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Degree Square Sum Energy of Graphs

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Abstract: Degree square sum matrix $DSS(G)$ of a graph G is a square matrix of order equal to the order of G with its $(i,j)^{th}$ entry as ${d_i}^2 + {d_j}^2$ if $i \neq j$ and zero otherwise, where d_i is the degree of the i^{th} vertex of G. In this paper, we obtain the characteristic polynomial of the degree square sum matrix of graphs obtained by some graph operations. In addition, bounds for largest degree square sum eigenvalue and for degree square sum energy of graphs are obtained.

MSC: 05C50.

Keywords: Degree square sum matrix, degree square sum polynomial, Eigenvalues, Graph operations.

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1. Introduction

In the literature of Graph Theory, there are several graph polynomials based on matrices such as adjacency matrix [7], Laplacian matrix [12], signless Laplacian matrix [8, 15], distance matrix [1], degree sum matrix [11, 16], seidel matrix [4], degree exponent matrix [17] etc. In this paper, we introduce one more such new matrix of a graph, called *degree square sum matrix*. We study the characteristic polynomial of the degree square sum matrix of graphs obtained by some graph operations. We also give bounds for the maximum eigenvalue of this matrix named as largest degree square sum eigenvalue. We extend our work by giving bounds for degree square sum energy of a graph. Let G be a simple, undirected graph with n vertices and m edges. Let $V(G)$ be the vertex set and $E(G)$ be an edge set of G. The *degree* $deg_G(v)$ (or $d_G(v)$) of a vertex $v \in V(G)$ is the number of edges incident to it in G. The graph G is *r-regular* if the degree of each vertex in G is r. Let v_1, v_2,..., v_n be the vertices of G and let $d_i = deg_G(v_i)$. For undefined terminologies we refer [10]. The degree square sum matrix of a graph G is an $n \times n$ matrix denoted by $DSS(G) = [dss_{ij}]$ and whose elements are defined as

$$dss_{ij} = \begin{cases} {d_i}^2 + {d_j}^2 & if\ i \neq j \\ 0 & otherwise. \end{cases}$$

Let I be an identity matrix and J be a matrix whose all entries are equal to 1. The *degree square sum polynomial* of a graph G is defined as

$$P_{DSS(G)}(\mu) = det(\mu I - DSS(G)).$$

The *eigenvalues* of $DSS(G)$, denoted by $\mu_1, \mu_2, ..., \mu_n$ are called the *degree square sum eigenvalues* of G and their collection is called the *degree square sum spectra* of G. It is easy to see that if G is an *r-regular* graph, then $DSS(G) = 2r^2 J - 2r^2 I$.

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Therefore, for an *r-regular* graph G of order n,

$$P_{DSS(G)}(\mu) = [\mu - 2r^2(n-1)][\mu + 2r^2]^{n-1}. \tag{1}$$

Example 1.1. *Let $G = P_4$ be a graph.*

Figure 1.

Then the degree square sum matrix, degree square sum polynomial and the degree square sum energy of G are as follows:

$$DSS(G) = \begin{bmatrix} 0 & 5 & 5 & 2 \\ 5 & 0 & 8 & 5 \\ 5 & 8 & 0 & 5 \\ 2 & 5 & 5 & 0 \end{bmatrix}, \ P_{DSS}(G;\mu) = \mu^4 - 168\mu^2 - 100\mu - 1344, \ E_{DSS}(G) = 20.$$

Example 1.2. *Let $H = C_4$ be a 2-regular graph.*

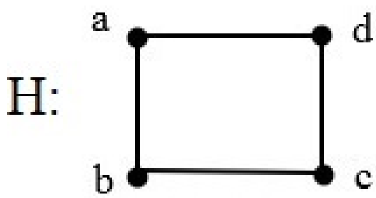

Figure 2.

Then the degree square sum matrix, degree square sum polynomial and the degree square sum energy of H are as follows:

$$DSS(H) = \begin{bmatrix} 0 & 8 & 8 & 8 \\ 8 & 0 & 8 & 8 \\ 8 & 8 & 0 & 8 \\ 8 & 8 & 8 & 0 \end{bmatrix}, \ P_{DSS}(H;\mu) = \mu^4 - 384\mu^2 - 4096\mu - 12288, \ E_{DSS}(H) = 4r^2(n-1) = 48,$$

where r is the degree of each vertex in H.

2. Degree Square Sum Polynomial of Graphs Obtained by Graph Operations

In this section, we obtain the degree square sum polynomial of graphs obtained by some graph operations. The *line graph* [10] $L(G)$ of a graph G is a graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G are adjacent. The k^{th} *iterated line graph* [5, 6, 10] of G is defined as $L^k(G) = L(L^{k-1}(G))$, $k = 1, 2, ...,$ where $L^0(G) \cong G$ and $L^1(G) \cong L(G)$.

Theorem 2.1. *Let G be an r-regular graph of order n and n_k be the order of $L^k(G)$. Then the degree square sum polynomial of $L^k(G)$, $k = 1, 2, \ldots$ is*

$$P_{DSS(L^k(G))}(\mu) = [\mu + 2(2^k r - 2^{k+1} + 2)^2]^{n_k - 1}[\mu - 2(n_k - 1)(2^k r - 2^{k+1} + 2)^2].$$

Proof. The line graph of a regular graph is a regular graph. In particular, the line graph of an r-regular graph G of order n is an $r_1 = (2r-2)$-regular graph of order $n_1 = \frac{1}{2}nr$. Thus, $L^k(G)$ is an r_k-regular graph of order n_k given by [5, 6]

$$n_k = \frac{n}{2^k}\prod_{i=0}^{k-1}(2^i r - 2^{i+1} + 2) \quad \text{and} \quad r_k = 2^k r - 2^{k+1} + 2.$$

Hence the result follows from Equation (1). □

The following lemma is useful for proving the forthcoming theorems.

Lemma 2.2 ([17])**.** *If a, b, c and d are real numbers, then the determinant of the form*

$$\begin{vmatrix} (\mu + a)I_{n_1} - aJ_{n_1} & -cJ_{n_1 \times n_2} \\ -dJ_{n_2 \times n_1} & (\mu + b)I_{n_2} - bJ_{n_2} \end{vmatrix} \tag{2}$$

of order $n_1 + n_2$ can be expressed in the simplified form as

$$(\mu + a)^{n_1 - 1}(\mu + b)^{n_2 - 1}\{[\mu - (n_1 - 1)a][\mu - (n_2 - 1)b] - n_1 n_2 cd\}.$$

The *subdivision graph* [10] $S(G)$ of a graph G is a graph with vertex set $V(G) \cup E(G)$ and is obtained by inserting a new vertex of degree 2 into each edge of G.

Theorem 2.3. *Let G be an r-regular graph of order n and size m. Then*

$$P_{DSS(S(G))}(\mu) = (\mu + 2r^2)^{n-1}(\mu + 8)^{m-1}\{\mu^2 - 2[4(m-1) - r^2(n-1)]\mu + 16r^2(n-1)(m-1) - mn(r^2+4)^2\}.$$

Proof. The subdivision graph of an r-regular graph has two types of vertices. The n vertices with degree r and m vertices are with degree 2. Hence

$$DSS(S(G)) = \begin{bmatrix} 2r^2(J_n - I_n) & (r^2 + 2^2)J_{n \times m} \\ (2^2 + r^2)J_{m \times n} & 2^3(J_m - I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(S(G))}(\mu) &= |\mu I - DSS(S(G))| \\ &= \begin{vmatrix} (\mu + 2r^2)I_n - 2r^2 J_n & -(r^2 + 2^2)J_{n \times m} \\ -(2^2 + r^2)J_{m \times n} & (\mu + 2^3)I_m - 2^3 J_m) \end{vmatrix} \end{aligned}$$

Now using Lemma 2.2, we get the desired result. □

The *semitotal point graph* [18] $T_2(G)$ of a graph G is a graph with vertex set $V(G) \cup E(G)$ where two vertices of $T_2(G)$ are adjacent if and only if they corresponds to two adjacent vertices of G or one is a vertex of G and another is an edge G incident with it in G.

Theorem 2.4. *Let G be an r-regular graph of order n and size m. Then*

$$P_{DSS(T_2(G))}(\mu) = (\mu+8r^2)^{n-1}(\mu+8)^{m-1}\{\mu^2 - 8[(m-1)-r^2(n-1)]\mu + 16[4r^2(m-1)(n-1) - mn(r^2+1)^2]\}.$$

Proof. The semitotal point graph of an r-regular graph has two types of vertices. The n vertices with degree $2r$ and m vertices are of degree 2. Hence

$$DSS(T_2(G)) = \begin{bmatrix} 8r^2(J_n - I_n) & 4(r^2+1)J_{n\times m} \\ 4(r^2+1)J_{m\times n} & 2^3(J_m - I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T_2(G))}(\mu) &= |\mu I - DSS(T_2(G))| \\ &= \begin{vmatrix} (\mu+8r^2)I_n - 8r^2 J_n & -4(r^2+1)J_{n\times m} \\ -4(r^2+1)J_{m\times n} & (\mu+8)I_m - 8J_m) \end{vmatrix} \end{aligned}$$

Now using Lemma 2.2, we get the expected result. □

The *semitotal line graph* [18] $T_1(G)$ of a graph G is a graph with vertex set $V(G)\cup E(G)$ where two vertices of $T_1(G)$ are adjacent if and only if they corresponds to two adjacent edges of G or one is a vertex of G and another is an edge G incident with it in G.

Theorem 2.5. *Let G be an r-regular graph of order n and size m. Then*

$$P_{DSS(T_1(G))}(\mu) = (\mu+2r^2)^{n-1}(\mu+8r^2)^{m-1}\{\mu^2 - 2r^2[4(m-1)+(n-1)]\mu + r^4[16(n-1)(m-1) - 25mn]\}.$$

Proof. The semitotal line graph of an r-regular graph has two types of vertices. The n vertices with degree r and the remaining m vertices are of degree $2r$. Hence

$$DSS(T_1(G)) = \begin{bmatrix} 2r^2(J_n - I_n) & 5r^2 J_{n\times m} \\ 5r^2 J_{m\times n} & 8r^2(J_m - I_m) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(T_1(G))}(\mu) &= |\mu I - DSS(T_1(G))| \\ &= \begin{vmatrix} (\mu+2r^2)I_n - 2r^2 J_n & -5r^2 J_{n\times m} \\ -5r^2 J_{m\times n} & (\mu+8r^2)I_m - 8r^2 J_m) \end{vmatrix} \end{aligned}$$

Now by using Lemma 2.2, we get the required result. □

The *total graph* [10] $T(G)$ of a graph G is the graph whose vertex set is $V(G)\cup E(G)$ and two vertices of $T(G)$ are adjacent if and only if the corresponding elements of G are either adjacent or incident.

Theorem 2.6. *Let G be an r-regular graph of order n and size m. Then*

$$P_{DSS(T(G))}(\mu) = (\mu+8r^2)^{n+m-1}[\mu - 8r^2(n+m-1)].$$

Proof. The total graph of an r-regular graph is a regular graph of degree $2r$ with $n+m$ vertices. Hence the result follows from Equation (1). $\square$

The graph G^{+k} [17] is a graph obtained from the graph G by attaching k pendant edges to each vertex of G. If G is a graph of order n and size m, then G^{+k} is graph of order $n+nk$ and size $m+nk$.

Theorem 2.7. *Let G be an r-regular graph of order n and size m. Then*

$$\begin{aligned} P_{DSS(G^{+k})}(\mu) &= [\mu+2(r+k)^2]^{n-1}(\mu+2)^{nk-1}\{\mu^2-2[(nk-1)+(n-1)(r+k)^2]\mu \\ &\quad +4(n-1)(nk-1)(r+k)^2-n^2k[1+(r+k)^2]^2\}. \end{aligned}$$

Proof. The graph G^{+k} of an r-regular graph has two types of vertices. The n vertices with degree $r+k$ and the remaining nk vertices are of degree 1. Hence

$$DSS(G^{+k}) = \begin{bmatrix} 2(r+k)^2(J_n-I_n) & [(r+k)^2+1]J_{n\times nk} \\ [(r+k)^2+1]J_{nk\times n} & 2(J_{nk}-I_{nk}) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS(G^{+k})}(\mu) &= |\mu I - DSS(G^{+k})| \\ &= \begin{vmatrix} [\mu+2(r+k)^2]I_n - 2(r+k)^2J_n & -[(r+k)^2+1]J_{n\times nk} \\ -[(r+k)^2+1]J_{nk\times n} & (\mu+2)I_{nk}-2J_{nk} \end{vmatrix} \end{aligned}$$

Next by using Lemma 2.2, we get the desired result. $\square$

The *union* [10] of the graphs G_1 and G_2 is a graph $G_1\cup G_2$ whose vertex set is $V(G_1\cup G_2)=V(G_1)\cup V(G_2)$ and the edge set $E(G_1\cup G_2)=E(G_1)\cup E(G_2)$.

Theorem 2.8. *Let G be an r_1-regular graph of order n_1 and H be an r_2-regular graph of order n_2. Then*

$$P_{DSS(G\cup H)}(\mu) = P_{DSS(G)}(\mu)P_{DSS(H)}(\mu) - (\mu+2r_1^2)^{n_1-1}(\mu+2r_2^2)^{n_2-1}n_1n_2(r_1^2+r_2^2)^2.$$

Proof. The degree square matrix of $G\cup H$ will be of the form

$$\begin{aligned} DSS(G\cup H) &= \begin{bmatrix} DSS(G) & (r_1^2+r_2^2)J_{n_1\times n_2} \\ (r_1^2+r_2^2)J_{n_2\times n_1} & DSS(H) \end{bmatrix} \\ &= \begin{bmatrix} 2r_1^2(J_{n_1}-I_{n_1}) & (r_1^2+r_2^2)J_{n_1\times n_2} \\ (r_1^2+r_2^2)J_{n_2\times n_1} & 2r_2^2(J_{n_2}-I_{n_2}) \end{bmatrix} \end{aligned}$$

Therefore,

$$\begin{aligned} P_{DSS(G\cup H)}(\mu) &= |\mu I - DSS(G\cup H)| \\ &= \begin{vmatrix} (\mu+2r_1^2)I_{n_1}-2r_1^2J_{n_1} & -(r_1^2+r_2^2)J_{n_1\times n_2} \\ -(r_1^2+r_2^2)J_{n_2\times n_1} & (\mu+2r_2^2)I_{n_2}-2r_2^2J_{n_2}) \end{vmatrix} \end{aligned}$$

Using Lemma 2.2, we get

$$P_{DSS(G\cup H)}(\mu)=(\mu+2r_1^2)^{n_1-1}(\mu+2r_2^2)^{n_2-1}\{[\mu-(n_1-1)2r_1^2][\mu-(n_2-1)2r_2^2]-n_1n_2(r_1^2+r_2^2)^2\} \tag{3}$$

As G and H are regular graphs of order n_1 and n_2 and of degree r_1 and r_2 respectively, by Equation (1) we have

$$P_{DSS(G)}(\mu)=(\mu+2r_1^2)^{n_1-1}[\mu-(n_1-1)2r_1^2] \tag{4}$$

and

$$P_{DSS(H)}(\mu)=(\mu+2r_2^2)^{n_2-1}[\mu-(n_2-1)2r_2^2] \tag{5}$$

The result follows by substituting the Equations (4) and (5) in Equation (3). □

The *join* [10] G_1+G_2 of two graphs G_1 and G_2 is the graph obtained from G_1 and G_2 by joining every vertex of G_1 to all vertices of G_2.

Theorem 2.9. *Let G be an r_1-regular graph of order n_1 and H be an r_2-regular graph of order n_2. Then*

$$\begin{aligned}P_{DSS(G+H)}(\mu) &= [\mu+2R_1^2]^{n_1-1}[\mu+2R_2^2]^{(n_2-1)}\{\mu^2-2[(n_2-1)R_2^2+(n_1-1)R_1^2]\mu\\ &\quad +4(n_1-1)(n_2-1)R_1^2R_2^2-n_1n_2[R_1^2+R_2^2]\}.\end{aligned}$$

Proof. If G is an r_1-regular graph of order n_1 and H is an r_2-regular graph of order n_2, then $G+H$ has two types of vertices, the n_1 vertices of degree $R_1=r_1+n_2$ and the remaining n_2 vertices are of degree $R_2=r_2+n_1$. Hence

$$DSS(G+H)=\begin{bmatrix} 2R_1^2(J_{n_1}-I_{n_1}) & (R_1^2+R_2^2)J_{n_1\times n_2}\\ (R_1^2+R_2^2)J_{n_2\times n_1} & 2R_2^2(J_{n_2}-I_{n_2})\end{bmatrix}$$

Therefore,

$$\begin{aligned}P_{DSS(G+H)}(\mu) &= |\mu I-DSS(G+H)|\\ &= \begin{vmatrix} (\mu+2R_1^2)I_{n_1}-2R_1^2J_{n_1} & -(R_1^2+R_2^2)J_{n_1\times n_2}\\ -(R_1^2+R_2^2)J_{n_2\times n_1} & (\mu+2R_2^2)I_{n_2}-2R_2^2J_{n_2}\end{vmatrix}\end{aligned}$$

Now by using Lemma 2.2, we get the required result. □

The *product* [10] $G\times H$ of two graphs G and H is defined as follows:
Consider any two points $u=(u_1,u_2)$ and $v=(v_1,v_2)$ in $V=V_1\times V_2$. Then u and v are adjacent in $G\times H$ whenever $[u_1=v_1$ and u_2 *adj* $v_2]$ or $[u_2=v_2$ and u_1 *adj* $v_1]$.

Theorem 2.10. *Let G be an r_1-regular graph of order n_1 and H be an r_2-regular graph of order n_2. Then*

$$P_{DSS(G\times H)}(\mu) = [\mu-2(n_1n_2-1)(r_1+r_2)^2][\mu+2(r_1+r_2)^2]^{(n_1n_2-1)}.$$

Proof. G is an r_1-regular graph of order n_1 and H is an r_2-regular graph of order n_2. Then $G\times H$ is an (r_1+r_2)-regular graph with n_1n_2 vertices. Hence the result follows from Equation (1). □

The *composition* [10] $G[H]$ of two graphs G and H is defined as follows:

Consider any two points $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V = V_1 \times V_2$. Then u and v are adjacent in $G[H]$ whenever $[u_1\ adj\ v_1\ ]$ or $[u_1 = v_1$ and $u_2\ adj\ v_2]$.

Theorem 2.11. *Let G be an r_1-regular graph of order n_1 and H be an r_2-regular graph of order n_2. Then*

$$P_{DSS(G[H])}(\mu) \ = \ [\mu - 2(n_1n_2 - 1)(n_2r_1 + r_2)^2][\mu + 2(r_1 + r_2)^2]^{(n_1n_2-1)}.$$

Proof. G is an r_1-regular graph of order n_1 and H is an r_2-regular graph of order n_2. Then $G[H]$ is an $(n_2r_1 + r_2)$-regular graph with n_1n_2 vertices. Hence the result follows from Equation (1). Similarly one can write degree square sum polynomial for $H[G]$. □

The *corona* [10] $G \circ H$ of graphs G and H is a graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and then joining by an edge each vertex of the i^{th} copy of H is named (H, i) with the i^{th} vertex of G.

Theorem 2.12. *Let G be an r_1-regular graph of order n_1 and H be an r_2-regular graph of order n_2. Then*

$$P_{DSS}(G \circ H; \mu) = [\mu + 2R_1^2]^{n_1-1}[\mu + 2R_2^2]^{n_1n_2-1}\big([\mu - 2(n_1 - 1)R_1^2][\mu - 2(n_1n_2 - 1)R_2^2] - n_1^2n_2[R_1^2 + R_2^2]\big).$$

Proof. If G is an r_1-regular graph of order n_1 and H is an r_2-regular graph of order n_2, then $G \circ H$ has two types of vertices, the n_1 vertices with degree $R_1 = r_1 + n_2$ and the remaining n_1n_2 vertices are of degree $R_2 = r_2 + 1$. Hence

$$DSS(G \circ H) = \begin{bmatrix} 2R_1^2(J_{n_1} - I_{n_1}) & (R_1^2 + R_2^2)J_{n_1 \times n_1n_2} \\ (R_1^2 + R_2^2)J_{n_1n_2 \times n_1} & 2R_2^2(J_{n_1n_2} - I_{n_1n_2}) \end{bmatrix}$$

Therefore,

$$\begin{aligned} P_{DSS}(G \circ H; \mu) \ &= \ |\mu I - DSS(G \circ H)| \\ &= \begin{vmatrix} (\mu + 2R_1^2)I_{n_1} - 2R_1^2J_{n_1} & -(R_1^2 + R_2^2)J_{n_1 \times n_1n_2} \\ -(R_1^2 + R_2^2)J_{n_1n_2 \times n_1} & (\mu + 2R_2^2)I_{n_1n_2} - 2R_2^2J_{n_1n_2}) \end{vmatrix} \end{aligned}$$

Using Lemma 2.2, we get the required result. □

The Cauchy-Schwarz inequality [2] states that if $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are n real vectors, then

$$\left(\sum_{i=1}^{n} a_ib_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \tag{6}$$

3. Bounds for the Largest Degree Square Sum Eigenvalue

Since $trace(DSS(G)) = 0$, the eigenvalues of $DSS(G)$ satisfies the following relations:

$$\sum_{i=1}^{n} \mu_i = 0 \tag{7}$$

Further,

$$\sum_{i=1}^{n} \mu_i^2 \ = \ trace([DSS(G)]^2)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}d_{ji} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2 \\
&= 2\sum_{i<j}(d_i^2+d_j^2)^2.
\end{aligned}
$$

$$
\sum_{i=1}^{n}\mu_i^2 = 2\mathcal{K},\ \textit{where}\ \mathcal{K} = \sum_{i<j}(d_i^2+d_j^2)^2. \tag{8}
$$

The following results are useful for proving further results in this paper.

Theorem 3.1 ([14])**.** *Let a_i and b_i are nonnegative real numbers, then*

$$
\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \le \frac{1}{4}\left(\sqrt{\frac{M_1M_2}{m_1m_2}}+\sqrt{\frac{m_1m_2}{M_1M_2}}\right)^2\left(\sum_{i=1}^{n} a_ib_i\right)^2. \tag{9}
$$

where, $M_1 = max_{1\le i\le n}(a_i)$; $M_2 = max_{1\le i\le n}(b_i)$; $m_1 = min_{1\le i\le n}(a_i)$; $m_2 = min_{1\le i\le n}(b_i)$.

Theorem 3.2 ([13])**.** *Let a_i and b_i are nonnegative real numbers, then*

$$
\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_ib_i\right)^2 \le \frac{n^2}{4}\left(M_1M_2-m_1m_2\right)^2. \tag{10}
$$

where, $M_1 = max_{1\le i\le n}(a_i)$; $M_2 = max_{1\le i\le n}(b_i)$; $m_1 = min_{1\le i\le n}(a_i)$; $m_2 = min_{1\le i\le n}(b_i)$.

Theorem 3.3 ([3])**.** *Let a_i and b_i are nonnegative real numbers, then*

$$
\left| n\sum_{i=1}^{n} a_ib_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \right| \le \alpha(n)(A-a)(B-b) \tag{11}
$$

where a, b, A and B are real constants such that $a \le a_i \le A$ and $b \le b_i \le B$ for each $i, 1 \le i \le n$. Further, $\alpha(n) = n\lfloor \frac{n}{2} \rfloor(1-\frac{1}{n}\lfloor \frac{n}{2} \rfloor)$.

Theorem 3.4 ([9])**.** *Let a_i and b_i are nonnegative real numbers, then*

$$
\sum_{i=1}^{n} b_i^2 + C_1C_2\sum_{i=1}^{n} a_i^2 \le (C_1+C_2)\left(\sum_{i=1}^{n} a_ib_i\right) \tag{12}
$$

where C_1 and C_2 are real constants such that $C_1a_i \le b_i \le C_2a_i$ for each $i, 1 \le i \le n$.

Theorem 3.5. *Let G be an r-regular graph of order n. Then G has only one positive degree square sum eigenvalue $\mu = 2r^2(n-1)$.*

Proof. Let G be a connected r-regular graph of order n and $\{v_1, v_2, ..., v_n\}$ be the vertex set of G. Let $d_i = r$ be the degree of $v_i, i = 1, 2, ..., n$. Then

$$
d_{ij} = \begin{cases} d_i^2 + d_j^2 = 2r^2 & if\ \ i \ne j \\ 0 & otherwise. \end{cases}
$$

Therefore, the characteristic polynomial of $DSS(G)$ is,

$$
\begin{aligned}
P_{DSS}(G;\mu) &= det(\mu I - DSS(G)) \\
\Longrightarrow det(\mu I - DSS(G)) &= det(\mu I - 2r^2A(K_n))
\end{aligned}
$$

$$
\begin{aligned}
&= (2r^2)^n \left| \frac{\mu}{2r^2} I - A(K_n) \right| \\
&= (2r^2)^n \left(\frac{\mu}{2r^2} - n + 1 \right) \left(\frac{\mu}{2r^2} + 1 \right)^{n-1} \\
&= (\mu - 2r^2(n-1))(\mu + 2r^2)^{n-1}
\end{aligned}
$$

Thus, $(\mu - 2r^2(n-1))(\mu + 2r^2)^{n-1} = 0$ will give

$$
\mu = \begin{cases} 2r^2(n-1) & 1 \ time \\ -2r^2 & (n-1) \ times. \end{cases}
$$

□

Theorem 3.6. *Let G be any graph of order n and μ_1 be the largest degree square sum eigenvalue. Then*

$$
\mu_1 \leq \sqrt{\frac{2\mathcal{K}(n-1)}{n}} \tag{13}
$$

Proof. Substituting $a_i = 1$ and $b_i = \mu_i$ for $i = 2, 3, ..., n$ in Equation (6), we get

$$
\left(\sum_{i=1}^{n} \mu_i \right)^2 \leq (n-1) \left(\sum_{i=1}^{n} \mu_i^2 \right) \tag{14}
$$

Again from Equations (7) and (8), we have

$$
\sum_{i=2}^{n} \mu_i = -\mu_1 and \sum_{i=2}^{n} \mu_i^2 = 2\mathcal{K} - \mu_1^2
$$

Therefore, Equation (14) becomes,

$$
(-\mu_1)^2 \leq (n-1)(2\mathcal{K} - \mu_1^2).
$$

Hence,

$$
\mu_1 \leq \sqrt{\frac{2\mathcal{K}(n-1)}{n}}
$$

Equality holds when G is a regular graph. □

4. Bounds for the Degree Square Sum Energy of Graphs

Theorem 4.1. *Let G be an r-regular graph of order n. Then $-2r^2$ and $2r^2(n-1)$ are degree square sum eigenvalues of G with multiplicities $(n-1)$ and 1 respectively and $E_{DSS}(G) = 4r^2(n-1)$.*

Proof.

$$
\begin{aligned}
|\mu I - DSS(G)| &= \begin{vmatrix} \mu & -2r^2 & -2r^2 & \dots & -2r^2 \\ -2r^2 & \mu & -2r^2 & \dots & -2r^2 \\ -2r^2 & -2r^2 & \mu & \dots & -2r^2 \\ \dots & \dots & \dots & \dots & \dots \\ -2r^2 & -2r^2 & -2r^2 & \dots & \mu \end{vmatrix} \\
&= (\mu + 2r^2)^{n-1} \begin{vmatrix} \mu & -2r^2 & -2r^2 & \dots & -2r^2 \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ -1 & 0 & 0 & \dots & 1 \end{vmatrix}
\end{aligned}
$$

$$= (\mu - 2r^2(n-1))(\mu + 2r^2)^{n-1}.$$

Thus,

$$E_{DSS}(G) = 4r^2(n-1).$$

□

Theorem 4.2. *Let G be a graph of order n and size m. Then*

$$E_{DSS}(G) \geq \sqrt{2n\mathcal{K} - \frac{n^2}{4}(|\mu_1| - |\mu_n|)^2} \tag{15}$$

where $|\mu_1|$ and $|\mu_2|$ are maximum and minimum of the absolute value of μ_i's.

Proof. Suppose $\mu_1, \mu_2, ..., \mu_n$ are the eigenvalues of $DSS(G)$. Substituting $a_i = 1$ and $b_i = |\mu_i|$ in Equation (10), we get

$$\begin{aligned}
\sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\mu_i|^2 - \left(\sum_{i=1}^{n} |\mu_i|\right)^2 &\leq \frac{n^2}{4}(|\mu_1| - |\mu_n|)^2 \\
2\mathcal{K}n - (E_{DSS}(G))^2 &\leq \frac{n^2}{4}(|\mu_1| - |\mu_n|)^2 \\
E_{DSS}(G) &\geq \sqrt{2n\mathcal{K} - \frac{n^2}{4}(|\mu_1| - |\mu_n|)^2}.
\end{aligned}$$

□

Corollary 4.3. *If G is an r-regular graph of order n, then*

$$E_{DSS}(G) \geq nr^2\sqrt{8(n-1) - n^2}$$

Theorem 4.4. *Let G be a graph of order n. Then*

$$\sqrt{2\mathcal{K}} \leq E_{DSS}(G) \leq \sqrt{2n\mathcal{K}}.$$

Proof. Upper bound: By substituting $a_i = 1$ and $b_i = \mu_i$ in Equation (6), we get

$$\begin{aligned}
\left(\sum_{i=1}^{n} |\mu_i|\right)^2 &\leq \sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} |\mu_i|^2 \\
(E_{DSS}(G))^2 &\leq 2\mathcal{K}n
\end{aligned}$$

which implies,

$$E_{DSS}(G) \leq \sqrt{2n\mathcal{K}}. \tag{16}$$

Lower bound: We have

$$(E_{DSS}(G))^2 = \left(\sum_{i=1}^{n} |\mu_i|\right)^2 \geq \sum_{i=1}^{n} |\mu_i|^2 = 2\mathcal{K}$$

which implies,

$$E_{DSS}(G) \geq \sqrt{2\mathcal{K}} \tag{17}$$

Combining Equations (16) and (17), we get the desired result. □

Theorem 4.5. *Let G be a graph of order n and let Δ be the absolute value of the determinant of $DSS(G)$. Then*

$$\sqrt{2\mathcal{K} + n(n-1)\Delta^{2/n}} \leq E_{DSS}(G) \leq \sqrt{2n\mathcal{K}}.$$

Proof. Lower bound: By definition of degree square sum energy, we have

$$\begin{aligned}
(E_{DSS}(G))^2 &= \left(\sum_{i=1}^{n}|\mu_i|\right)^2 \\
&= \sum_{i=1}^{n}\mu_i^2 + 2\sum_{i<j}|\mu_i||\mu_j| \\
&= 2\mathcal{K} + 2\sum_{i<j}|\mu_i||\mu_j| \\
&= 2\mathcal{K} + \sum_{i\neq j}|\mu_i||\mu_j| \qquad (18)
\end{aligned}$$

Since we know for nonnegative numbers, the arithmetic mean is always greater than or equal to the geometric mean

$$\begin{aligned}
\frac{1}{n(n-1)}\sum_{i\neq j}|\mu_i||\mu_j| &\geq \left(\prod_{i\neq j}|\mu_i||\mu_j|\right)^{\frac{1}{n(n-1)}} \\
&= \left(\prod_{i=1}^{n}|\mu_i|^{2(n-1)}|\right)^{\frac{1}{n(n-1)}} \\
&= \prod_{i=1}^{n}|\mu_i|^{2/n)} \\
&= \Delta^{2/n}
\end{aligned}$$

Therefore,

$$\sum_{i\neq j}|\mu_i||\mu_j| \geq n(n-1)\Delta^{2/n} \qquad (19)$$

Combining Equations (18) and (19), we get

$$E_{DSS}(G) \geq \sqrt{2\mathcal{K} + n(n-1)\Delta^{2/n}} \qquad (20)$$

Upper bound: Consider a nonnegative quantity

$$Y = \sum_{i=1}^{n}\sum_{j=1}^{n}(|\mu_i| - |\mu_j|)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(|\mu_i|^2 - |\mu_j|^2 - 2|\mu_i||\mu_j|)$$

By direct expansion, we get

$$Y = n\sum_{i=1}^{n}|\mu_i|^2 + n\sum_{j=1}^{n}|\mu_j|^2 - 2\left(\sum_{i=1}^{n}|\mu_i|\right)\left(\sum_{j=1}^{n}|\mu_j|\right)$$

Now, by definition of degree square sum energy of a graph and Equation (8) we have

$$Y = 4n\mathcal{K} - 2(E_{DSS}(G))^2$$

Since $Y \geq 0$,

$$4n\mathcal{K} - 2(E_{DSS}(G))^2 \geq 0$$

$$E_{DSS}(G) \leq \sqrt{2n\mathcal{K}} \qquad (21)$$

Combining Equations (20) and (21), we get the desired result. □

Corollary 4.6. *If G is an r-regular graph of order n, then*

$$E_{DSS}(G) \leq 2nr^2\sqrt{n-1}.$$

Theorem 4.7. *Let G be a graph of order n and size m. Let $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$ be a non-increasing arrangement of degree square sum eigenvalues. Then*

$$E_{DSS}(G) \geq \sqrt{2n\mathcal{K} - \alpha(n)(|\mu_1| - |\mu_n|)^2} \tag{22}$$

where $\alpha(n) = n\lfloor \frac{n}{2} \rfloor (1 - \frac{1}{n}\lfloor \frac{n}{2} \rfloor)$.

Proof. Suppose $\mu_1, \mu_2, ..., \mu_n$ are the degree square sum eigenvalues of G. Then by substituting $a_i = |\mu_i| = b_i,\ a = |\mu_n| = b$ and $A = |\mu_1| = B$ in Equation (11), we get

$$\left| n \sum_{i=1}^{n} |\mu_i|^2 - \left(\sum_{i=1}^{n} |\mu_i| \right)^2 \right| \leq \alpha(n)(|\mu_1| - |\mu_n|)^2 \tag{23}$$

Since $E_{DSS}(G) = \sum_{i=1}^{n} |\mu_i|,\ \sum_{i=1}^{n} |\mu_i|^2 = 2\mathcal{K}$, we get the required result. □

Remark 4.8. *Since $\alpha(n) \leq \frac{n^2}{4}$, from Equations (15) and (22) one can easily observe that the inequality in Equation (22) is sharper than the inequality in Equation (15).*

Theorem 4.9. *Let G be a graph of order n and size m. Let $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$ be a non-increasing arrangement of degree square sum eigenvalues. Then*

$$E_{DSS}(G) \geq \frac{|\mu_1||\mu_n|n + 2\mathcal{K}}{|\mu_1| + |\mu_n|} \tag{24}$$

where $|\mu_1|$ and $|\mu_2|$ are maximum and minimum of the absolute value of μ_i's.

Proof. Suppose $\mu_1, \mu_2, ..., \mu_n$ are the degree square sum eigenvalues of G. Then by substituting $b_i = |\mu_i|, a_i = 1, C_1 = |\mu_n|$ and $C_2 = |\mu_1|$ in Equation (12), we get

$$\sum_{i=1}^{n} |\mu_i|^2 + |\mu_1||\mu_n| \sum_{i=1}^{n} 1^2 \leq (|\mu_1| + |\mu_n|) \left(\sum_{i=1}^{n} |\mu_i| \right) \tag{25}$$

Since $E_{DSS}(G) = \sum_{i=1}^{n} |\mu_i|$ and $\sum_{i=1}^{n} |\mu_i|^2 = 2\mathcal{K}$, we get the required result. □

5. Conclusion

In this paper, we have obtained the characteristic polynomial of the degree square sum matrix of graphs obtained by some graph operations. Also, bounds for both largest degree square sum eigenvalue and degree square sum energy of graphs are established. It can be observed that the lower bound for degree square sum energy given in Theorem 4.7 is sharper.

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