, where $B = \cup_{i\in I} A_i$, $\beta_{\mathcal{B}} = \vee S$.

Procedure:

/* First compute $B = \cup_{i\in I} A_i$ */

Set $B = A_0$, where 0 is the least element in I

For each $i \in I \setminus \{0\}$

For each $a \in A_i$

if $\{a \notin B\}$ then $\{B = B \cup \{a\}\}$

Continue

Continue

/* compute $T = \vee S$ */

Set $T = \phi$

For each $(f, g) \in S$

For each $a \in B$

For each $b \in B$

For each $i \in I$

if $\{fi = a \wedge gi = b \wedge (a, b) \notin T\}$ then $\{T = T \cup \{(a, b)\}\}$

Continue

Continue

Continue

Continue

end

Aliter:

Procedure:

For each $(f, g) \in S$

For each $j \in J$

if $\{(fj, gj) \notin T\}$ then $\{T = T \cup \{(fj, gj)\}\}$

Continue

Continue

end

Algorithm 5: To compute i_0-factor

Input: Well ordered set I such that $\exists\ i+\ \forall i \in I$, $(A_i)_{i\in I}$ and $S \subseteq \Pi_{i\in I} A_i \times \Pi_{i\in I} A_i$ and $i_0 \in I$.

Output: The i_0-factor $\mathcal{B}$, where $B = A_{i_0}$, $\beta_{\mathcal{B}} = (S)_{i_0}$

Procedure:

/* compute $T = (S)_{i_0}$ */

Set $B = A_{i_0}$

For each $(f, g) \in S$

For each $a \in B$

For each $b \in B$

if $\{fi_0 = a \wedge gi_0 = b \wedge (a, b) \notin T\}$ then $\{T = T \cup \{(a, b)\}\}$

Continue

Continue

Continue

end

Aliter:

Procedure:

Set $B = A_{i_0}$, $T = \phi$

If $\{S = \phi\}$ then {exit}

For each $(f, g) \in S$

if $\{(fi_0, gi_0) \notin T\}$ then $\{T = T \cup \{(fi_0, gi_0)\}\}$

Continue

end.

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