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Oscillation of Third Order Nonlinear Difference Equations With Several Neutral Terms

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Abstract: This paper deals with the oscillatory properties of third order delay difference equation with several neutral terms. Some new sufficient conditions are derived which are very useful to study the given equation. Examples are provided to illustrate the main results.

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1. Introduction

In this paper, we are concerned with the third order delay difference equation with several neutral terms of the form

$$\Delta(a(n)\Delta(b(n)z(n))) + q(n)x^{\alpha}(n-k) = 0, \quad n \geq n_0 \geq 0 \tag{1}$$

where $z(n) = x(n) + \sum_{i=1}^{m} p_i(n)x(n-\ell_i)$, m is a positive integer and we assume that

(H_1) $\{a(n)\}$, $\{b(n)\}$, $\{p_i(n)\}$ and $q(n)$ are positive real sequences with $0 \leq p_i(n) \leq p_i < \infty$ for $i = 1, 2, \ldots, m$;

(H_2) ℓ_i and k are positive integers for $i = 1, 2, \ldots, m$ and α is a ratio of odd positive integers.

Let $\theta = \max\{\ell_1, \ell_2, \ldots, \ell_m, k\}$. By a solution of equation (1), we mean a real sequence $\{x(n)\}$ defined for $n \geq n_0 - \theta$ and satisfies equation (1) for all $n \geq n_0$. We consider only those solutions $\{x(n)\}$ of equation (1) which satisfy $\sup\{|x(n)| : n \geq N\} > 0$ for all $n \geq N$, and assume that the equation (1) possesses such solutions. A solution of equation (1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. Equation (1) is said to be almost oscillatory if all its solutions are oscillatory or convergent to zero asymptotically. Recently, great attention has been devoted to the oscillation problem of third order difference equations, see for example [1, 3–5, 7, 9–14], and the references cited therein. In the following, we present some background details that motive our study. In [4], the authors considered the equation

$$\Delta\left(c(n)\Delta(d(n)\Delta x(n))\right) + q(n)f(x(n-\sigma+1)) = 0 \tag{2}$$

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and investigated the oscillatory and asymptotic behavior of solutions of equation (2). In [3], the authors considered the equation

$$\Delta\left(c(n)(\Delta^2 x(n))^{\alpha}\right)+q(n)f(x(\sigma(n)))=0 \tag{3}$$

and studied the oscillatory behavior of equation (3) under the condition $\sum\limits_{n=n_0}^{\infty}\frac{1}{c^{\frac{1}{\alpha}}(n)}<\infty$. In [9], the authors considered the following equation

$$\Delta\left(c(n)\Delta\left(d(n)\Delta(x(n)+p(n)x(n-k))\right)\right)+q(n)f(x(n-m))=0 \tag{4}$$

and established criteria for the oscillation of all solutions of equation (4) under the condition $\sum\limits_{n=n_0}^{\infty}\frac{1}{c(n)}=\sum\limits_{n=n_0}^{\infty}\frac{1}{d(n)}=\infty$. In [11], the authors considered the equation

$$\Delta\left(a(n)\left(\Delta^2(x(n)+p(n)x(n-\delta))\right)^{\alpha}\right)+q(n)x^{\alpha}(n-\tau)=0 \tag{5}$$

and derived several criteria for the almost oscillation of equation (5). Motivated be the above observation, in this paper we shall further the investigation of the oscillation behavior of solutions of equation (1) under the following two cases:

$$\sum_{n=n_0}^{\infty}\frac{1}{a(n)}=\infty,\qquad \sum_{n=n_0}^{\infty}\frac{1}{b(n)}=\infty, \tag{6}$$

and

$$\sum_{n=n_0}^{\infty}\frac{1}{a(n)}<\infty,\qquad \sum_{n=n_0}^{\infty}\frac{1}{b(n)}=\infty. \tag{7}$$

The results obtained here reduced to those presented in [9, 11] for the particular case $m=1$. In Section 2, we obtain some sufficient conditions for the almost oscillatory of equation (1) and in Section 3 we provide some examples to illustrate the main results.

2. Oscillation Results

In this section, we obtain some new oscillation criteria for the equation (1). We begin with a useful lemma which will be used later. Without loss of generality, we can deal only with the positive solutions of equation (1) since the proof for the opposite case is similar.

Lemma 2.1. *Assume that $y_i\ge 0$ for $i=1,\ 2,\ \ldots,\ m$. Then*

(a). $\sum\limits_{i=1}^{m}y_i^{\alpha}\ge\frac{1}{m^{\alpha-1}}\left(\sum\limits_{i=1}^{m}y_i\right)^{\alpha}$ for $\alpha\ge 1$;

(b). $\sum\limits_{i=1}^{m}y_i^{\alpha}\ge\left(\sum\limits_{i=1}^{m}y_i\right)^{\alpha}$ for $0<\alpha<1$.

Proof. The proof is similar to that of in [11] and hence the details are omitted. □

Theorem 2.2. *Assume that (6) holds and $\alpha\ge 1$. If*

$$\sum_{n=n_0}^{\infty}\frac{1}{b(n)}\sum_{s=n}^{\infty}\frac{1}{a(s)}\sum_{t=s}^{\infty}q(t)=\infty \tag{8}$$

and the first order difference inequality

$$\Delta\left(y(n)+\sum_{i=1}^{m}p_i^{\alpha}y(n-\ell_i)\right)+\frac{Q(n)B^{\alpha}(n-k)}{(m+1)^{\alpha-1}}y^{\alpha}(n-k)\le 0 \tag{9}$$

where $Q(n)=\min\{q(n),\ q(n-\ell),\ \ldots,\ q(n-\ell_m)\}$ and $B(n)=\sum\limits_{s=n_2}^{n-1}\frac{\sum\limits_{t=n_1}^{s-1}\frac{1}{a(t)}}{b(s)}$ for $n\ge n_2\ge n_1$, has no positive decreasing solution, then equation (1) is almost oscillatory.

Proof. Assume that $\{x(n)\}$ is a positive solution of equation (1). Based on the condition (6) there exist two possible cases:

(I) $z(n) > 0,\ \Delta z(n) > 0,\ \Delta(b(n)\Delta z(n)) > 0,\ \Delta(a(n)\Delta(b(n)\Delta z(n))) < 0,$

(II) $z(n) > 0,\ \Delta z(n) < 0,\ \Delta(b(n)\Delta z(n)) > 0,\ \Delta(a(n)\Delta(b(n)\Delta z(n))) < 0,$

for $n \geq n_1$ where n_1 is large enough. Assume that Case (I) holds. From equation (1), we have

$$\sum_{i=1}^{m} p_i^{\alpha} \Delta\left(a(n-\ell_i)\Delta(b(n-\ell_i)\Delta z(n-\ell_i))\right) + \sum_{i=1}^{m} p_i^{\alpha} q(n-\ell_i) x^{\alpha}(n-\ell_i-k) = 0. \tag{10}$$

Combining (1) and (10), we obtain

$$\Delta(a(n)\Delta(b(n)\Delta z(n))) + \sum_{i=1}^{m} p_i^{\alpha} \Delta\left(a(n-\ell_i)\Delta(b(n-\ell_i)\Delta z(n-\ell_i))\right) + Q(n)\left(x_{n-k}^{\alpha} + \sum_{i=1}^{m} p_i^{\alpha} x_{n-\ell_i-k}^{\alpha}\right) \leq 0. \tag{11}$$

Using Lemma 2.1(a), we get

$$\Delta(a(n)\Delta(b(n)\Delta z(n))) + \sum_{i=1}^{m} p_i^{\alpha} \Delta\left(a(n-\ell_i)\Delta(b(n-\ell_i)\Delta z(n-\ell_i))\right) + \frac{Q(n)}{(m+1)^{\alpha-1}} z_{n-k}^{\alpha} \leq 0. \tag{12}$$

Now

$$b(n)\Delta z(n) \geq \sum_{s=n_1}^{n-1} \frac{a(s)\Delta(b(s)\Delta z(s))}{a(s)} \geq a(n)\Delta(b(n)\Delta z(n)) \sum_{s=n_1}^{n-1} \frac{1}{a(s)} \tag{13}$$

we have that

$$\Delta\left(\frac{b(n)\Delta z(n)}{\sum\limits_{s=n_1}^{n-1} \frac{1}{a(s)}}\right) \leq 0. \tag{14}$$

Thus, we have

$$\begin{aligned} z(n) &= z(n_2) + \sum_{s=n_2}^{n-1} \frac{b(s)\Delta z(s)}{\sum\limits_{t=n_1}^{s-1} \frac{1}{a(t)}} \frac{\sum\limits_{t=n_1}^{s-1} \frac{1}{a(t)}}{b(s)} \\ &\geq \frac{b(n)\Delta z(n)}{\sum\limits_{s=n_1}^{n-1} \frac{1}{a(s)}} \sum_{s=n_2}^{n-1} \frac{\sum\limits_{t=n_1}^{s-1} \frac{1}{a(t)}}{b(s)} \end{aligned} \tag{15}$$

where we have used (14). From (13) and (15), we obtain

$$z(n) \geq B(n)a(n)\Delta(b(n)\Delta z(n)). \tag{16}$$

Let $y(n) = a(n)\Delta(b(n)\Delta z(n)) > 0$ be decreasing and from (12) and (16), we see that

$$\Delta\left(y(n) + \sum_{i=1}^{m} p_i^{\alpha} y(n-\ell_i)\right) + \frac{Q(n)B^{\alpha}(n-k)}{(m+1)^{\alpha-1}} y_{n-k}^{\alpha} \leq 0. \tag{17}$$

Hence $\{y(n)\}$ is a positive decreasing solution of (9), which is a contradiction.

Case (II) Assume that Case (II) holds. Since $z(n) > 0$ and $\Delta z(n) < 0$, we have $\lim_{n\to\infty} z(n) = L \geq 0$. If $L > 0$

then $\lim_{n\to\infty}\left(x(n)+\sum_{i=1}^{m}p_i(n)x(n-k)\right)=L$ or $\lim_{n\to\infty}x(n)=L_1\leq L$. Then there exists $n_2\geq n_1\geq n_0$ such that $L_1<x(n)<L_1+\epsilon$ for $n\geq n_2$. Hence from equation (1), we have

$$\Delta\left(a(n)\Delta(b(n)\Delta z(n))\right)\leq -L_1q(n),\quad n\geq n_2. \tag{18}$$

Summing (18) from $n\geq n_2$ to ∞ and using the fact that $a(n)\Delta(b(n)\Delta z(n))$ is positive and decreasing, we obtain

$$a(n)\Delta(b(n)\Delta z(n))\geq L_1\sum_{s=n}^{\infty}q(s).$$

Summing again gives

$$b(n)\Delta z(n)\leq -L_1\sum_{s=n}^{\infty}\frac{1}{a(s)}\sum_{t=s}^{\infty}q(t)$$

and a finial summation yields

$$z(n_2)\geq L_1\sum_{n=n_2}^{\infty}\frac{1}{b(n)}\sum_{s=n}^{\infty}\frac{1}{a(s)}\sum_{t=s}^{\infty}q(t).$$

This contradicts (8) and shows that $L=0$, that is, $z(n)\to 0$. Since $z(n)>x(n)>0$, we have $x(n)\to 0$ as $n\to\infty$. This completes the proof of theorem. □

Theorem 2.3. *Assume that (6) holds and $\alpha\geq 1$. Let $\ell=\max\{\ell_1,\ ...,\ \ell_m\}$ with $\ell<k$. If (8) holds and the first order difference inequality*

$$\Delta w(n)+\frac{Q(n)B^{\alpha}(n-k)}{(m+1)^{\alpha-1}\left(1+\sum_{i=1}^{m}p_i^{\alpha}\right)^{\alpha}}w^{\alpha}(n+\ell-k)\leq 0 \tag{19}$$

has no positive decreasing solution, then equation (1) is almost oscillatory.

Proof. Proceeding as in the proof of Theorem 2.2, wee see that $\{z(n)\}$ satisfies Case (I) and Case (II) for all $n\geq n_1$. Let Case (I) holds. Then as in the proof of Case (I) of Theorem 2.2, we obtain (17). We now define

$$w(n)=y(n)+\sum_{i=1}^{m}p_i^{\alpha}y(n-\ell_i).$$

Then $w(n)>0$ and in view of $\ell=\max\{\ell_1,\ ...,\ \ell_m\}$, we have

$$w(n)\leq\left(1+\sum_{i=1}^{m}p_i^{\alpha}\right)y(n-\ell).$$

Substituting the last inequality into (17), we see that $\{w(n)\}$ is a positive decreasing solution of inequality (19) which is a contradiction. The proof for the Case (II) is similar to that of Case (II) of Theorem 2.2. Now the proof is complete. □

Remark 2.4. *Theorem 2.2 complements to that of in [11] when $m=1$. Theorem 2.3 extends some results in [9] in the case $m=1$ and $\alpha=1$.*

Next, by adding additional assumption on α, one can derive explicit oscillation criteria for the equation (1) from Theorem 2.3.

Corollary 2.5. *In addition to assumption of Theorem 2.3, let $\alpha=1$. If*

$$\liminf_{n\to\infty}\sum_{s=n-k+\ell}^{n-1}Q(s)B(s-k)>\left(1+\sum_{i=1}^{m}p_i\right)\left(\frac{k-\ell}{k-\ell+1}\right)^{k-\ell+1} \tag{20}$$

then equation (1) is almost oscillatory.

Proof. By Theorem 2 of [6], assumption (20) ensure that the inequality (19) has no positive solutions. The conclusion now follows from Theorem 2.3. □

Corollary 2.6. *In addition to assumptions of Theorem 2.3, let $\alpha > 1$. If there exists a $\lambda > \frac{1}{k-\ell} ln\alpha$ such that*

$$\lim_{n\to\infty} \inf \left[Q(n)B^{\alpha}(n-k)exp(-e^{\lambda n})\right] > 0 \tag{21}$$

then equation (1) is almost oscillatory.

Proof. By Theorem 2 of [8], condition (21) ensures that the inequality (19) has no positive solutions. The conclusion now follows from Theorem 2.3. □

Next, we turn our attention to the case $0 < \alpha < 1$.

Theorem 2.7. *Assume that (6) holds and $0 < \alpha < 1$. If (8) holds and the first order difference inequality*

$$\Delta\left(y(n) + \sum_{i=1}^{m} p_i^{\alpha} y(n-\ell_i)\right) + Q(n)B^{\alpha}(n-k)y^{\alpha}(n-k) \le 0 \tag{22}$$

has no positive decreasing solution, then equation (1) is almost oscillatory.

Proof. The proof is exactly same as that of Theorem 2.2 except by using Lemma 2.1(b) instead Lemma 2.1(a). Hence the details are omitted. □

Theorem 2.8. *Assume that (6) holds and $0 < \alpha < 1$. Let $\ell = \max\{\ell_1, \ldots, \ell_m\}$ with $\ell < k$. If (8) holds and the first order difference inequality*

$$\Delta w(n) + \frac{Q(n)B^{\alpha}(n-k)}{\left(1 + \sum\limits_{i=1}^{m} p_i^{\alpha}\right)^{\alpha}} w^{\alpha}(n+\ell-k) \le 0 \tag{23}$$

has no positive decreasing solution, then equation (1) is almost oscillatory.

Proof. The proof is similar to that of Theorem 2.3 and hence the details are omitted. □

Corollary 2.9. *In addition to assumption of Theorem 2.8, let*

$$\sum_{n=n_0}^{\infty} Q(n)B^{\alpha}(n-k) = \infty. \tag{24}$$

Then equation (1) is almost oscillatory.

Proof. By Theorem 1 of [8], the condition (24) ensure that the difference inequality (23) has no positive solution. The results now follows from Theorem 2.8. □

In the following result we assume that condition (7) holds and $0 \le \sum\limits_{i=1}^{m} p_i(n) < 1$ for all $n \ge n_0$, and $\alpha = 1$.

Theorem 2.10. *Assume that condition (7) holds, $0 \le \sum\limits_{i=1}^{m} p_i(n) < 1$ and $\alpha = 1$. If condition (8) holds and*

$$\sum_{n=n_0}^{\infty} \left[q(n)\left(1 - \sum_{i=1}^{m} p_i(n-k)\right) \frac{B(n-k)}{\sum\limits_{s=n_0}^{n-1} \frac{1}{a(s)}} \right] = \infty \tag{25}$$

and

$$\lim_{n\to\infty} \sup \sum_{s=n_2}^{n-1} \left[\delta(s)q(s)\left(1 - \sum_{i=1}^{m} p_i(s-k)\right) \sum_{t=n_1}^{s-k-1} \frac{1}{b(t)} - \frac{1}{4a(s)\delta(s)}\right] = \infty \tag{26}$$

where $\delta(n) = \sum\limits_{s=n}^{\infty} \frac{1}{a(s)}$, then equation (1) is almost oscillatory.

Proof. Assume that $\{x(n)\}$ is a positive solution of equation (1). Based on the condition (7), there exist three possible Cases (I), (II) (as in Theorem 2.2), and

Case (III) $z(n) > 0,\ \Delta z(n) > 0,\ \Delta(b(n)\Delta z(n)) > 0,\ \Delta\left(a(n)\Delta(b(n)\Delta z(n))\right) < 0$ for all $n \geq n_1$, n_1 is large enough. Assume that Case (I) holds. Define

$$w(n) = \frac{a(n)\Delta(b(n)\Delta z(n))}{b(n)\Delta z(n)}, \quad n \geq n_1. \tag{27}$$

Then $w(n) > 0$ for $n \geq n_1$. Using $\Delta z(n) > 0$, we have

$$x(n) \geq \left(1 - \sum_{i=1}^{m} p_i(n)\right) z(n). \tag{28}$$

Since

$$b(n)\Delta z(n) \geq a(n)\Delta(b(n)\Delta z(n)) \sum_{s=n_1}^{n-1} \frac{1}{a(s)}, \tag{29}$$

we have

$$\Delta\left(\frac{b(n)\Delta z(n)}{\sum_{s=n_1}^{n-1} \frac{1}{a(s)}}\right) \leq 0. \tag{30}$$

Thus, we get

$$z(n) \geq \frac{b(n)\Delta z(n)}{\sum_{s=n_1}^{n-1} \frac{1}{a(s)}} B(n), \quad n \geq n_2 > n_1. \tag{31}$$

From (27), we obtain

$$\begin{aligned} \Delta w(n) &= \frac{\Delta(a(n)\Delta(b(n)\Delta z(n)))}{b(n)\Delta z(n)} - \frac{a(n+1)\Delta(b(n+1)\Delta z(n+1))}{b(n+1)\Delta z(n+1) b(n)\Delta z(n)} \Delta(b(n)\Delta z(n)) \\ &\leq -\frac{q(n)\left(1 - \sum_{i=1}^{m} p_i(n-k)\right) z(n-k)}{b(n)\Delta z(n)}. \end{aligned} \tag{32}$$

From (31) we have

$$z(n-k) \geq \frac{b(n-k)\Delta z(n-k)}{\sum_{s=n_1}^{n-k-1} \frac{1}{a(s)}} B(n-k) \geq \frac{b(n)\Delta z(n)}{\sum_{s=n_1}^{n-1} \frac{1}{a(s)}} B(n-k) \tag{33}$$

where we have used (30). Using (33) in (32) we obtain

$$\Delta w(n) \leq -q(n)\left(1 - \sum_{i=1}^{m} p_i(n-k)\right) \frac{B(n-k)}{\sum_{s=n_1}^{n-1} \frac{1}{a(s)}}, \quad n \geq n_2 \geq n_1.$$

Summing the last inequality from n_2 to n, we obtain

$$\sum_{s=n_2}^{n} \left[q(s)\left(1 - \sum_{i=1}^{m} p_i(s-k)\right) \frac{B(s-k)}{\sum_{s=n_1}^{s-1} \frac{1}{a(t)}} \right] \leq w(n_2) < \infty$$

which contradicts. Assume Case (II) holds. Then as in the proof of Theorem 2.2(Case II)) we see that $\lim_{n\to\infty} x(n) = 0$. Assume that Case (III) holds. Since $a(n)\Delta(b(n)\Delta z(n))$ is negative decreasing, we have

$$a(s)\Delta(b(s)\Delta z(s)) \leq a(n)\Delta(b(n)\Delta z(n)), \ s \geq n \geq n_1. \tag{34}$$

Dividing the last inequality by $a(s)$ and then summing it from n to $\ell-1$, we obtain

$$b(\ell)\Delta z(\ell) \leq b(n)\Delta z(n) + a(n)\Delta(b(n)\Delta z(n)) \sum_{s=n}^{\ell-1} \frac{1}{a(s)}.$$

Letting $\ell \to \infty$, we get

$$0 \leq b(n)\Delta z(n) + a(n)\Delta(b(n)\Delta z(n)) \sum_{s=n}^{\infty} \frac{1}{a(s)},$$

that is,

$$-\frac{a(n)\Delta(b(n)\Delta z(n))}{b(n)\Delta z(n)} \sum_{s=n}^{\infty} \frac{1}{a(s)} \leq 1. \tag{35}$$

Define

$$v(n) = \frac{a(n)\Delta(b(n)\Delta z(n))}{b(n)\Delta z(n)}, \; n \geq n_1. \tag{36}$$

Then $v(n) < 0$ for $n \geq n_1$ and by (35) and (36), we obtain

$$-\delta(n)v(n) \leq 1. \tag{37}$$

From (36), we have

$$\Delta v(n) = \frac{\Delta(a(n)\Delta(b(n)\Delta z(n)))}{b(n)\Delta z(n)} - \frac{a(n+1)\Delta(b(n+1)\Delta z(n+1))\Delta(b(n)\Delta z(n))}{b(n+1)\Delta z(n+1)b(n)\Delta z(n)}. \tag{38}$$

Using $\Delta z(n) > 0$, we have (28). From equation (1) and (38), we have

$$\Delta v(n) \leq -q(n)\left(1 - \sum_{i=1}^{m} p_i(n-k)\right) \frac{z(n-k)}{b(n)\Delta z(n)} - \frac{v^2(n)}{a(n)}, \tag{39}$$

where we have used $a(n)\Delta(b(n)\Delta z(n))$ is negative and decreasing and $b(n)\Delta z(n)$ is positive and decreasing. Also

$$z(n) \geq b(n)\Delta z(n) \sum_{s=n_1}^{n-1} \frac{1}{b(s)}, \tag{40}$$

and hence

$$\Delta\left(\frac{z(n)}{\sum_{s=n_1}^{n-1}} \frac{1}{b(s)}\right) \leq 0,$$

which implies that

$$\frac{z(n-k)}{z(n)} \geq \frac{\sum_{s=n_1}^{n-k-1} \frac{1}{b(s)}}{\sum_{s=n_1}^{n-1} \frac{1}{b(s)}}. \tag{41}$$

By (39), (40) and (41), we obtain

$$\Delta v(n) \leq -q(n)\left(1 - \sum_{i=1}^{m} p_i(n-k)\right) \sum_{s=n_1}^{n-k-1} \frac{1}{b(s)} - \frac{v^2(n+1)}{a(n)}.$$

Multiplying the last inequality $\delta(n)$ and then summing it from n_2 to $n-1$, we have

$$\delta(n)v(n) - \delta(n_2)v(n_2) + \sum_{s=n_2}^{n-1} \delta(s)q(s)\left(1 - \sum_{i=1}^{m} p_i(s-k)\right) \sum_{s=n_1}^{s-k-1} \frac{1}{b(t)} - \sum_{s=n_2}^{n-1} \frac{v(s+1)}{a(s)} + \sum_{s=n_2}^{n-1} \delta(s)\frac{v^2(s+1)}{a(s)} \leq 0$$

which on completing the square yields

$$\sum_{s=n_2}^{n-1} \left[\delta(s)q(s)\left(1 - \sum_{i=1}^{m} p_i(s-k)\right) \sum_{t=n_1}^{s-k-1} \frac{1}{b(t)} - \frac{1}{4a(s)\delta(s)}\right] \leq -1 + \delta(n_2)v(n_2)$$

when using (37), which contradicts (26). This completes the proof. □

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. *Consider the third order neutral delay difference equation*

$$\Delta\left(n\Delta^2\left(x(n)+\frac{1}{2}x(n-1)+2x(n-2)\right)\right)+\frac{1}{(n+1)}x(n-3)=0,\quad n\geq 1. \tag{42}$$

Here $a(n)=n,\ b(n)=1,\ p_1(n)=\frac{1}{2},\ p_2(n)=2,\ \ell_1=1,\ \ell_2=2,\ q(n)=\frac{1}{(n+1)},\ k=3$ *and* $\alpha=1$. *The condition* (6) *clearly satisfied and condition* (8) *becomes*

$$\sum_{n=1}^{\infty}\frac{1}{b(n)}\sum_{s=n}^{\infty}\frac{1}{a(s)}\sum_{t=s}^{\infty}q(t)=\sum_{n=1}^{\infty}\sum_{s=n}^{\infty}\frac{1}{s}\sum_{t=s}^{\infty}\frac{1}{(t+1)}=\infty.$$

Further the condition (20) *becomes*

$$\lim_{n\to\infty}\inf\sum_{s=n-1}^{n-1}\frac{1}{s+1}\sum_{t=2}^{s-1}\left(\sum_{u=1}^{t-1}\frac{1}{u}\right)>\lim_{n\to\infty}\inf\sum_{s=n-1}^{n-1}\frac{s-2}{s+1}=1>\frac{7}{8}.$$

Hence all conditions of Corollary 2.5 are satisfied and so the equation (42) *is almost oscillatory.*

Example 3.2. *Consider the third order neutral delay difference equation*

$$\Delta\left(\frac{1}{n}\Delta^2\left(x(n)+\frac{1}{2}x(n-1)+x(n-2)\right)\right)+\frac{1}{(n+1)}x^{\frac{1}{3}}(n-3)=0,\quad n\geq 1. \tag{43}$$

Here $a(n)=\frac{1}{n},\ b(n)=1,\ p_1(n)=\frac{1}{2},\ p_2(n)=1,\ \ell_1=1,\ \ell_2=2,\ q(n)=\frac{1}{(n+1)},\ k=3$ *and* $\alpha=\frac{1}{3}$. *It is easy to see that all conditions of Corollary 2.9 are satisfied and so the equation* (43) *is almost oscillatory.*

Example 3.3. *Consider the third order neutral delay difference equation*

$$\Delta\left(n^2\Delta^2\left(x(n)+\frac{1}{4}x(n-1)+\frac{1}{2}x(n-2)\right)\right)+nx(n-3)=0,\quad n\geq 1. \tag{44}$$

Here $a(n)=n^2,\ b(n)=1,\ p_1(n)=\frac{1}{4},\ p_2(n)=\frac{1}{2},\ \ell_1=1,\ \ell_2=2,\ q(n)=n,\ k=3$ *and* $\alpha=1$. *It is easy to see that all conditions of Theorem 2.10 are satisfied and hence equation* (44) *is almost oscillatory.*

We conclude this paper with the following remark.

Remark 3.4. *In this paper, we have established some new oscillation theorems for the equation* (1) *by reducing to the oscillation of first order delay difference equation. The obtained results complement and generalize some of the results in [3–5, 7, 9–14].*

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