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General Weak Contraction and Some Fixed Point Results on Cone Random Metric Space

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Abstract: We define weak contraction and general weak contraction on cone random metric space. Then we prove some fixed point results of weak contraction on cone random metric space. The results are verified with the help of suitable example.

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1. Introduction and Mathematical Preliminaries

The study of random fixed point theory was initiated by the Prague school of probability in the 1950's (see [9, 10, 26]). The concept of randomness leads to several new questions of measurability of solution, probabilistic and statistical aspects of random solutions. Random fixed point theory provides a convenient way of modeling many problems arising from economic theory (see [21]). The machinery of random methods has revolutionized the financial markets. The survey article by Bharucha-Reid [2] in 1976 attracted the attention of several mathematician and gave wings to the theory, Itoh [13] extended Spacek's and Hans's theorem to multivalued contraction mappings. Now this theory has become the full fledged research area (see [8, 14, 16–18]). In 1997, the concept of weak contraction, a generalization of Banach's contraction principle was introduced by Alber and Guerre-Delabriere [1]. Actually in [1], the authors defined such mapping for single valued maps on Hilbert spaces and proved the existence of fixed point. Rhoades [22] showed that most of the results of [1] are still true for any Banach space. There are a number of works in which weakly contractive mapping has been considered, some of these works are noted in [3, 6, 15, 20, 27]. In 2007, Huang and Zhang [11] introduced the concept of cone metric space, where they generalized metric space by replacing the set of real numbers with an ordering Banach space. Thus the cone naturally induces a partial order in Banach space. Some of the works are noted in [4, 5, 11, 12], etc. The aim of this paper is to extend general weak contraction and establish some random fixed point results under this condition in random cone metric space. Some Definitions and Results are as Follows:

Definition 1.1. *Let E always be a real Banach space and P a subset of E. P is called a cone if and only if:*

(1). P is nonempty, closed, and $P \neq \{0\}$

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(2). $a, b \in R, \quad a, b \geq 0, \quad x, y \in P \Longrightarrow ax + by \in P$

(3). $P \cap (-P) = \{0\}$

Given a cone $P \subset E$, a partial ordering $\leq$ with respect to P is naturally defined by $x \leq y$ if and only if $x - y \in P$ for $x, y \in E$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denote the interior of P. The cone P is said to be normal if there exists a real number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \longrightarrow \|x\| \leq k\|y\|$$

The least positive number K satisfying the above statement is called the normal constant of P. The cone P is called regular if every increasing sequence which is bounded from above is convergent; that is, if $\{x_n\}$ is a sequence such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y,$$

for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

Definition 1.2 (Measurable function). *Let (Ω, Σ) be a measurable space with Σ a sigma algebra of subset of Ω and M be a non-empty subset of a metric space $X = (X, d)$. Let 2^M be the family of all non-empty subset of M and $C(M)$ the family of nonempty closed subset of M . A mapping G: $\Omega \rightarrow 2^M$ is called measurable if for each open set U of M, $G^{-1}(U) \in \Sigma$, where $G^{-1}(U) = \{\omega \in \Omega : G(\Omega) \cap U \neq \emptyset\}$.*

Definition 1.3 (Measurable selector). *A mapping $\xi : \Omega \rightarrow M$ is called measurable selector of a measurable mapping G: $\Omega \rightarrow 2^M$ if ξ is measurable and $\xi(\omega) \in G(\omega)$ for all $\omega \in \Omega$.*

Definition 1.4 (Random operator). *A mapping $T : \Omega \times M \rightarrow X$ is said to be a random operator if for each fixed $x \in M$, $T(., x) : \Omega \rightarrow X$ is measurable.*

Definition 1.5 (Continuous Random operator). *A random operator $T : \Omega \times M \rightarrow X$ is said to be continuous random operator if for each fixed $x \in M$, $T(., x) : \Omega \rightarrow X$ is continuous.*

Definition 1.6 (Random fixed point). *A measurable mapping $\xi : \Omega \rightarrow M$ is a random fixed point of a random operator $T : \Omega \times M \rightarrow X$ if $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$.*

Definition 1.7. *Let M be a nonempty set and the mapping $T : \Omega \times M \rightarrow X$ and $P \subset X$ be a cone, $\omega \in \Omega$ be a selector , satisfies the following conditions:*

(1). $d(x(\omega), y(\omega)) > 0$ and $d(x(\omega), y(\omega)) = 0 \Leftrightarrow x(\omega) = y(\omega)$ for all $x(\omega), y(\omega) \in \Omega \times M$.

(2). $d(x(\omega), y(\omega)) = d(y(\omega), x(\omega))$ for all x, $y \in M$, $\omega \in \Omega$ and $x(\omega), y(\omega) \in \Omega \times M$.

(3). $d(x(\omega), y(\omega)) \leq d(y(\omega), z(\omega)) + d(z(\omega), y(\omega))$ for all x, y, $z \in M$ and $\omega \in \Omega$ be a selector.

(4). for any x, $y \in M$, $\omega \in \Omega, d(x(\omega), y(\omega))$ is nonincreasing and left continuous.

Then d is called cone random metric on M and (M, d) is called cone random metric space.

Lemma 1.8. *Let M be a nonempty set and the mapping $T : \Omega \times M \rightarrow X$ and $P \subset X$ be a cone, $\omega \in \Omega$ be a selector and let $\{x_n(\omega)\}$ be a sequence in $\Omega \times X$. We have*

(a). $\{x_n(\omega)\}$ converges to $x(\omega) \in \Omega \times X$ if and only if $d(x_n(\omega), x(\omega)) \rightarrow 0$ as $n \rightarrow \infty$.

(b). $\{x_n(\omega)\}$ is a Cauchy sequence if and only if $d(x_n(\omega), x_m(\omega)) \rightarrow 0$ as $n, m \rightarrow \infty$.

2. Main Section

Definition 2.1. *Let (X, d) be a cone random metric space and M be a nonempty subset of X and $P \subset X$ be a cone and the mapping $T : \Omega \times M \to X$, is said to be weakly contractive if x, y $\in M$, $\omega \in \Omega$*

$$d\Big(T(x(\omega)), T(y(\omega))\Big) \leq l\Big[d(x(\omega), y(\omega)) - \phi(d(x(\omega), y(\omega)))\Big] \tag{1}$$

where $0 \leq l < 1$ and $\phi : intP \cup \{0\} \to intP \cup \{0\}$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$

(1) is more general than that satisfying

$$d\Big(T(x(\omega)), T(y(\omega))\Big) \leq d\Big(x(\omega), y(\omega)\Big) - \phi\Big(d(x(\omega), y(\omega))\Big) \tag{2}$$

because (2) is derived from (1) by taking $l = 1$.

Theorem 2.2. *Let (X, d) be a complete cone random metric space and M be a nonempty subset of X and $P \subset X$ be a cone and the mapping $T : \Omega \times M \to X$ is a weakly contractive mapping. Then T has a unique random fixed point in $\Omega \times M$.*

Proof. For each $x_0(\omega) \in \Omega \times X$ and $n \geq 1$, we choose $x_1(\omega), x_2(\omega) \in \Omega \times X$ such that $x_1(\omega) = T(x_0(\omega))$ and $x_2(\omega) = T(x_1(\omega))$. In general we define a sequence of elements of X such that $x_{n+1}(\omega) = T(x_n(\omega))$ and $x_{n+2}(\omega) = T(x_{n+1}(\omega))$. Now substituting $x(\omega) = x_n(\omega)$ and $y(\omega) = x_{n+1}(\omega)$ in (1) we obtain

$$\begin{aligned}
d\Big(T(x_n(\omega)), T(x_{n+1}(\omega))\Big) &= d\Big(x_{n+1}(\omega), x_{n+2}(\omega)\Big) \\
&\leq l\Big[d(x_n(\omega), x_{n+1}(\omega)) - \phi(d(x_n(\omega), x_{n+1}(\omega)))\Big] \\
&\leq l^2\Big[d(x_{n-1}(\omega), x_n(\omega)) - \phi(d(x_{n-1}(\omega), x_n(\omega)))\Big] - l\phi\Big(d((x_n(\omega), x_{n+1}(\omega))\Big) \\
&\leq l^2\Big[d(x_{n-1}(\omega), x_n(\omega)) - \phi(d((x_{n-1}(\omega), x_n(\omega)))\Big] \\
&\vdots \\
&\leq l^n\Big[d(x_1(\omega), x_2(\omega)) - \phi(d((x_1(\omega), x_2(\omega)))\Big]
\end{aligned} \tag{3}$$

Now if we take

$$d\Big(x_n(\omega), x_{n+1}(\omega)\Big) - \phi\Big(d((x_n(\omega), x_{n+1}(\omega))\Big) = \beta_n$$

then,

$$d\Big(x_{n+1}(\omega), x_{n+2}(\omega)\Big) \leq l\beta_n \leq l^2\beta_{n-1} \leq \cdots \leq l^n\beta_1$$

For $m > n$ we have

$$\begin{aligned}
d\Big(x_m(\omega), x_n(\omega)\Big) &\leq d\Big(x_m(\omega), x_{m-1}(\omega)\Big) + d\Big(x_{m-1}(\omega), x_{m-2}(\omega)\Big) + \cdots + d\Big(x_{n+1}(\omega), x_n(\omega)\Big) \\
&\leq l\beta_{m-1} + l\beta_{m-2} + \cdots + l\beta_n \\
&\leq (l^{m-1} + l^{m-2} + \cdots + l^n)\beta_1 \\
&= l^n(1 + l + \cdots + l^{m-n-1})\beta_1 \\
&= \frac{l^n(1 - l^{m-n})}{1 - l}\beta_1
\end{aligned}$$

$$
\begin{aligned}
&= \frac{l^n - l^m}{1-l}\beta_1 \\
&\leq \frac{l^n}{1-l}\beta_1 \qquad (4)
\end{aligned}
$$

Let $0 \ll c$ be given. Choose a natural number N such that $\frac{l^n}{1-l}\beta_1 \ll c$ for n > N. Thus $d(x_n(\omega), x_m(\omega)) \ll c$ for m > n.Therefore the sequence$\{x_n(\omega)\}$ is a Cauchy sequence. From the completeness of X, there exists $p(\omega) \in \Omega \times X$ such that

$$\lim_{n \to \infty} x_n(\omega) = p(\omega)$$

Now we prove that $p(\omega)$ is a random fixed point of T i.e., $T(\omega, p(\omega)) = p(\omega)$. Choose a natural number N_1 such that

$$d\Big(x_{n+1}(\omega), p(\omega)\Big) \ll \frac{c}{m}$$

and

$$\phi\Big(d(x_{n+1}(\omega), p(\omega))\Big) \ll \frac{c}{m}$$

for every $n \geq N_1$ and $m \geq 1$. Hence for $n \geq N_1$ we have

$$
\begin{aligned}
d\Big(T(p(\omega)), p(\omega)\Big) &\ll d\Big(T(p(\omega)), x_{n+1}(\omega)\Big) + d\Big(x_{n+1}(\omega), p(\omega)\Big) \\
&= d\Big(T(p(\omega)), T(x_n(\omega))\Big) + d\Big(x_{n+1}(\omega), p(\omega)\Big) \\
&\leq l\Big[d(x_{n+1}(\omega), p(\omega)) - \phi(d(x_{n+1}(\omega), p(\omega)))\Big] + d\Big(x_{n+1}(\omega), p(\omega)\Big) \\
&\leq l\Big[\frac{c}{m} - \frac{c}{m}\Big] + \frac{c}{m} \\
&= \frac{c}{m}
\end{aligned}
$$

Thus $d\Big(T(p(\omega)), p(\omega)\Big) \ll \frac{c}{m}$ for all m≥ 1. So $\frac{c}{m} - d\Big(T(p(\omega)), p(\omega)\Big) \in P$ for all m≥ 1. Since $\frac{c}{m} \longrightarrow 0$ as $m \longrightarrow \infty$ and P is closed so $-d\Big(T(p(\omega)), p(\omega)\Big) \in P$. But $d\Big(T(p(\omega)), p(\omega)\Big) \in P$. Therefore $d\Big(T(p(\omega)), p(\omega)\Big) = 0$ and so $T(p(\omega)) = p(\omega)$. For uniqueness let $q(\omega)$ be another random fixed point of T i.e $T(q(\omega)) = q(\omega)$. Now for $n \geq N_1$ and m≥ 1 we have

$$
\begin{aligned}
d\Big(p(\omega), q(\omega)\Big) &= d\Big(T(p(\omega)), T(q(\omega))\Big) \\
&\leq d\Big(T(p(\omega)), x_{n+1}(\omega)\Big) + d\Big(x_{n+1}(\omega), T(q(\omega))\Big) \\
&= d\Big(T(p(\omega)), T(x_n(\omega))\Big) + d\Big(T(x_n(\omega)), T(q(\omega))\Big) \\
&\leq l\Big[d(p(\omega), x_n(\omega)) - \phi(d(p(\omega), x_n(\omega))\Big] + l\Big[d(x_{n+1}(\omega), q(\omega)) - \phi(d(x_{n+1}(\omega), q(\omega))\Big] \\
&\leq l\Big[\frac{c}{m} - \frac{c}{m}\Big] + l[\frac{c}{m} - \frac{c}{m}\Big] \\
&\leq 0
\end{aligned}
$$

a contradiction. Hence $p(\omega) = q(\omega)$ and so $p(\omega)$ is a unique random fixed point of T. □

Definition 2.3 (Altering distance function)**.** *A function $\psi : intP \cup \{0\} \to intP \cup \{0\}$ is said to be altering distance function if the following properties are satisfied*

(1). ψ is monotone nondecreasing and continuous

(2). $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.4. *Let M be a nonempty set and $P \subset X$ be a cone and the mapping $T : \Omega \times M \to X$, where (X, d) is a cone random metric space is said to be generalized weakly contractive if $x,\ y \in M,\ \omega \in \Omega$*

$$\psi\Big[d\{T(x(\omega)), T(y(\omega))\}\Big] \leq \psi\Big(m(x(\omega), y(\omega))\Big) - \phi\Big[max\{d(x(\omega), y(\omega)), d(y(\omega), T(y(\omega))\}\Big] \tag{5}$$

where

$$m\Big(x(\omega), y(\omega)\Big) = \max\Big[d(x(\omega), y(\omega)), d(x(\omega), T(x(\omega))), d(y(\omega), T(y(\omega))), \frac{1}{2}\{d(x(\omega), T(y(\omega))) + d(y(\omega), T(x(\omega)))\}\Big]$$

and ψ is an linear altering distance function and $\phi : intP \cup \{0\} \to intP \cup \{0\}$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$.

Theorem 2.5. *Let (X, d) is a complete cone random metric space and M be a nonempty subset of X and $P \subset X$ be a cone and the mapping $T : \Omega \times M \to X$ is a generalized weakly contractive mapping. Then T has a unique random fixed point in $\Omega \times M$.*

Proof. For each $x_0(\omega) \in \Omega \times X$ and $n \geq 1$, we choose $x_1(\omega), x_2(\omega) \in \Omega \times X$ such that $x_1(\omega) = T(x_0(\omega))$ and $x_2(\omega) = T(x_1(\omega))$. In general we define a sequence of elements of X such that $x_{n+1}(\omega) = T(x_n(\omega))$ and $x_{n+2}(\omega) = T(x_{n+1}(\omega))$. If there exists a positive integer N such that $x_N(\omega) = x_{N+1}(\omega)$, then $x_N(\omega)$ is a random fixed point of T, hence we shall assume that $x_N(\omega) \neq x_{N+1}(\omega)$ for all $n \geq 1$. Now substituting $x(\omega) = x_n(\omega)$ and $y(\omega) = x_{n+1}(\omega)$ in (5) we obtain

$$\begin{aligned}\psi\Big(d(x_{n+1}(\omega), x_{n+2}(\omega))\Big) &= \psi\Big(d(T(x_n(\omega)), T(x_{n+1}(\omega)))\Big)\\ &\leq \psi\Big(m(x_n(\omega), x_{n+1}(\omega))\Big) - \phi\Big[max\{d(x_n(\omega), x_{n+1}(\omega)), d(x_{n+1}(\omega), T(x_{n+1}(\omega)))\}\Big]\\ &= \psi\Big(m(x_n(\omega), x_{n+1}(\omega))\Big) - \phi\Big[max\{d(x_n(\omega), x_{n+1}(\omega)), d(x_{n+1}(\omega), x_{n+2}(\omega))\}\Big]\end{aligned}$$

Since $\frac{1}{2}d\Big(x_n(\omega), x_{n+2}(\omega)\Big) \leq max\Big\{d(x_n(\omega), x_{n+1}(\omega)), d(x_{n+1}(\omega), x_{n+2}(\omega))\Big\}$. So

$$\begin{aligned}m\Big(x_n(\omega), x_{n+1}(\omega)\Big) &= \max\Big(d(x_n(\omega), x_{n+1}(\omega)), d(x_n(\omega), x_{n+1}(\omega)), d(x_{n+1}(\omega), x_{n+1}(\omega)), \frac{1}{2}[d(x_n(\omega), x_{n+2}(\omega)) + 0]\Big)\\ &= \max\Big\{d(x_n(\omega), x_{n+1}(\omega)), d(x_{n+1}(\omega), x_{n+2}(\omega))\Big\}\end{aligned}$$

So

$$\begin{aligned}\psi\Big(d(x_{n+1}(\omega), x_{n+2}(\omega))\Big) &\leq \psi\Big(max\{d(x_n(\omega), x_{n+1}(\omega)), d(x_{n+1}(\omega), x_{n+2}(\omega))\}\Big)\\ &\quad - \phi\Big(max\{d(x_n(\omega), x_{n+1}(\omega)), d(x_{n+1}(\omega), x_{n+2}(\omega))\}\Big)\end{aligned} \tag{6}$$

Suppose $d\Big(x_n(\omega), x_{n+1}(\omega)\Big) \leq d\Big(x_{n+1}(\omega), x_{n+2}(\omega)\Big)$ for some positive integer n. then

$$\psi\Big(d(x_{n+1}(\omega), x_{n+2}(\omega))\Big) \leq \psi\Big(d(x_{n+1}(\omega), x_{n+2}(\omega))\Big) - \phi\Big(d(x_{n+1}(\omega), x_{n+2}(\omega))\Big)$$

Hence it follows that $\phi\Big(d(x_{n+1}(\omega), x_{n+2}(\omega)\Big) \leq 0$, which implies that $d(x_{n+1}(\omega), x_{n+2}(\omega)) = 0$, contradicting our assumption that $x_N(\omega) \neq x_{N+1}(\omega)$ for each n. Therefore, $d\Big(x_{n+1}(\omega), x_{n+2}(\omega)\Big) \leq d\Big(x_n(\omega), x_{n+1}(\omega)\Big)$ for all $n \geq 1$ and $\{d(x_n(\omega), x_{n+1}(\omega))\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there exists an $r \in intP \cup \{0\}$ such that

$$d\Big(x_n(\omega), x_{n+1}(\omega)\Big) \longrightarrow r \quad \text{as} \quad n \longrightarrow \infty \tag{7}$$

In view of the above fact, from (6) we have for all $n \geq 1$

$$\psi\Big(d(x_{n+1}(\omega), x_{n+2}(\omega))\Big) \leq \psi\Big(d(x_n(\omega), x_{n+1}(\omega))\Big) - \phi\Big(d(x_n(\omega), x_{n+1}(\omega))\Big) \tag{8}$$

Taking limit as $n \longrightarrow \infty$ in both sides of the above inequality and using the continuity property of ψ, ϕ, we get

$$\psi(r) \leq \psi(r) - \phi(r)$$

That is, $\phi(r) \leq 0$ which implies that $r = 0$. Hence, we have

$$d\Big(x_n(\omega), x_{n+1}(\omega)\Big) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty \tag{9}$$

Let $0 \ll c$ be given. Choose a natural number N such that $d(x_n(\omega), x_{n+1}(\omega)) \ll \frac{c}{m-n}$ for $m > n > N$. For $m > n$ we have

$$\begin{aligned} d\Big(x_m(\omega), x_n(\omega)\Big) &\leq d\Big(x_m(\omega), x_{m-1}(\omega)\Big) + d\Big(x_{m-1}(\omega), x_{m-2}(\omega)\Big) + \cdots + d\Big(x_{n+1}(\omega), x_n(\omega)\Big) \\ &\leq \frac{c}{m-n} + \frac{c}{m-n} + \cdots + \frac{c}{m-n} \\ &= (m-n)\frac{c}{(m-n)} \\ &\leq c \end{aligned}$$

Thus, $d(x_n(\omega), x_m(\omega)) \ll c$ for m $>$ n.Therefore the sequence$\{x_n(\omega)\}$ is a Cauchy sequence. From the completeness of X, there exists $p(\omega) \in \Omega \times X$ such that

$$\lim_{n \to \infty} x_n(\omega) = p(\omega)$$

Now we prove that $p(\omega)$ is a random fixed point of T i.e., $T(\omega, p(\omega)) = p(\omega)$. Choose a natural number N_1 such that

$$\psi\Big(d(x_{n+1}(\omega), p(\omega))\Big) \ll \frac{c}{m}$$

and

$$\phi\Big(d(x_{n+1}(\omega), p(\omega))\Big) \ll \frac{c}{2m}$$

for every $n \geq N_1$ and $m \geq 1$. Hence for $n \geq N_1$, in (8) we put $x_{n+1}(\omega) = p(\omega)$, then we get

$$\begin{aligned} \psi\Big(d(T(p(\omega)), p(\omega))\Big) &\leq \psi\Big(d(p(\omega), x_n(\omega))\Big) - \phi\Big(d(x_n(\omega), p(\omega))\Big) \\ &\leq \frac{c}{m} - \frac{c}{2m} \\ &= \frac{c}{m} \end{aligned}$$

Thus $\psi\Big(d(T(p(\omega)), p(\omega))\Big) \ll \frac{c}{m}$ for all m$\geq$ 1. So $\frac{c}{m} - \psi\Big(d(T(p(\omega)), p(\omega))\Big) \in P$ for all m$\geq$ 1. Since $\frac{c}{m} \longrightarrow 0$ as $m \longrightarrow \infty$ and P is closed so $-\psi\Big(d(T(p(\omega)), p(\omega))\Big) \in P$. But $\psi\Big(d(T(p(\omega)), p(\omega))\Big) \in P$. Therefore $\psi\Big(d(T(p(\omega)), p(\omega))\Big) = 0$ that is $d\Big(T(p(\omega)), p(\omega)\Big) = 0$ and so $T(p(\omega)) = p(\omega)$. For uniqueness let $q(\omega)$ be another random fixed point of T i.e., $T(q(\omega)) = p(\omega)$. Now in (8) we put $x_{n+1}(\omega) = q(\omega)$ and $x_{n+2}(\omega) = p(\omega)$ then we get

$$\psi\Big(d(q(\omega), p(\omega))\Big) \leq \psi\Big(d(q(\omega), x_n(\omega))\Big) - \psi\Big(d(x_n(\omega), q(\omega))\Big)$$

Taking limit as $n \longrightarrow \infty$ in both sides of the above inequality and using the continuity property of ψ, ϕ we get

$$\psi\Big(d(q(\omega), p(\omega))\Big) \leq \psi\Big(d(q(\omega), p(\omega))\Big) - \psi\Big(d(p(\omega), q(\omega))\Big)$$

That is $\psi\Big(d(p(\omega), q(\omega))\Big) \leq 0$ a contradiction. Hence $d(p(\omega), q(\omega)) = 0$ that is $p(\omega) = q(\omega)$ and so $p(\omega)$ is a unique random fixed point of T. □

3. Example

Example 3.1. *Let $E = R$ and $M = [0,\infty)$ and $P = \{x \in M : x \geq 0\}$, also $\Omega = [0,1]$ and Σ be the sigma algebra of Lebesgue's measurable subset of $[0,1]$. Define a mapping $d : (\Omega \times X) \times (\Omega \times X) \to M$ by $d(x(\omega), y(\omega)) = \frac{|x(\omega)-y(\omega)|}{2}$. Then (d,X) is a cone random metric space.Define a random operator $T : (\Omega \times X) \to X$ as $T(\omega, x) = \frac{1-\omega+x}{2}$ and the mapping $\phi : intP \cup \{0\} \to intP \cup \{0\}$ is defined by $\phi(t) = \frac{t}{8}$ then*

$$d\Big(T(x(\omega)), T(y(\omega))\Big) = \frac{|x(\omega) - y(\omega)|}{4}$$

Also by taking $l = \frac{6}{7}$ we have

$$\begin{aligned} l\Big[d(x(\omega), y(\omega)) - \phi(d(x(\omega), y(\omega)))\Big] &= \frac{6}{7} \times \frac{7|x(\omega) - y(\omega)|}{16} \\ &= \frac{3|x(\omega) - y(\omega)|}{8} \end{aligned}$$

Since

$$\frac{|x(\omega) - y(\omega)|}{4} \leq \frac{3|x(\omega) - y(\omega)|}{8}$$

So

$$d\Big(T(x(\omega)), T(y(\omega))\Big) = l\Big[d(x(\omega), y(\omega)) - \phi(d(x(\omega), y(\omega)))\Big]$$

Therefore, the condition of the theorem (2.1) is satisfied. Now we define a measurable mapping $\xi : \Omega \to X$ as $\xi(\omega) = 1 - \omega$ for every $\omega \in \Omega$.Then clearly $1 - \omega$ is a random fixed point of the random operator T.

Example 3.2. *Let $E = R$ and $M = [0,\infty)$ and $P = \{x \in M : x \geq 0\}$, also $\Omega = [0,1]$ and Σ be the sigma algebra of Lebesgue's measurable subset of $[0,1]$. Define a mapping $d : (\Omega \times X) \times (\Omega \times X) \to M$ by*

$$\begin{aligned} d(x(\omega), y(\omega)) &= x(\omega) - y(\omega) \ \textit{if} \ \ x \geq y \\ &= y(\omega) - x(\omega) \ \textit{if} \ \ y > x \end{aligned}$$

Then (d, X) is a cone random metric space.Define a random operator $T : (\Omega \times X) \to X$ as $T(\omega, x) = \omega^2 - x$ and the mapping $\psi, \phi : intP \cup \{0\} \to intP \cup \{0\}$ is defined by $\psi(t) = \frac{t}{2}$ and $\phi(t) = \frac{t}{4}$. Now the following cases arise

Case 1: *If $x > y$ then*

$$\psi\Big(d(T(x(\omega)), T(y(\omega)))\Big) = \frac{x - y}{2}$$

and

$$\begin{aligned} m\Big(x(\omega), y(\omega)\Big) &= max\Big[d(x(\omega), y(\omega)), d(x(\omega), T(x(\omega))), d(y(\omega), T(y(\omega))), \frac{1}{2}\{d(x(\omega), T(y(\omega))) + d(y(\omega), T(x(\omega)))\}\Big] \\ &= max\Big[(x - y), (2x - \omega^2), (2y - \omega^2), \Big\{\frac{(y - \omega^2 + x)}{2} + \frac{(x - \omega^2 + y)}{2}\Big\}\Big] \\ &= max\Big[(x - y), (2x - \omega^2), (2y - 1 + \omega^2), (x + y - \omega^2)\Big] \\ &= (2x - \omega^2) \end{aligned}$$

Then

$$\psi\Big(m(x(\omega), y(\omega))\Big) = \frac{(2x-\omega^2)}{2}$$

and

$$\begin{aligned}
\phi\Big[max\{d(x(\omega), T(y(\omega))), d(y(\omega), T(x(\omega)))\}\Big] &= max\Big[\frac{(x+y-\omega^2)}{4}, \frac{(x+y-\omega^2)}{4}\Big] \\
&= \frac{(x+y-\omega^2)}{4}) \\
\psi\Big(m(x(\omega), y(\omega))\Big) - \phi\Big[max\{d(x(\omega), T(y(\omega))), d(y(\omega), T(x(\omega)))\}\Big] &= \frac{(2x-\omega^2)}{2} - \frac{(x+y-\omega^2)}{4} \\
&= \frac{3x-y-\omega^2}{4}
\end{aligned}$$

Since $x > y$ *and* $\omega \in \Omega$ *so* $\omega^2 \leq 1$ *so*

$$\frac{x-y}{2} \leq \frac{3x-y-\omega^2}{4}$$

Therefore,

$$\psi\Big(d(T(x(\omega)), T(y(\omega)))\Big) \leq \psi\Big(m(x(\omega), y(\omega))\Big) - \phi\Big[max\{d(x(\omega), T(y(\omega))), d(y(\omega), T(x(\omega)))\}\Big]$$

Case 2: *If* $y > x$ *then*

$$\psi\Big(d(T(x(\omega)), T(y(\omega)))\Big) = \frac{y-x}{2}$$

and

$$\begin{aligned}
m\Big(x(\omega), y(\omega)\Big) &= max\Big[d(x(\omega), y(\omega)), d(x(\omega), T(x(\omega))), d(y(\omega), T(y(\omega))), \frac{1}{2}\{d(x(\omega), T(y(\omega))) + d(y(\omega), T(x(\omega)))\}\Big] \\
&= max\Big[(y-x), (2x-\omega^2), (2y-\omega^2), \Big\{\frac{(x-\omega^2+y)}{2} + \frac{(y-\omega^2+x)}{2}\Big\}\Big] \\
&= max\Big[(y-x), (2x-\omega^2), (2y-1+\omega^2), (x+y-\omega^2)\Big] \\
&= (2y-\omega^2)
\end{aligned}$$

Then

$$\psi\Big(m(x(\omega), y(\omega))\Big) = \frac{(2y-\omega^2)}{2}$$

and

$$\begin{aligned}
\phi\Big[max\{d(x(\omega), T(y(\omega))), d(y(\omega), T(x(\omega)))\}\Big] &= max\Big[\frac{(x+y-\omega^2)}{4}, \frac{(x+y-\omega^2)}{4}\Big] \\
&= \frac{(x+y-\omega^2)}{4} \\
\psi\Big(m(x(\omega), y(\omega))\Big) - \phi\Big[max\{d(x(\omega), T(y(\omega))), d(y(\omega), T(x(\omega)))\}\Big] &= \frac{(2y-\omega^2)}{2} - \frac{(x+y-\omega^2)}{4} \\
&= \frac{3y-x-\omega^2}{4}
\end{aligned}$$

Since $y > x$ *and* $\omega \in \Omega$ *so* $\omega^2 \leq 1$ *so*

$$\frac{x-y}{2} \leq \frac{3y-x-\omega^2}{4}$$

Therefore,

$$\psi\Big(d(T(x(\omega)), T(y(\omega)))\Big) \leq \psi\Big(m(x(\omega), y(\omega))\Big) - \phi\Big[max\{d(x(\omega), T(y(\omega))), d(y(\omega), T(x(\omega)))\}\Big]$$

Therefore, the condition of the theorem (2.5) is satisfied. Now we define a measurable mapping $\xi : \Omega \to X$ *as* $\xi(\omega) = \frac{\omega^2}{2}$ *for every* $\omega \in \Omega$*.Then clearly* $\frac{\omega^2}{2}$ *is a random fixed point of the random operator* T*.*

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