

Domination Number of A Line Graph Formed from the Cartesian Product of Cycle and Path Graphs

L. Jethruth Emelda Mary¹ and T. Monica^{1,*}

1 PG & Research Department of Mathematics, St. Joseph's College of Arts and Science (Autonomous), Cuddalore, Tamilnadu, India.

Abstract: In this paper we discussed the domination of line graph formed from the Cartesian product of cycle and path graphs. Also we have introduced disconnected total irregular dominating set and disconnected inverse irregular dominating set and various parameters of domination are applied on line graph and some relation between them are discussed with some examples.

Keywords: Domination number, Inverse Domination number, Total Domination number, disconnected inverse irregular dominating set, disconnected total irregular dominating set Line graph, Cartesian product, Path graph, Cycle graph.

© JS Publication.

Accepted on: 30.04.2018

1. Introduction

In this paper we have taken the graphs to be undirected, finite and simple graph. One of the important concepts in the field of research is the domination in graphs. In 1958, the concept of domination in graph was defined by Claude Berge and Ore [1]. He also called domination number the Coefficient of External Stability. The first paper Optimal Domination in Graphs was published in 1975 by E. J. Cockayne and S. T. Hedetniemi [2] and the notation $\gamma(G)$ was first introduced by these two mathematicians. Domination has many applications which is used in chessboard, communication network, wireless Ad-hoc network, maps etc. also it has many parameters and in this paper we discussed about total domination number and inverse domination number. The concept of inverse domination Kulli and Sigarkanti [3] was the first to publish the paper on the Inverse Domination in Graphs. Cockayne, Dawes and Hedetniemi formalized total domination in graphs [4]. The term domination for undirected graphs was first published by Ore in 1962. Cartesian product of graphs have been described by Whitehead and Russel in 1912 according to Imrich and Klavzar [5]. The concept of line graph was invented by H. Whitney in 1932. We begin with some basic definitions and notations :

Definition 1.1. The Cartesian product of simple graphs G and H is denoted by $G \times H$ whose vertex set is $V(G) \times V(H)$ and $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if $u_1 = v_1$ and u_2 is adjacent to v_2 in H or u_1 is adjacent to v_1 in G and $u_2 = v_2$ in H .

Definition 1.2. The line graph of a simple graph G is obtained by means of associating a vertex with every edge of the graph and connecting two vertices with an edge iff the corresponding edges of G have a vertex in common. The Line graph of G is denoted by $L(G)$.

* E-mail: monicamaths3595@gmail.com (Research scholar)

Definition 1.3. A subset D of $V(G)$ is said to be a dominating set of graph G if every $v \in V - D$ is adjacent to atleast one vertex in D . The minimum cardinality of a dominating set D is called domination number. It is denoted by $\gamma(G)$.

Definition 1.4. If D is a dominating set in G and if a dominating set exists in V/D say D' then D' is called the inverse dominating set. The inverse domination number denoted by $\gamma'(G)$ is the smallest size of the inverse dominating set.

Definition 1.5. If every vertex of V is adjacent to some vertex in D then the dominating set D is said to be total dominating set. The minimum cardinality of total dominating set is called the total domination number and is denoted by $\gamma_t(G)$.

Definition 1.6. If the induced subgraph $\langle D \rangle$ is disconnected then the total dominating set D is said to be disconnected total dominating set. The disconnected total domination number is the minimum size of the disconnected total dominating set and is denoted by $\gamma_{dct}(G)$.

Definition 1.7. A disconnected total dominating set is called disconnected total irregular dominating set if the induced subgraph $\langle D \rangle$ is not regular graph. The disconnected total irregular domination number is the minimum size of the disconnected total irregular dominating set. It is denoted by $\gamma_{dctir}(G)$.

Definition 1.8. A disconnected inverse dominating set is called disconnected inverse irregular dominating set if the induced subgraph $\langle D \rangle$ is not regular graph. The disconnected inverse irregular domination number is the minimum size of the disconnected inverse irregular dominating set. It is denoted by $\gamma'_{dctir}(G)$.

2. Relation Between $\gamma(G)$, $\gamma_t(G)$, $\gamma_{tdcir}(G)$, $\gamma'(G)$ and $\gamma'_{tdcir}(G)$ of $L(C_m \times C_n)$

Theorem 2.1. Let C_m and C_n be two cycle graphs with m and n vertices respectively and let $G = C_m \times C_n$ be the Cartesian product of two cycle graphs. Let $L(G)$ be the line graph and $\gamma(G)$, $\gamma'(G)$, $\gamma_{dctir}(G)$ are the domination number, total domination number, disconnected total irregular domination number, inverse domination number, disconnected inverse irregular domination number of G then $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{dctir}[L(G)] < p$ where p is the no. of vertices in $L(G) \forall m, n > 2$.

Proof. The vertex set of C_m be $U = \{u_1, u_2, \dots, u_m\}$ and the vertex set of C_n be $W = \{w_1, w_2, \dots, w_n\}$. The graph G has nm vertices and $2mn$ edges and the line graph $L(G)$ has $2mn$ vertices. To find the domination number, total domination number, disconnected total irregular domination number, inverse domination number and inverse disconnected irregular domination number we consider the following cases:

Case 1: When m and n are even and distinct (i. e) $m = 4, n = 6$ the graph $G = C_4 \times C_6$ has 24 vertices and 48 edges in G and $L(G)$ has 48 vertices.

Let $D = \{v_1, v_2, v_5, v_{21}, v_{24}, v_{25}, v_{28}, v_{29}, v_{30}, v_{33}, v_{37}, v_{48}\}$ be the dominating set of $L(G)$ and so $\gamma[L(G)] = 12$. Let $D' = \{v_3, v_4, v_6, v_7, v_{10}, v_{23}, v_{26}, v_{27}, v_{32}, v_{34}, v_{40}, v_{41}\}$ be the inverse dominating set and $\gamma'[L(G)] = 12$. By definition of total dominating set we get $\gamma_t[L(G)] = 12$ and since the induced subgraph of D is disconnected and not regular graph we have disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 12$. And also since the induced subgraph of D' is disconnected and it is not regular graph so by definition of disconnected inverse irregular domination number we have $\gamma'_{dctir}[L(G)] = 12$. Hence $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{dctir}[L(G)] = 12$ which is less than $p = 48$.

Case 2: When m is even and n is odd (i. e) $m = 6, n = 3$. The graph $G = C_6 \times C_3$ has 18 vertices and 36 edges and $L(G)$ has 36 vertices.

Let $D = \{v_3, v_7, v_8, v_{15}, v_{19}, v_{20}, v_{27}, v_{28}, v_{29}\}$ be the dominating set of $V[L(G)]$ which is the minimum dominating set of (G)

and hence $\gamma[L(G)] = 9$. Let $D' = \{v_4, v_9, v_{11}, v_{16}, v_{23}, v_{25}, v_{26}, v_{31}, v_{35}\}$ be the inverse dominating set and so $\gamma'[L(G)] = 9$. By definition of total dominating set we get $\gamma_t(G) = 9$. Since the induced subgraph of D and D' is disconnected and it is not regular graph so by definition we have disconnected inverse irregular domination number $\gamma'_{dcir}[L(G)] = 9$ and disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 9$. Thus $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{dcir}[L(G)] = 9$ which is less than $p = 36$.

Case 3: When m and n are odd and equal (i. e) $m=5, n=5$. The graph $G = C_5 \times C_5$ has 25 vertices and 50 edges and $L(G)$ has 50 vertices.

Let $D = \{v_1, v_3, v_{11}, v_{12}\}$ be the dominating set of $L(G)$ and so $\gamma[L(G)] = 4$. Let $D' = \{v_4, v_5, v_6, v_{10}\}$ be the inverse dominating set of $V[L(G)]$ and hence $\gamma'[L(G)] = 4$. By definition of total dominating set we get $\gamma_t(G) = 4$ and since the induced subgraph of D is disconnected and not regular graph we have disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 4$. Since the induced subgraph of D' is disconnected and it is not regular graph so we have $\gamma'_{dcir}[L(G)] = 4$. Therefore $\gamma[L(G)] = \gamma_t(G) = \gamma_{tdcir}(G) = \gamma'[L(G)] = \gamma'_{dcir}[L(G)] = 4$ which is less than $p = 50$.

Case 4: When m and n are even and equal (i. e) $m = 4, n = 4$. The graph $G = C_4 \times C_4$ has 16 vertices and 32 edges and $L(G)$ has 32 vertices.

Let $D = \{v_6, v_{17}, v_{22}, v_{25}, v_{27}, v_{29}, v_{30}, v_{31}\}$ is the dominating set of $L(G)$ and so $\gamma[L(G)] = 8$. Let $D' = \{v_2, v_3, v_8, v_{11}, v_{12}, v_{18}, v_{23}, v_{24}\}$ be the inverse dominating set of $V[L(G)]$ and hence $\gamma'[L(G)] = 8$. By definition of total dominating set we get $\gamma_t[L(G)] = 8$ and since the induced subgraph of D is disconnected and not regular graph we have disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 8$. Since the induced subgraph of D' is disconnected and it is irregular graph so by definition of disconnected inverse irregular domination number we get $\gamma'_{dcir}[L(G)] = 8$. Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{dcir}[L(G)] = 8$ which is less than $p=32$.

Case 5: When m and n are odd and distinct (i. e) $m = 3, n = 5$. The graph $G = C_3 \times C_5$ has 15 vertices and 30 edges and $L(G)$ has 30 vertices.

Since $D = \{v_2, v_4, v_{10}, v_{14}, v_{16}, v_{27}, v_{29}\}$ is the dominating set of $L(G)$ and hence $\gamma[L(G)] = 7$. Let $D' = \{v_5, v_7, v_{12}, v_{16}, v_{19}, v_{28}, v_{30}\}$ be the inverse dominating set and hence $\gamma'[L(G)] = 7$. By definition of total dominating set we get $\gamma_t[L(G)] = 7$. Since the induced subgraph of D and D' is disconnected and it is irregular graph so by definition we have disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 7$ and disconnected inverse irregular domination number $\gamma'_{dcir}[L(G)] = 7$. Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{dcir}[L(G)] = 7$ which is less than $p = 30$.

From all these cases we generalized that for all $m, n > 2$ we have $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{dcir}[L(G)] < p$ where p is the no. of vertices in $L(G)$. □

Example 2.2. When $m = 4, n = 4$. The Cartesian product of $G = C_4 \times C_4$ is given in figure (1).

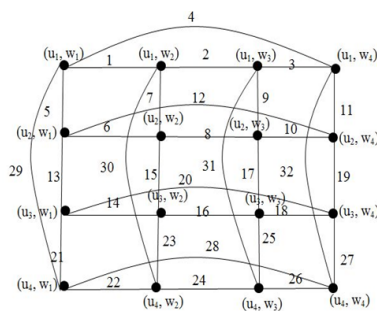


Figure 1. $C_4 \times C_4$

The line graph of figure (1) is shown in figure (2).

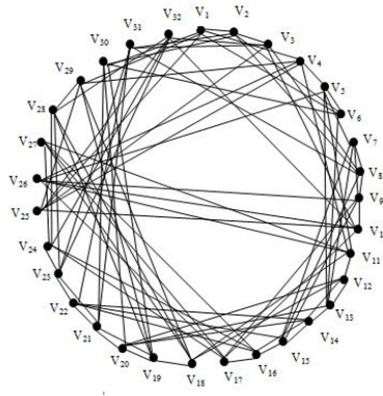


Figure 2. $L(G)$

In figure (2) we get the dominating set $D = \{v_6, v_{17}, v_{22}, v_{25}, v_{27}, v_{29}, v_{30}, v_{31}\}$ and so $\gamma[L(G)] = 8$. Then the inverse dominating set is $D' = \{v_2, v_3, v_8, v_{11}, v_{12}, v_{18}, v_{23}, v_{24}\}$ and hence $\gamma'[L(G)] = 8$ and by definition of disconnected total irregular domination number and disconnected inverse irregular domination number we have $\gamma_{tdcir}[L(G)] = 8$ and $\gamma'_{dcir}[L(G)] = 8$.

3. Relation Between $\gamma(G)$, $\gamma_t(G)$, $\gamma_{tdcir}(G)$, $\gamma'(G)$ and $\gamma'_{tdcir}(G)$ of $L(P_m \times P_n)$

Theorem 3.1. Let P_m and P_n be two path graphs with m and n vertices respectively and let $G = P_m \times P_n$ be the Cartesian product of two path graphs. Let $L(G)$ be the line graph and “ $\gamma(G), \gamma_t(G), \gamma_{tdcir}(G), \gamma'(G), \gamma'_{dcir}(G)$ are the domination number, total domination number, disconnected total irregular domination number, inverse domination number, disconnected inverse irregular domination number of G then $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma_{dcir}[L(G)] < p$ where p is the no. of vertices in $L(G)$ and $\forall m, n \geq 2$.

Proof. Let $U = \{u_1, u_2, \dots, u_m\}$ be the vertex set in P_m and the vertex set of P_n be $W = \{w_1, w_2, \dots, w_n\}$. The graph G has nm vertices and $2mn - (m + n)$ edges and the line graph $L(G)$ has $2mn(m + n)$ vertices. To find the domination number, total domination number, disconnected total irregular domination number, inverse domination number and disconnected inverse irregular domination number we consider the following cases:

Case 1: When m and n are even and distinct (i. e) $m = 8, n = 2$ the graph $G = P_8 \times P_2$ has 16 vertices and 22 edges in G and $L(G)$ has 22 vertices.

Let $D = \{v_2, v_3, v_8, v_{19}, v_{13}, v_{17}, v_{18}, v_{22}\}$ and hence $\gamma[L(G)] = 4$. If $D' = \{v_1, v_4, v_6, v_{10}, v_{14}, v_{15}, v_{20}, v_{21}\}$ then $\gamma'[L(G)] = 4$. By definition of total dominating set we get $\gamma_t(G) = 4$. Since the induced subgraph of D and D' is disconnected and it is not regular graph so by definition we have disconnected inverse irregular domination number $\gamma'_{dcir}[L(G)] = 4$ and disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 4$. Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{dcir}[L(G)] = 4$ which is less than p .

Case 2: When m is odd and n is even (i. e) $m = 3, n = 6$. The graph $G = P_3 \times P_6$ has 18 vertices and 27 edges and $L(G)$ has 27 vertices.

If $D = \{v_4, v_7, v_8, v_{12}, v_{20}, v_{21}, v_{26}, v_{27}\}$ then $\gamma[L(G)] = 8$. Let $D' = \{v_2, v_{10}, v_{14}, v_{15}, v_{17}, v_{18}, v_{23}$ and $v_{24}\}$ then $\gamma'[L(G)] = 8$. By definition of total dominating set we get $\gamma_t[L(G)] = 8$ and since the induced subgraph of D is disconnected and not regular graph we have disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 8$. Since the induced subgraph of D is disconnected and it is not regular graph so we have $\gamma'_{dcir}[L(G)] = 8$. Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] =$

$$\gamma'[L(G)] = \gamma'_{dcir}[L(G)] = 8 < 27.$$

Case 3: When m and n are odd and equal (i. e) $m = 3, n = 3$. The graph $G = P_3 \times P_3$ has 9 vertices and 12 edges and $L(G)$ has 12 vertices.

Let $D = \{v_1, v_3, v_{11}, v_{12}\}$ and thus $\gamma[L(G)] = 4$. Let $D' = \{v_4, v_5, v_6, v_{10}\}$ and hence $\gamma'[L(G)] = 4$. By definition of total dominating set we get $\gamma_t[L(G)] = 4$. Since the induced subgraph of D and D' is disconnected and it is not regular graph so by definition we have disconnected inverse irregular domination number $\gamma_{dcir}[L(G)] = 4$ and disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 4$. Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma_{dcir}[L(G)] = 4$ which is less than $p = 12$.

Case 4: When m and n are even and equal (i. e) $m = 4, n = 4$. The graph $G = P_4 \times P_4$ has 16 vertices and 24 edges and $L(G)$ has 24 vertices.

Let $D = \{v_2, v_3, v_7, v_{11}, v_{13}, v_{18}, v_{23}, v_{24}\}$ be the dominating set and $\gamma[L(G)] = 8$. Let $D' = \{v_5, v_6, v_8, v_9, v_{12}, v_{17}, v_{20}, v_{21}\}$ and thus $\gamma'[L(G)] = 8$. By definition of total dominating set we get $\gamma_t[L(G)] = 8$ and since the induced subgraph of D is disconnected and not regular graph we have disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 8$. Since the induced subgraph of D' is disconnected and it is not regular graph so by definition of disconnected inverse irregular domination number so we have $\gamma_{dcir}[L(G)] = 8$. Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma_{dcir}[L(G)] = 8 < 24$.

Case 5: When m and n are odd and distinct (i. e) $m = 5, n = 3$. The graph $G = P_5 \times P_3$ has 15 vertices and 22 edges and $L(G)$ has 22 vertices.

Let $D = \{v_4, v_5, v_{11}, v_{12}, v_{17}, v_{18}, v_{19}\}$ be the dominating set of $V[L(G)]$ and hence $\gamma[L(G)] = 7$. If $D' = \{v_2, v_7, v_8, v_9, v_{14}, v_{16}, v_{20}\}$ then $\gamma'[L(G)] = 7$. By definition of total dominating set we get $\gamma_t[L(G)] = 7$. Since the induced subgraph of D and D' is disconnected and it is not regular graph so by definition we have disconnected inverse irregular domination number $\gamma'_{dcir}[L(G)] = 7$ and disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 7$. Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{dcir}[L(G)] = 7$ which is less than $p = 22$.

From all these cases we conclude that for all $m, n > 2$ we have $\gamma[L(G)] = \gamma_t(G) = \gamma_{tdcir}(G) = \gamma'[L(G)] = \gamma'_{dcir}[L(G)] < p$ where p is the no. of vertices in $L(G)$. □

Example 3.2. The Cartesian product of $G = P_3 \times P_6$ is given in figure (3).

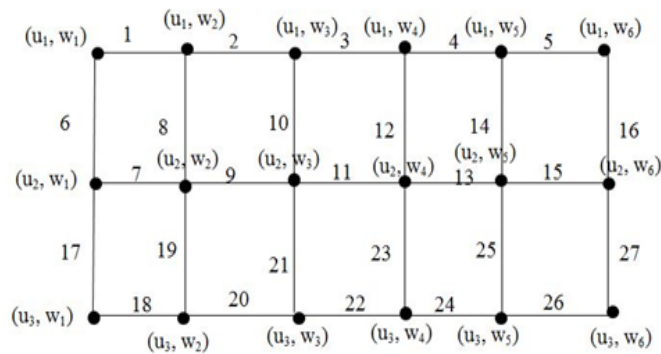


Figure 3. $G = P_3 \times P_6$

The line graph of $G = P_3 \times P_6$ is shown in figure (4).

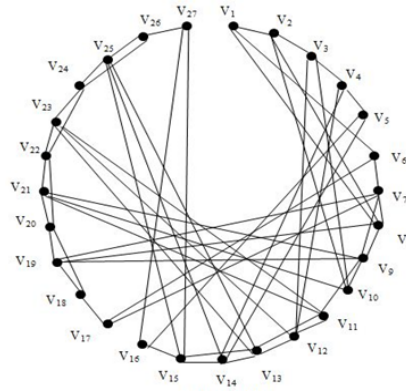


Figure 4. $L(G)$

From figure 3. we have the dominating set $D = \{v_4, v_7, v_8, v_{12}, v_{20}, v_{21}, v_{26}, v_{27}\}$ and so $\gamma[L(G)] = 8$. By definition of inverse dominating set, $D' = \{v_2, v_{10}, v_{14}, v_{15}, v_{17}, v_{18}, v_{23}$ and $v_{24}\}$ and hence $\gamma'[L(G)] = 8$ and the total dominating set $\gamma_t[L(G)] = 8$. By definition of disconnected total irregular domination number and disconnected inverse irregular domination number we have $\gamma_{tdcir}[L(G)] = 8$ and $\gamma'_{dcir}[L(G)] = 8$.

4. Relation Between $\gamma(G)$, $\gamma_t(G)$, $\gamma_{tdcir}(G)$, $\gamma'(G)$ and $\gamma'_{tdcir}(G)$ of $L(P_m \times C_n)$

Theorem 4.1. Let P_m and C_n be two graphs with m and n vertices respectively and let $G = P_m \times C_n$ be the Cartesian product of two cycle graphs. Let $L(G)$ be the line graph and $\gamma(G), \gamma_t(G), \gamma_{tdcir}(G), \gamma'(G), \gamma'_{dcir}(G)$ are the domination number, total domination number, disconnected total irregular domination number, inverse domination number, disconnected inverse irregular domination number of G then $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma_{dcir}[L(G)] < p$ where p is the no. of vertices in $L(G) \forall m \geq 2$ and $n > 2$.

Proof. The vertex set of P_m be $U = \{u_1, u_2, \dots, u_m\}$ and the vertex set of C_n be $W = \{w_1, w_2, \dots, w_n\}$. The graph G has mn vertices and $2mn - n$ edges and the line graph $L(G)$ has $2mn - n$ vertices. To find the domination number, total domination number, disconnected total irregular domination number, inverse domination number and inverse disconnected irregular domination number we consider the following cases:

Case 1: When m and n are even and distinct (i. e) $m = 4, n = 6$ the graph $G = P_4 \times C_6$ has 24 vertices and 42 edges in G and $L(G)$ has 42 vertices.

Since $D = \{v_5, v_6, v_{11}, v_{12}, v_{15}, v_{21}, v_{26}, v_{29}, v_{30}, v_{33}, v_{36}, v_{37}, v_{41}\}$ is the dominating set of $L(G)$ and hence $\gamma[L(G)] = 13$. Let $D' = \{v_3, v_8, v_9, v_{13}, v_{14}, v_{17}, v_{18}, v_{24}, v_{28}, v_{31}, v_{32}, v_{35}, v_{39}\}$ be the inverse dominating set and $\gamma'[L(G)] = 13$. By definition of total dominating set we get $\gamma_t[L(G)] = 13$ and since the induced subgraph of D is disconnected and not regular graph we have disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 13$. Since the induced subgraph of D' is disconnected and it is not regular graph so by definition of disconnected inverse irregular domination number we have $\gamma'_{dcir}[L(G)] = 13$. Hence $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{dcir}[L(G)] = 13 < 42$.

Case 2: When m is even and n is odd (i. e) $m = 6, n = 3$. The graph $G = P_6 \times C_3$ has 18 vertices and 33edges and $L(G)$ has 33 vertices.

Let $D = \{v_4, v_6, v_7, v_{10}, v_{12}, v_{21}, v_{22}, v_{24}, v_{30}, v_{31}\}$ be the dominating set of $V[L(G)]$ which is the minimum dominating set of G and hence $\gamma[L(G)] = 10$. Let $D' = \{v_2, v_9, v_8, v_{15}, v_{19}, v_{17}, v_{23}, v_{27}, v_{28}, v_{29}\}$ be the inverse dominating set and so $\gamma'[L(G)] = 10$. By definition of total dominating set we get $\gamma_t[L(G)] = 10$. Since the induced sub-

graph of D and D' is disconnected and it is not regular graph so by definition we have disconnected inverse irregular domination number $\gamma'_{dcir}[L(G)] = 10$ and disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 10$. Thus $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{dcir}[L(G)] = 10$ which is less than $p = 33$.

Case 3: When m and n are odd and equal (i. e) $m=3, n=3$. The graph $G = P_3 \times C_3$ has 9 vertices and 15edges and $L(G)$ has 15 vertices.

Since $D = \{v_2, v_5, v_8, v_{10}, v_{12}\}$ is the dominating set of $L(G)$ and hence $\gamma[L(G)] = 5$. Let $D' = \{v_1, v_6, v_7, v_{11}, v_{13}\}$ be the inverse dominating set of $V[L(G)]$ and hence $\gamma'[L(G)] = 5$. By definition of total dominating set we get $\gamma_t[L(G)] = 5$ and since the induced subgraph of D is disconnected and not regular graph we have disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 5$. Since the induced subgraph of D' is disconnected and it is not regular graph so we have $\gamma'_{dcir}[L(G)] = 5$. Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{dcir}[L(G)] = 5$ which is less than $p = 50$.

Case 4: When m and n are even and equal (i. e) $m = 4, n = 4$. The graph $G = P_4 \times C_4$ has 16 vertices and 28 edges and $L(G)$ has 28 vertices.

Let $D = \{v_1, v_4, v_5, v_{12}, v_{14}, v_{17}, v_{20}, v_{25}, v_{28}\}$ is the dominating set of $L(G)$ and so $\gamma[L(G)] = 9$. Let $D' = \{v_3, v_6, v_7, v_{11}, v_{13}, v_{16}, v_{18}, v_{23}, v_{27}\}$ be the inverse dominating set of $V[L(G)]$ and hence $\gamma'[L(G)] = 9$. By definition of total dominating set we get $\gamma_t[L(G)] = 9$. Since the induced subgraph of D and D' is disconnected and it is not regular graph so by definition we have disconnected inverse irregular domination number $\gamma'_{dcir}[L(G)] = 9$ and disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 9$. Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{dcir}[L(G)] = 9$ which is less than $p = 28$.

Case 5: When m and n are odd and distinct (i. e) $m = 5, n = 3$. The graph $G = P_5 \times C_3$ has 15 vertices and 27 edges and $L(G)$ has 27 vertices.

Since $D = \{v_1, v_4, v_9, v_{13}, v_{20}, v_{22}, v_{23}\}$ is the dominating set of $L(G)$ and hence $\gamma[L(G)] = 7$. Let $D' = \{v_3, v_8, v_{11}, v_{12}, v_{21}, v_{25}, v_{26}\}$ be the inverse dominating set and hence $\gamma'[L(G)] = 7$. By definition of total dominating set we get $\gamma_t[L(G)] = 7$ and since the induced subgraph of D is disconnected and not regular graph we have disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 7$. Since the induced subgraph of D' is disconnected and it is irregular graph so by definition of disconnected inverse irregular domination number we have $\gamma'_{dcir}[L(G)] = 7$. Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{dcir}[L(G)] = 7 < 27$.

From all these cases we generalized that for all $m, n > 2$ we have $\gamma[L(G)] = \gamma'[L(G)] = \gamma'_{dcir}[L(G)] < p$ where p is the no. of vertices in $L(G)$. □

Example 4.2. When $m=4, n=4$. The Cartesian product of $G = P_4 \times C_4$ is given in figure (5).

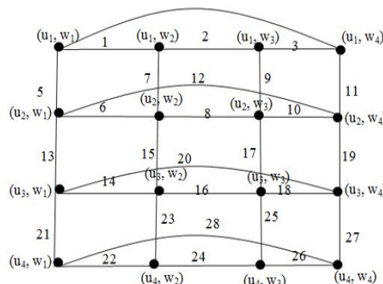


Figure 5. $G = P_3 \times P_4$

The line graph of $G = P_4 \times C_4$ is shown in figure (6).

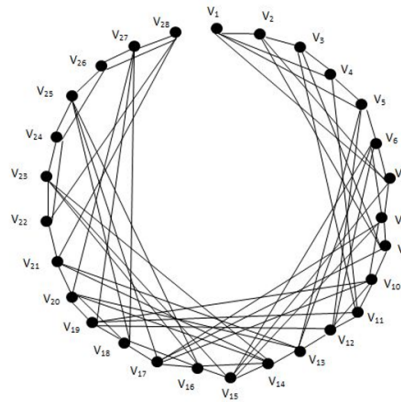


Figure 6. $L(G)$

By definition of domination set, $D = \{v_1, v_4, v_5, v_{12}, v_{14}, v_{17}, v_{20}, v_{25}, v_{28}\}$ and so $\gamma[L(G)] = 9$. By definition of inverse domination set, $D = \{v_3, v_6, v_7, v_{11}, v_{13}, v_{16}, v_{18}, v_{23}, v_{27}\}$ and hence $\gamma'[L(G)] = 9$. By definition of total domination set we get $\gamma_t[L(G)] = 9$. Then by definition of disconnected total irregular domination number and disconnected inverse irregular domination number we have $\gamma_{tdcir}[L(G)] = 9$ and $\gamma'_{dcir}[L(G)] = 9$.

Observation: When $G_1 = (C_m \times C_n)$, $G_2 = (P_m \times P_n)$ and $G_2 = (P_m \times C_n) \forall n, m > 2$ then $L(G_1)$ and $L(G_2)$ has the following properties:

- It has no accurate dominating set.
- It has no global dominating set.

5. Domination Number of Complement of $L(P_m \times P_n)$

In this section we obtain the domination number of complement of line graph formed from the Cartesian product of two path graphs.

Theorem 5.1. Let P_m and P_n be two path graphs with m and n vertices respectively and let $G = P_m \times P_n$ be the Cartesian product of two path graphs. Let $L(G)$ be the line graph and $L(\overline{G})$ be the complement of line graph then the domination number then $\gamma[\overline{L(G)}] = \gamma'[\overline{L(G)}] = 2\forall m, n > 2$.

Proof. Let $U = \{u_1, u_2, \dots, u_m\}$ be the vertex set in P_m and the vertex set of P_n be $W = \{w_1, w_2, \dots, w_n\}$. The graph G has nm vertices and $2mn - (m + n)$ edges and the line graph $L(G)$ has $2mn(m + n)$ vertices. To find the domination number, inverse domination number and inverse disconnected irregular domination number we consider the following cases:

Case 1: When m and n are odd (i. e) $m = 5, n = 5$. The graph $G = P_5 \times P_5$ has 25 vertices and 40 edges and $L(G)$ has 40 vertices.

The dominating set of $[L(G)]$ denoted is $D = \{v_1, v_4\}$ (i. e) $\gamma[L(G)] = 2$. The dominating set of V-D denoted by $D' = \{v_2, v_{12}\}$ (i. e) $\gamma'[L(G)] = 2$. Therefore $\gamma[\overline{L(G)}] = \gamma'[\overline{L(G)}] = 2 < 40$.

Case 2: When $m = n =$ even and distinct (i. e) $m = 4, n = 6$. The graph $G = P_4 \times P_6$ has 24 vertices and 38 edges and $L(G)$ has 38 vertices.

The dominating set of $[L(G)]$ is $D = \{v_1, v_5\}$. (i. e) $\gamma[L(G)] = 2$. The dominating set of V-D denoted by $D' = \{v_{15}, v_{34}\}$. (i. e) $\gamma'[L(G)] = 2$. Therefore $\gamma[\overline{L(G)}] = \gamma'[\overline{L(G)}] = 2 < 38$.

Case 3: When $m = n =$ odd and distinct (i. e) $m = 5, n = 3$. The graph $G = P_5 \times P_3$ has 15 vertices and 22 edges and

$L(G)$ has 22 vertices.

The dominating set of $[L(G)]$ denoted by D is v_1, v_9 . (i. e) $\gamma[L(G)] = 2$. The dominating set of $V - D$ denoted by D' is v_2 and v_8 . (i. e) $\gamma'[L(G)] = 2$. Therefore $\gamma[\overline{L(G)}] = \gamma'[\overline{L(G)}] = 2$ which is less than $p = 22$.

Case 4: When m and n are even and equal(i. e) $m = 6, n = 6$. The graph $G = P_6 \times P_6$ has 36 vertices and 60 edges and $L(G)$ has 60 vertices.

The dominating set of $[L(G)]$ is $D = \{v_2, v_5\}$. (i. e) $\gamma[L(G)] = 2$. The dominating set of $V-D$ is denoted by $D' = \{v_{40}, v_{47}\}$. (i. e) $\gamma'[L(G)] = 2$. Therefore $\gamma[\overline{L(G)}] = \gamma'[\overline{L(G)}] = 2 < 60$.

Case 5: When m is even and n is odd(i. e) $m = 5, n = 4$. The graph $G = P_5 \times P_4$ has 20 vertices and 31 edges and $L(G)$ has 31 vertices.

The dominating set of $[L(G)]$ is denoted by $D = \{v_4, v_9\}$. (i. e) $\gamma[L(G)] = 2$. The dominating set of $V-D$ is denoted by $D' = \{v_3, v_5\}$. (i. e) $\gamma'[L(G)] = 2$. Therefore $\gamma[\overline{L(G)}] = \gamma'[\overline{L(G)}] = 2 < p = 31$.

From all these cases we can generalize that $\gamma[\overline{L(G)}] = \gamma'[\overline{L(G)}] = 2 \forall m, n > 2$. □

Example 5.2. When $m = 3, n = 3$. The Cartesian product of $G = P_3 \times P_3$ is given in figure (7).

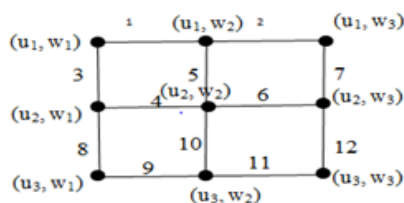


Figure 7. $G = P_3 \times P_4$

The complement of $L(G)$ is shown in figure (8).

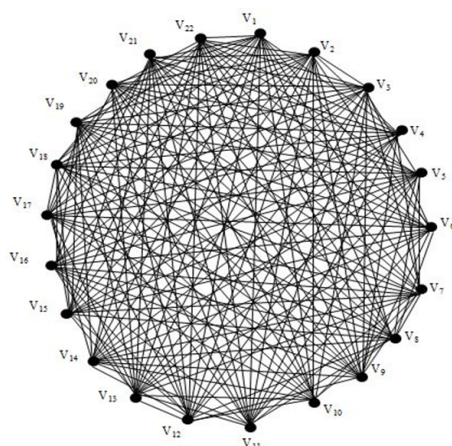


Figure 8. $\overline{L(G)}$

Theorem 5.3. Let C_m and C_n be two cycle graphs with m and n vertices respectively and let $G = C_m \times C_n$ be the Cartesian product of two cycle graphs. Let $L(G)$ be the line graph and $\overline{L(G)}$ be the complement of line graph γ be the domination number then $\gamma[\overline{L(G)}] = 2\forall m, n > 2$.

Proof. The proof of this theorem follows from the Theorem 5. 1. □

Theorem 5.4. Let P_m and C_n be two graphs with m and n vertices respectively and let $G = P_m \times C_n$ be the Cartesian product of two cycle graphs. Let $L(G)$ be the line graph and $\overline{L(G)}$ be the complement of line graph γ be the domination number then $\gamma[\overline{L(G)}] = 2\forall m \geq 2$ and $n > 2$.

Proof. The proof of this theorem follows from the Theorem 5. 1. □

6. Conclusion

The concept of domination number of line graphs formed from the Cartesian product of path and cycle graphs are defined and the relation between many domination parameters are characterized and described with few examples. This work can be used for further research to any other simple graph products.

References

-
- [1] C. Berge, *Theory of graphs and its applications*, Methuen, London, (1962).
 - [2] E. J. Cockayne and S. T. Hedetniemi, *Theory Of Domination in Graphs*, Networks, (1977), 247-261.
 - [3] E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi, *Total domination in graphs*, Networks, 10(1980), 211-219.
 - [4] W. Imrich and S. Klavzar, *Product Graphs: Structure and Recognition*, Wiley, (2000).
 - [5] O. Ore, *Theory of Graphs*, American Mathematical Society, Providence, R. I , (1962).
 - [6] T. TamizhChelvan, T. Asir and G. S. Grace Prema, *Inverse Domination in graphs*, Lambert Academic Publishing, (2013).
 - [7] V. R. Kulli and S. C. Sigarkanti, *Inverse domination in graphs*, Nat. Acad. Sci. Letters, 14(1991), 473-475.