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Domination Parameters in Shadow Graph and Path Connected Graph

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Abstract: A locating-dominating set (LDS) S of a graph G is a dominating set S of G such that for every two vertices u and v in $V(G) \setminus S$, $N(u) \cap S \neq N(v) \cap S$. The locating-domination number $\gamma^L(G)$ is the minimum cardinality of a LDS of G. Further, if S is a total dominating set then S is called a locating-total dominating set (LTDS). In this paper, we determine the locating-domination and locating-total domination numbers of the shadow graph and for a special class of path connected graph.

Keywords: Dominating set, total dominating set, locating-dominating set, locating-total dominating set, shadow graph, path connected graph.

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1. Introduction

A set S of vertices in a graph G is called a dominating set of G if every vertex in $V(G) \setminus S$ is adjacent to some vertex in S. The set S is said to be a total dominating set of G if every vertex in $V(G)$ is adjacent to some vertex in S. The minimum cardinality of a dominating set and a total dominating set of G is denoted as $\gamma(G)$ and $\gamma_t(G)$, respectively. Domination arises in facility location problems, where the number of facilities such as hospitals or fire stations are fixed and one attempts to minimize the distance that a person needs to travel to get to the closest facility. Domination has also been widely used in areas like locating radar station problem, coding theory, modelling biological networks, nuclear power plants and so on [1–4]. Total domination plays a role in the problem of placing monitoring devices in a system in such a way that every site in the system, including the monitors, is adjacent to a monitor site so that, if a monitor goes down, then an adjacent monitor can still protect the system. Installing minimum number of expensive sensors in the system which will transmit a signal at the detection of faults and uniquely determining the location of the faults motivates the concept of locating sets and locating-total dominating sets [5].

In a parallel computer, the processors and interconnection networks are modeled by the graph $G = (V, E)$, where each processor is associated with a vertex of G and a direct communication link between two processors is indicated by the existence of an edge between the associated vertices. Suppose we have limited resources such as disks, input-output connections, or software modules, and we want to place a minimum number of these resource units at the processors, so that every processor is adjacent to at least one resource unit, then finding such a placement involves constructing a minimum dominating set for the graph G. Installing minimum number of expensive sensors in a system which will transmit a signal at the detection of

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faults and uniquely determining the location of the faults give rise to the concept of locating-dominating sets and locating-total dominating sets [5]. Determining if an arbitrary graph has a locating-dominating set and locating-total dominating set of a given size are well-known NP-complete problems [6, 7].

In this paper, we obtain a lower bound for locating-dominating set and locating-total dominating set for the shadow graph and for a special class of path connected graph. We also give some graphs for which the lower bound attained for a special class of path connected graph is sharp.

2. Basic Concepts and Known Results

In this section, we give the preliminaries that are required for this study. All graphs considered in this paper are simple and connected. Here, P_m and C_m denotes a path and a cycle on m vertices, respectively.

Definition 2.1. *A locating-dominating set* (LDS) *in a graph* G *is a dominating set* S *of* G *such that for every pair of vertices* u *and* v *in* $V(G) \setminus S$, $N(u) \cap S \neq N(v) \cap S$. *The minimum cardinality of a locating-dominating set of* G *is called the locating-domination number* $\gamma^L(G)$.

Definition 2.2. *A locating-total dominating set* $(LTDS)$ *in a graph* G *is a total dominating set* S *of* G *such that for every pair of vertices* u *and* v *in* $V(G) \setminus S$, $N(u) \cap S \neq N(v) \cap S$. *The minimum cardinality of a locating total-dominating set of* G *is called the locating-total domination number* $\gamma_t^L(G)$.

Every locating-total dominating set of a graph G is also a locating-dominating set of the graph G, and so $\gamma_t^L(G) \geq \gamma_t^L(G)$ for every graph G. We say a vertex $u \in V \setminus S$ is *located* by S ($S \subseteq V(G)$), if $N(u) \cap S$ is distinct from any $N(v) \cap S$, for all $v \in V(G) \setminus S$.

Definition 2.3. *The shadow graph* $D_2(G)$ *of a graph* G *is constructed by taking two copies of* G, *say* G_1 *and* G_2. *Join each vertex* u *in* G_1 *to the neighbors of the corresponding vertex* u' *in* G_2.

A shadow graph $D_2(P_5)$ and $D_2(S_5)$ is shown in Figure 1, where P_5 and S_5 is path and star graph on 5 vertices, respectively.

Definition 2.4. *A path connected graph* $P(G, P_m, k)$ *of* G *is obtained by taking* k *copies of* G, *say* $G_1, G_2, \ldots, G_k$. G_i *is connected with* G_{i+1} *by a path* P_m *(call it as, binding path), such that left and right end vertices of* P_m *is joined by an edge to a vertex of* G_i *and* G_{i+1}, $1 \leq i \leq k-1$, *respectively. Moreover, the distance between two consecutive binding paths must be three. See Figure* 2.

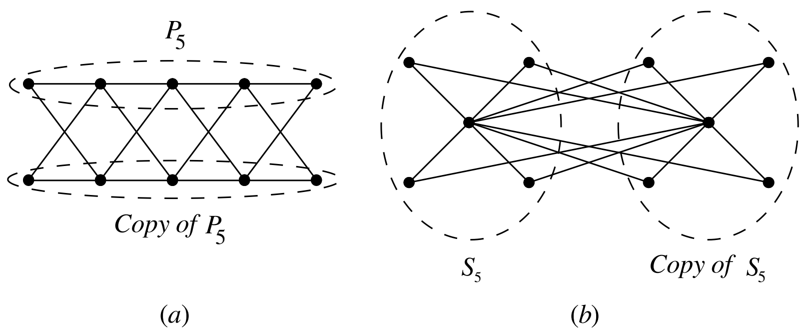

Figure 1. $(a) D_2(P_5)$ **and** $(b) D_2(S_5)$

Lemma 2.5 ([8])**.** *Let* G *be a graph of order* n *and maximum degree* Δ. *Then* $\gamma^L(G) \geq \frac{2n}{\Delta+3}$.

Lemma 2.6 ([9])**.** *If* G *is a graph of order* $n \geq 3$ *and maximum degree* $\Delta \geq 2$ *with no isolated vertex, then* $\gamma_t^L(G) \geq \frac{2n}{\Delta+2}$.

3. Main Results

In this section, we discuss about the locating-domination number and locating-total domination number for shadow graph and path connected graph.

3.1. Shadow graph $D_2(G)$

Lemma 3.1. *Let S be a locating-dominating set of G. If there exists vertices u and v in G such that $N(u) = N(v)$, then either u or v must be in S.*

Proof. Let u and v be the vertices in G such that $N(u) = N(v)$. Suppose S does not contain u and v. Then $N(u) \cap S = N(v) \cap S$, since every neighbor of u is a neighbor of v. This is a contradiction to S. Thus S contains either u or v. □

Theorem 3.2. *Let $D_2(G)$ be a shadow graph of G and $o(G) = n$, $n \geq 2$. Then $\gamma^L(D_2(G)) = \gamma_t^L(D_2(G)) \geq n$.*

Proof. Let S be a locating-dominating set of $D_2(G)$. Let G_1 and G_2 be the graphs isomorphic to G in $D_2(G)$. Let u be a vertex in G_1 which is connected to the neighbors of the corresponding vertex u' in G_2. By the definition of shadow graph, we have $N(u) = N(u')$ in $D_2(G)$. Thus by Lemma 3.1, either u or u' must be in S. Moreover, in $D_2(G)$ there are at least n such pairs like u and u' such that $N(u) = N(u')$. Therefore, $|S| \geq n$. As S contains no isolated vertex, S is also a locating-total dominating set of $D_2(G)$. Therefore, $\gamma_t^L(D_2(G)) \geq n$. □

Corollary 3.3. *Let $D_2(G)$ be a shadow graph of G and $o(G) = n$, $n \geq 2$. If there exist no vertices u and v in G such that $N(u) = N(v)$, then $\gamma^L(D_2(G)) = \gamma_t^L(D_2(G)) = n$.*

Corollary 3.4. *Let $D_2(G)$ be a shadow graph of G and $o(G) = n$, $n \geq 2$. If there exist m vertices, say $u_1, u_2, \ldots, u_m$ in G such that $N(u_i) = N(u_j)$, $1 \leq i, j \leq m$, then $\gamma^L(D_2(G)) = \gamma_t^L(D_2(G)) = n + m - 1$.*

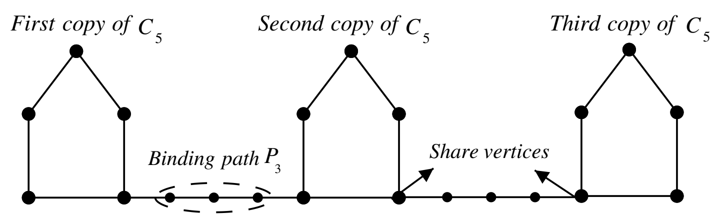

Figure 2. $P(C_5, P_3, 3)$

3.2. Path connected graph

The following notions will be helpful for proving the results. Two cycles, say C_n and C_n' of G are said to be distinct if they don't share a vertex in common. A vertex u of C_n in G is called a *share vertex* if $N(u)$ has a member outside C_n. See Figure 2.

Theorem 3.5. *Let G be a path connected graph of $P(C_n, P_m, k)$, $k \geq 2$ and $n, m \geq 1$. Then $\gamma^L(G) \geq k \left\lceil \frac{2(n-2)}{5} \right\rceil + (k - 1) \left\lceil \frac{2m}{5} \right\rceil$.*

Proof. Let S be a locating-dominating set of G. Let $G_1, G_2, \ldots, G_k$ be the k copies of C_n, and P_m be a binding path in G. Consider the worst case when one copy of C_n, say G_i, contains two share vertices. Now to locate the remaining vertices of G_i, by Lemma 2.5, we need at least $2(n-2)/5$ vertices from G_i in S, since the remaining vertices of G_i induces a path on $k-2$ vertices. Hence, S contains at least $k\left\lceil\frac{2(n-2)}{5}\right\rceil$ vertices, since G_i is taken as a random copy. It is clear that, if G contains k copies of C_n, then G contains $k-1$ copies of binding path P_m. By Lemma 2.5, we need at least $2m/5$ vertices from every copy of P_m. Hence, S contains at least $(k-1)\left\lceil\frac{2m}{5}\right\rceil$ vertices from $k-1$ copies of P_m. From the above arguments we can conclude that $|S| \geq k\left\lceil\frac{2(n-2)}{5}\right\rceil + (k-1)\left\lceil\frac{2m}{5}\right\rceil$. □

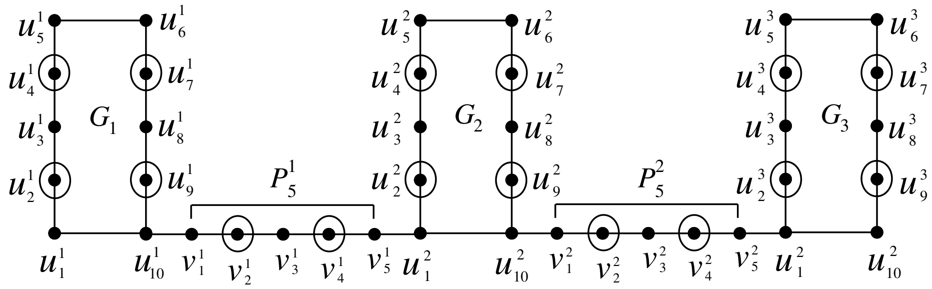

Figure 3. **Vertices in a locating-dominating set of $P(C_{10}, P_5, 3)$ are circled**

Theorem 3.6. *Let G be a path connected graph of $P(C_{5n}, P_{5m}, k)$, $k \geq 2$ and $n, m \geq 1$. Then $\gamma^L(G) = k\left\lceil\frac{2(n-2)}{5}\right\rceil + (k-1)\left\lceil\frac{2m}{5}\right\rceil$.*

Proof. Label the vertices of G_i as u_j^i where $1 \leq j \leq 5n$ and $1 \leq i \leq k$ in G. Let $P_m^1, P_m^2, \ldots, P_m^{k-1}$ be the $k-1$ copies of binding path P_m in G. Label the vertices of P_m as v_p^q, where $1 \leq p \leq 5m$ and $1 \leq q \leq k-1$ in G. See Figure 3. Let $S = \cup_{1\leq i\leq k}\cup_{1\leq j\leq n}\cup_{1\leq q\leq k-1}\cup_{1\leq p\leq m}\left\{u_{5j-1}^i, u_{5j-3}^i, v_{5p-1}^q, v_{5p-3}^q\right\}$. We claim that S is a locating-dominating set of G. Clearly S is a dominating set. We have only to prove that S is a locating-dominating set of G. Let $x, y \in V \setminus S$. If x and y are in different cycles, then $N(x) \cap S \neq N(y) \cap S$. Suppose x and y are in the same cycle, say G_i. Now $V(G) \setminus S = \{u_{5j}^i, u_{5j-2}^i, u_{5j-4}^i, v_{5p}^q, v_{5p-2}^q, v_{5p-4}^q\}$. If $x = u_{5j}^i$ and $y = u_{5j-2}^i$, $1 \leq i \leq k$ and $1 \leq j \leq n$, then $N(u_{5j}^i) \cap S = \{u_{5j-1}^i\} \neq \{u_{5j-1}^i, u_{5j-3}^i\} = N(u_{5j-2}^i) \cap S$. If $x = u_{5j}^i$ and $y = u_{5j-4}^i$, $1 \leq i \leq k$ and $1 \leq j \leq n$, then $N(u_{5j}^i) \cap S = \{u_{5j-1}^i\} \neq \{u_{5j-3}^i\} = N(u_{5j-4}^i) \cap S$. If $x = u_{5j-2}^i$ and $y = u_{5j-4}^i$, $1 \leq i \leq k$ and $1 \leq j \leq n$, then $N(u_{5j-2}^i) \cap S = \{u_{5j-1}^i, u_{5j-3}^i\} \neq \{u_{5j-3}^i\} = N(u_{5j-4}^i) \cap S$. Thus $N(x) \cap S \neq N(y) \cap S$. Now consider if x and y are in different binding paths, then $N(x) \cap S \neq N(y) \cap S$. Suppose x and y are in the same binding path, then by the similar argument as in the case of x and y are in different cycles, we can conclude that $N(x) \cap S \neq N(y) \cap S$. Thus S is a locating-dominating set of G. Therefore $|S| \leq k\left\lceil\frac{2(n-2)}{5}\right\rceil + (k-1)\left\lceil\frac{2m}{5}\right\rceil$. By Theorem 3.5, $|S| = k\left\lceil\frac{2(n-2)}{5}\right\rceil + (k-1)\left\lceil\frac{2m}{5}\right\rceil$. □

Theorem 3.7. *Let G be a path connected graph of $P(C_n, P_m, k)$, $k \geq 2$ and $n, m \geq 1$. Then $\gamma_t^L(G) \geq k\left\lceil\frac{2(n-2)}{4}\right\rceil + (k-1)\left\lceil\frac{2m}{4}\right\rceil$*

Proof. Let S be a locating-total dominating set of G. Let $G_1, G_2, \ldots, G_k$ be the k copies of C_n and P_m be a binding path in G. Consider the worst case when one copy of C_n say, G_i, contains two share vertex. Now to locate the remaining vertices of G_i, by Lemma 2.6, we need at least $2(n-2)/4$ vertices from G_i in S, since the remaining vertices of G_i induces a path on $k-2$ vertices. Hence, S contains at least $k\left\lceil\frac{2(n-2)}{4}\right\rceil$ vertices, since G_i is taken as a random copy. It is clear that, if G contains k copies of C_n then G contains $k-1$ copies of binding path P_m. By Lemma 2.6, we need at least $2m/4$ vertices from every copy of P_m. Hence, S contains at least $(k-1)\left\lceil\frac{2m}{4}\right\rceil$ vertices. From the above arguments we can conclude that $|S| \geq k\left\lceil\frac{2(n-2)}{4}\right\rceil + (k-1)\left\lceil\frac{2m}{4}\right\rceil$. □

Theorem 3.8. *Let G be a path connected graph of $P(C_{4n}, P_{4m}, k)$, $k \geq 2$ and $n, m \geq 1$. Then $\gamma_t^L(G) = k\left\lceil\frac{2(n-2)}{4}\right\rceil + (k-1)\left\lceil\frac{2m}{4}\right\rceil$.*

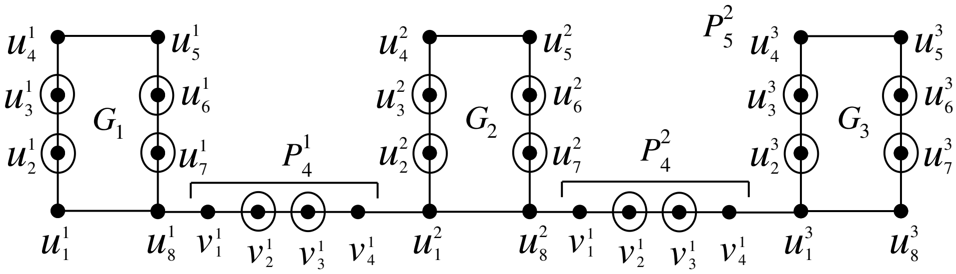

Figure 4. **Vertices in a locating-total dominating set of $P(C_8, P_4, 3)$ are circled**

Proof. Label the vertices of G_i as u_j^i where $1 \leq j \leq 45n$ and $1 \leq i \leq k$ in G. Let $P_m^1, P_m^2, \ldots, P_m^{k-1}$ be the $k-1$ copies of binding path P_m in G. Label the vertices of P_m as v_p^q where $1 \leq p \leq 4m$ and $1 \leq q \leq k-1$ in G. See Figure 4. Let $S = \cup_{1\leq i\leq k} \cup_{1\leq j\leq n} \cup_{1\leq q\leq k-1} \cup_{1\leq p\leq m} \left\{u_{4j-1}^i, u_{4j-2}^i, v_{4p-1}^q, v_{4p-2}^q\right\}$. We claim that S is a locating-total dominating set of G. Clearly S is a total dominating set. We have only to prove that S is a locating-total dominating set of G. Let $x, y \in V \setminus S$. If x and y are in different cycles, then $N(x) \cap S \neq N(y) \cap S$. Suppose x and y are in the same cycle, say G_i. Now $V(G) \setminus S = \{u_{4j}^i, u_{4j-3}^i, v_{4p}^q, v_{4p-3}^q\}$. If $x = u_{4j}^i$ and $y = u_{4j-3}^i$, $1 \leq i \leq k$ and $1 \leq j \leq n$, then $N(u_{4j}^i) \cap S = \{u_{4j-1}^i\} \neq \{u_{4j-2}^i\} = N(u_{4j-3}^i) \cap S$. Thus $N(x) \cap S \neq N(y) \cap S$. Now consider if x and y are in different binding paths, then $N(x) \cap S \neq N(y) \cap S$. Suppose x and y are in the same binding path, then by the similar argument as in the case of x and y are in different cycles, we can conclude that $N(x) \cap S \neq N(y) \cap S$. Thus S is a locating-total dominating set of G. Therefore $|S| \leq k\left\lceil \frac{2(n-2)}{4} \right\rceil + (k-1)\left\lceil \frac{2m}{4} \right\rceil$. By Theorem 3.7, $|S| = k\left\lceil \frac{2(n-2)}{4} \right\rceil + (k-1)\left\lceil \frac{2m}{4} \right\rceil$. $\square$

4. Conclusion

In this paper, we have determined a lower bound for the locating-dominating set and locating-total dominating set for the shadow graph and path connected graph $P(C_n, P_m, k), k \geq 2$. Moreover, the bound attained for the locating-domination and locating-total domination problems for a path connected graph are sharp for $P(C_{5n}, P_{5m}, k)$ and $P(C_{4n}, P_{4m}, k)$, respectively. It would be interesting to focus our study on finding the locating-domination number and locating-total domination number for $P(C_n, P_m, k)$.

References

[1] E. J. Cockayne and S. T. Hedetniemi, *Towards a theory of domination in graphs*, Networks, 7(3)(1977), 247-261.

[2] C. Berge, *Graphs and hypergraphs*, North Holland Publisher, Amsterdam, Netherlands, (1973).

[3] T. Haynes, D. Knisley, E. Seier and Y. Zou, *A quantitative analysis of secondary RNA structure using domination based parameters on trees*, BMC Bioinformatics, 7(2006), 108-118.

[4] J. G. Kalbeisch, R. G. Stanton and J. D. Horton, *On covering sets and error-correcting codes*, Journal of Combinatorial Theory, Series A, 11(3)(1971), 233-250.

[5] T. W. Haynes, M. A. Henning and J. Howard, *Locating and total dominating sets in trees*, Discrete Applied Mathematics, 154(8)(2006), 1293-1300.

[6] I. Charon, O. Hudry and A. Lobstein, *Minimizing the size of an identifying or locating-dominating code in a graph is NP-hard*, Theoretical Computer Science, 290(3)(2006), 2109-2120.

[7] Mirka Miller, R. Sundara Rajan, R. Jayagopal, Indra Rajasingh and Paul Manuel, *A Note on the Locating-Total Domi-*

nation in Graphs, Discussiones Mathematicae Graph Theory, 37(2017), 745-754.

[8] P. J. Slater, *Dominating and reference sets in a graph*, Journal of Mathematical Physics, 22(1988), 445-455.

[9] J. McCoy and M. A. Henning, *Locating and paired-dominating sets in graphs*, Discrete Applied Mathematics, 157(2009), 3268-3280.