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Unified Finite Integral Associated with Generalized I-function

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Abstract: In this paper, the authors established new results by applying definite integrals on the product of $\bar{I}$-function with the hypergeometric function. Several other new and known results can also be obtained from our main theorems.

Keywords: $\bar{I}$-function, hypergeometric function, Definite integrals.

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1. Introduction and Preliminaries

During the last four decays or so, several many interesting and useful extensions of the familiar special functions (such as I-function which is extension of H-function and G- function and so on) have been considered by several authors (see, for example, Satyanarayana [10], Saxena [11], Saxena [12–14], Shantha [15] and Vyas and Rathie [19]. The $\bar{I}$-function, introduced by Rathie [9], however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral:

$$\bar{I}(z)=\bar{I}_{p,\,q}^{m,\,n}\left(z\left|\begin{matrix} {}_1\left(a_j,\alpha_j;A_j'\right)_n,\ {}_{n+1}\left(a_j,\alpha_j;A_j''\right)_p \\ {}_1\left(b_j,\beta_j;B_j''\right)_m,\ {}_{m+1}\left(b_j,\beta_j;B_j'\right)_q\end{matrix}\right.\right)=\frac{1}{2\pi\omega}\int_L \chi(s)z^s ds, \tag{1}$$

For all z different from zero and

$$\chi(s)=\frac{\prod\limits_{j=1}^{m}\Gamma^{B_j''}\left(b_j-\beta_j s\right)\prod\limits_{j=1}^{n}\Gamma^{A_j'}\left(1-a_j+\alpha_j s\right)}{\prod\limits_{j=n+1}^{p}\Gamma^{A_j''}\left(a_j-\alpha_j s\right)\prod\limits_{j=m+1}^{q}\Gamma^{B_j'}\left(1-b_j+\beta_j s\right)}, \tag{2}$$

The integral (1) converges when $|\arg z|<\frac{1}{2}\Delta\pi$, if $\Delta>0$, where

$$\Delta=\sum_{j=1}^{m}B_j''\beta_j-\sum_{j=m+1}^{q}B_j'\beta_j+\sum_{j=1}^{n}A_j'\alpha_j-\sum_{j=n+1}^{p}A_j''\alpha_j \tag{3}$$

Recently, the following formulas are defined Qureshi [8].

$$\int_0^\infty\left[\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right]^{-u-1}dx\ =\ \frac{\sqrt{\pi}}{2\rho\left(4\rho\tau+\gamma\right)^{u+1/2}}\frac{\Gamma\left(u+1/2\right)}{\Gamma\left(u+1\right)}, \tag{4}$$

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$(\rho > 0;\ \tau \geq 0;\ \gamma + 4\rho\tau > 0;\ (\Re(u) + 1/2) > 0)$.

$$\int_0^\infty \frac{1}{x^2}\left[\left(\rho\, x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-u-1} dx = \frac{\sqrt{\pi}}{2\tau\,(4\rho\tau + \gamma)^{u+1/2}}\,\frac{\Gamma\,(u+1/2)}{\Gamma\,(u+1)}, \tag{5}$$

$(\rho \geq 0;\ \tau > 0;\ \gamma + 4\rho\tau > 0;\ (\Re(u) + 1/2) > 0)$.

$$\int_0^\infty \left(\rho + \frac{\tau}{x^2}\right)\left[\left(\rho\, x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-u-1} dx = \frac{\sqrt{\pi}}{(4\rho\tau + \gamma)^{u+1/2}}\,\frac{\Gamma\,(u+1/2)}{\Gamma\,(u+1)}, \tag{6}$$

$(\rho > 0;\ \tau > 0;\ \gamma + 4\rho\tau > 0;\ (\Re(u) + 1/2) > 0)$.

The following formulas [16] will be required in our investigation

$${}_2F_1\left(a,\, b,\, c + \frac{1}{2};\, x\right)\ {}_2F_1\left(c - a,\, c - b,\, c + \frac{1}{2};\, x\right) = \sum_{r=0}^{\infty} \frac{(c)_r}{\left(c + \frac{1}{2}\right)_r} a_r\, x^r. \tag{7}$$

2. Main Results

Theorem 2.1. *If $\rho > 0$; $\tau \geq 0$; $\gamma + 4\rho\tau > 0$; $(\Re(\lambda) + 1/2) > 0$, $-\frac{1}{2} < (a - b - c) < \frac{1}{2}$; $|\arg z| < \frac{1}{2}\Delta$, where $\Delta > 0$ and given by equation (3), then the following formula holds true:*

$$\begin{aligned}
&\int_0^\infty \left[\left(\rho\, x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-\lambda-1} {}_2F_1\left(a,\, b,\, c + \frac{1}{2};\, y\left\{\left(\rho\, x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\mu}\right) \\
&\qquad \times {}_2F_1\left(c - a,\, c - b,\, c + \frac{1}{2};\, y\left\{\left(\rho\, x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\mu}\right) \bar{I}^{m,n}_{p,q}\left[z\left\{\left(\rho\, x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\omega}\right] dx \\
&\qquad = \frac{\sqrt{\pi}}{2\rho\,(4\rho\tau + \gamma)^{\lambda+1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{\left(c + \frac{1}{2}\right)_r} \frac{a_r\, y^r}{(4\rho\tau + \gamma)^{\mu\, r}} \\
&\qquad \times\ \bar{I}^{m,n+1}_{p+1,q+1}\left[\frac{z}{(4\rho\tau + \gamma)^{\omega}}\left|\begin{matrix} \left(\frac{1}{2} - \lambda - \mu\, r, \omega;\ 1\right),\ {}_1\left(a_j, \alpha_i;\ A'_j\right)_n,\ {}_{n+1}\left(a_j, \alpha_i;\ A''_j\right)_p \\ {}_1\left(b_j, \beta_j;\ B''_j\right)_m,\ {}_{m+1}\left(b_j, \beta_j;\ B'_j\right)_q,\ (-\lambda - \mu\, r, \omega;\ 1) \end{matrix}\right.\right].
\end{aligned} \tag{8}$$

Proof. By virtue of equation (1), (4) and (7), we have

$$= \int_0^\infty \left[\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-\lambda-1} \sum_{r=0}^{\infty} \frac{(c)_r}{\left(c + \frac{1}{2}\right)_r} a_r y^r \left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\mu r} \times \frac{1}{(2\pi\omega)} \int_L \chi(s)\,(z)^s \left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\omega s} ds dx \tag{9}$$

Interchanging the order of integration and summation, under the valid conditions, we get

$$= \sum_{r=0}^{\infty} \frac{(c)_r}{\left(c + \frac{1}{2}\right)_r} a_r\, y^r \frac{1}{(2\pi\,\omega)} \int_L \chi(s)\,(z)^s \left\{\int_0^\infty \left[\left(\rho\, x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-\lambda-\mu\, r-\omega s-1} dx\right\} ds, \tag{10}$$

With the help of equation (4), the above integral becomes

$$= \frac{\sqrt{\pi}}{2\rho\,(4\rho\tau + \gamma)^{\lambda+1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{\left(c + \frac{1}{2}\right)_r} \frac{a_r\, y^r}{(4\rho\tau + \gamma)^{\mu\, r}} \frac{1}{(2\pi\,\omega)} \int_L \chi(s) \left(\frac{z}{(4\rho\tau + \gamma)^{\omega}}\right)^s \frac{\Gamma\,(\lambda + \mu\, r + \omega\, s + 1/2)}{\Gamma\,(\lambda + \mu\, r + \omega\, s + 1)} ds,$$

Now using the above equation and equation (1), we arrive at (8). This is the completed proof of Theorem 2.1. □

Theorem 2.2. *If $\rho \geq 0$; $\tau > 0$; $\gamma + 4\rho\tau > 0$; $(\Re(\lambda) + 1/2) > 0$, $-\frac{1}{2} < (a - b - c) < \frac{1}{2}$; $|\arg z| < \frac{1}{2}\Delta$, where $\Delta > 0$ and given by equation (3), then the following formula holds true:*

$$\int_0^\infty \frac{1}{x^2}\left[\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right]^{-\lambda-1} {}_2F_1\left(a, b, c + \frac{1}{2}; y\left\{\left(\rho x + \frac{\tau}{x}\right)^2 + \gamma\right\}^{-\mu}\right)$$

$$\times\,{}_2F_1\left(c-a,c-b,c+\frac{1}{2};y\left\{\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right\}^{-\mu}\right)\bar{I}^{m,n}_{p,q}\left[z\left\{\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right\}^{-\omega}\right]dx$$

$$=\frac{\sqrt{\pi}}{2\tau(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r y^r}{(4\rho\tau+\gamma)^{\mu r}}$$

$$\times\,\bar{I}^{m,n+1}_{p+1,q+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\left|\begin{matrix}\left(\frac{1}{2}-\lambda-\mu r,\omega;1\right),{}_1\left(a_j,\alpha_i;A'_j\right)_n,{}_{n+1}\left(a_j,\alpha_i;A''_j\right)_p\\ {}_1\left(b_j,\beta_j;B''_j\right)_m,{}_{m+1}\left(b_j,\beta_j;B'_j\right)_q,(-\lambda-\mu r,\omega;1)\end{matrix}\right.\right]. \tag{11}$$

Proof. By virtue of equation (1), (5) and (7), we obtain

$$=\int_0^{\infty}\frac{1}{x^2}\left[\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right]^{-\lambda-1}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}a_r\,y^r\left\{\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right\}^{-\mu r}$$
$$\times\,\frac{1}{(2\pi\,\omega)}\int_L\chi(s)\,(z)^s\left\{\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right\}^{-\omega s}dsdx, \tag{12}$$

Now interchanging the order of integration and summation, we get

$$=\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}a_r\,y^r\,\frac{1}{(2\pi\,\omega)}\int_L\chi(s)\,(z)^s\left\{\int_0^{\infty}\frac{1}{x^2}\left[\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right]^{-\lambda-\mu r-\omega s-1}dx\right\}ds, \tag{13}$$

With the help of equation (5), the above integral becomes

$$=\frac{\sqrt{\pi}}{2\tau\,(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu\,r}}\frac{1}{(2\pi\,\omega)}\int_L\chi(s)\left(\frac{z}{(4\rho\tau+\gamma)^{\omega}}\right)^s\frac{\Gamma\left(\lambda+\mu\,r+\omega\,s+1/2\right)}{\Gamma\left(\lambda+\mu\,r+\omega\,s+1\right)}ds$$

Now using the above equation in view of equation (1), we arrive at (11). This is the completed proof of Theorem 2.2. □

Theorem 2.3. *If* $\rho>0$; $\tau>0$; $\gamma+4\rho\tau>0$; $(\Re(\lambda)+1/2)>0$, $-\frac{1}{2}<(a-b-c)<\frac{1}{2}$; $|\arg z|<\frac{1}{2}\Delta$, *where* $\Delta>0$ *and given by equation* (3)*, then the following formula holds true:*

$$\int_0^{\infty}\left(\rho+\frac{\tau}{x^2}\right)\left[\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right]^{-\lambda-1}{}_2F_1\left(a,b,c+\frac{1}{2};y\left\{\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right\}^{-\mu}\right)$$
$$\times\,{}_2F_1\left(c-a,c-b,c+\frac{1}{2};y\left\{\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right\}^{-\mu}\right)\bar{I}^{m,n}_{p,q}\left[z\left\{\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right\}^{-\omega}\right]dx$$
$$=\frac{\sqrt{\pi}}{(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r y^r}{(4\rho\tau+\gamma)^{\mu r}}$$
$$\times\,\bar{I}^{m,n+1}_{p+1,q+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\left|\begin{matrix}\left(\frac{1}{2}-\lambda-\mu r,\omega;1\right),{}_1\left(a_j,\alpha_i;A'_j\right)_n,{}_{n+1}\left(a_j,\alpha_i;A''_j\right)_p\\ {}_1\left(b_j,\beta_j;B''_j\right)_m,{}_{m+1}\left(b_j,\beta_j;B'_j\right)_q,(-\lambda-\mu r,\omega;1)\end{matrix}\right.\right] \tag{14}$$

Proof. By virtue of equation (1), (6) and (7), we have

$$=\int_0^{\infty}\left(\rho+\frac{\tau}{x^2}\right)\left[\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right]^{-\lambda-1}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}a_r\,y^r\left\{\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right\}^{-\mu r}$$
$$\times\,\frac{1}{(2\pi\,\omega)}\int_L\chi(s)\,(z)^s\left\{\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right\}^{-\omega s}dsdx, \tag{15}$$

On interchanging the order of integration and summation under the valid assumption, we get

$$=\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}a_r\,y^r\,\frac{1}{(2\pi\,\omega)}\int_L\chi(s)\,(z)^s\left\{\int_0^{\infty}\left(\rho+\frac{\tau}{x^2}\right)\left[\left(\rho x+\frac{\tau}{x}\right)^2+\gamma\right]^{-\lambda-\mu r-\omega s-1}dx\right\}ds, \tag{16}$$

With the help of equation (6), the above integral becomes

$$=\frac{\sqrt{\pi}}{(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu\,r}}\frac{1}{(2\pi\,\omega)}\int_L\chi(s)\left(\frac{z}{(4\rho\tau+\gamma)^{\omega}}\right)^s\frac{\Gamma\left(\lambda+\mu\,r+\omega\,s+1/2\right)}{\Gamma\left(\lambda+\mu\,r+\omega\,s+1\right)}ds,$$

Now using the above equation in view of (1), we arrive at (14). This is the completed proof of Theorem 2.3. □

3. Special Cases

In this section, we derive some new integral formulae by using some known $\bar{H}$-function, H-function, G-function which are given in Corollaries (17) to (19), (20) to (22) and (23) to (25) respectively.

(I). If we use the same method as in getting Theorem 2.1 to 2.3, we obtain the following three corollaries which is well known $\bar{H}$- function, due to Inayat-Hussain [5], for giving value $A_j'' = B_j'' = 1$in equation (1), we get

$$\bar{I}^{m,\,n}_{p,\,q}\left(z\,\middle|\,\begin{matrix}{}_1\left(a_j,\alpha_j;A_j'\right)_n,\ {}_{n+1}\left(a_j,\alpha_j;A_j''\right)_p\\ {}_1\left(b_j,\beta_j;B_j''\right)_m,\ {}_{m+1}\left(b_j,\beta_j;B_j'\right)_q\end{matrix}\right)=\bar{H}^{m,\,n}_{p,\,q}\left(z\,\middle|\,\begin{matrix}{}_1\left(a_j,\alpha_j;A_j'\right)_n,\ {}_{n+1}\left(a_j,\alpha_j;1\right)_p\\ {}_1\left(b_j,\beta_j;1\right)_m,\ {}_{m+1}\left(b_j,\beta_j;B_j'\right)_q\end{matrix}\right)$$

Now apply above identity, on the Theorems 2.1-2.3 reduces respectively as:

Corollary 3.1. *Let the condition of Theorem 2.1 be satisfied, we have*

$$\begin{aligned}\int_0^\infty &\Phi^{-\lambda-1}{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\left\{\Phi\right\}^{-\mu}\right){}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\left\{\Phi\right\}^{-\mu}\right)\bar{H}^{m,n}_{p,q}\left[z\left\{\Phi\right\}^{-\omega}\right]dx\\ &=\frac{\sqrt{\pi}}{2\rho\left(4\rho\tau+\gamma\right)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{\left(4\rho\tau+\gamma\right)^{\mu\,r}}\\ &\times\bar{H}^{m,n+1}_{p+1,q+1}\left[\frac{z}{\left(4\rho\tau+\gamma\right)^{\omega}}\,\middle|\,\begin{matrix}\left(\frac{1}{2}-\lambda-\mu\,r,\omega;\ 1\right),{}_1\left(a_j,\alpha_i;A_j'\right)_n,\ {}_{n+1}\left(a_j,\alpha_i;\ 1\right)_p\\ {}_1\left(b_j,\beta_j;\ 1\right)_m,\ {}_{m+1}\left(b_j,\beta_j;B_j'\right)_q,\left(-\lambda-\mu\,r,\omega;\ 1\right)\end{matrix}\right],\end{aligned}\tag{17}$$

where $\Phi=\left[\left(\rho\,x+\frac{\tau}{x}\right)^2+\gamma\right]$.

Corollary 3.2. *Let the condition of Theorem 2.2 be satisfied, we obtain*

$$\begin{aligned}\int_0^\infty &\frac{1}{x^2}\,\Phi^{-\lambda-1}{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\left\{\Phi\right\}^{-\mu}\right){}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\left\{\Phi\right\}^{-\mu}\right)\bar{H}^{m,n}_{p,q}\left[z\left\{\Phi\right\}^{-\omega}\right]dx\\ &=\frac{\sqrt{\pi}}{2\tau\left(4\rho\tau+\gamma\right)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{\left(4\rho\tau+\gamma\right)^{\mu\,r}}\\ &\times\bar{H}^{m,n+1}_{p+1,q+1}\left[\frac{z}{\left(4\rho\tau+\gamma\right)^{\omega}}\,\middle|\,\begin{matrix}\left(\frac{1}{2}-\lambda-\mu\,r,\ \omega;\ 1\right),{}_1\left(a_j,\alpha_i;\ A_j'\right)_n,\ {}_{n+1}\left(a_j,\alpha_i;\ 1\right)_p\\ {}_1\left(b_j,\beta_j;\ 1\right)_m,\ {}_{m+1}\left(b_j,\beta_j;\ B_j'\right)_q,\left(-\lambda-\mu\,r,\ \omega;\ 1\right)\end{matrix}\right].\end{aligned}\tag{18}$$

Corollary 3.3. *Let the condition of Theorem 2.3 be satisfied, we get*

$$\begin{aligned}\int_0^\infty &\left(\rho+\frac{\tau}{x^2}\right)\Phi^{-\lambda-1}{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\left\{\Phi\right\}^{-\mu}\right){}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\left\{\Phi\right\}^{-\mu}\right)\bar{H}^{m,n}_{p,q}\left[z\left\{\Phi\right\}^{-\omega}\right]dx\\ &=\frac{\sqrt{\pi}}{\left(4\rho\tau+\gamma\right)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{\left(4\rho\tau+\gamma\right)^{\mu\,r}}\\ &\times\bar{H}^{m,n+1}_{p+1,q+1}\left[\frac{z}{\left(4\rho\tau+\gamma\right)^{\omega}}\,\middle|\,\begin{matrix}\left(\frac{1}{2}-\lambda-\mu\,r,\omega;\ 1\right),{}_1\left(a_j,\alpha_i;\ A_j'\right)_n,\ {}_{n+1}\left(a_j,\alpha_i;\ 1\right)_p\\ {}_1\left(b_j,\beta_j;\ 1\right)_m,\ {}_{m+1}\left(b_j,\beta_j;\ B_j'\right)_q,\left(-\lambda-\mu\,r,\omega;\ 1\right)\end{matrix}\right].\end{aligned}\tag{19}$$

(II). If we use $A_j'' = B_j'' = A_j' = B_j' = 1$ in Theorem 2.1 to 2.3, then $\bar{I}$-function reduces in to well known H- function defined by Fox [4], Braaksma [2] and Mathai and Saxena [7].

$$\bar{I}^{m,\,n}_{p,\,q}\left(z\,\middle|\,\begin{matrix}{}_1\left(a_j,\alpha_j;A_j'\right)_n,\ {}_{n+1}\left(a_j,\alpha_j;A_j''\right)_p\\ {}_1\left(b_j,\beta_j;B_j''\right)_m,\ {}_{m+1}\left(b_j,\beta_j;B_j'\right)_q\end{matrix}\right)=H^{m,\,n}_{p,\,q}\left(z\,\middle|\,\begin{matrix}{}_1\left(a_j,\alpha_j;1\right)_p\\ {}_1\left(b_j,\beta_j;1\right)_q\end{matrix}\right)$$

Corollary 3.4. *Let the condition of Theorem 2.1 be satisfied, we have*

$$\int_0^\infty \Phi^{-\lambda-1}\,{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,H_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]dx$$
$$=\frac{\sqrt{\pi}}{2\rho\,(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu\,r}}H_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\left|\begin{matrix}\left(\frac{1}{2}-\lambda-\mu\,r,\omega;\,1\right),\,{}_1(a_j,\alpha_i;\,1)_p\\ {}_1(b_j,\beta_j;\,1)_q,\,(-\lambda-\mu\,r,\omega;\,1)\end{matrix}\right.\right].\tag{20}$$

Corollary 3.5. *Let the condition of Theorem 2.2 be satisfied, we obtain*

$$\int_0^\infty \frac{1}{x^2}\,\Phi^{-\lambda-1}\,{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,H_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]dx$$
$$=\frac{\sqrt{\pi}}{2\tau\,(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu\,r}}H_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\left|\begin{matrix}\left(\frac{1}{2}-\lambda-\mu\,r,\,\omega;\,1\right),\,{}_1(a_j,\alpha_i;\,1)_p\\ {}_1(b_j,\beta_j;\,1)_q,\,(-\lambda-\mu\,r,\,\omega;\,1)\end{matrix}\right.\right].\tag{21}$$

Corollary 3.6. *Let the condition of Theorem 2.3 be satisfied, we get*

$$\int_0^\infty \left(\rho+\frac{\tau}{x^2}\right)\,\Phi^{-\lambda-1}\,{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,H_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]dx$$
$$=\frac{\sqrt{\pi}}{(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu\,r}}H_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\left|\begin{matrix}\left(\frac{1}{2}-\lambda-\mu\,r,\omega;\,1\right),\,{}_1(a_j,\alpha_i;\,1)_p\\ {}_1(b_j,\beta_j;\,1)_q,\,(-\lambda-\mu\,r,\omega;\,1)\end{matrix}\right.\right].\tag{22}$$

(III). If we set $A''_j = B''_j = A'_j = B'_j = 1$ and $\alpha_j = \beta_j = 1$ in Theorem 2.1 to 2.3, then $\bar{I}$-function reduces in to G- function. (see Luke [6]).

$$\bar{I}_{p,\,q}^{m,\,n}\left(z\left|\begin{matrix}{}_1\left(a_j,\alpha_j;A'_j\right)_n,\;{}_{n+1}\left(a_j,\alpha_j;A''_j\right)_p\\ {}_1\left(b_j,\beta_j;B''_j\right)_m,\;{}_{m+1}\left(b_j,\beta_j;B'_j\right)_q\end{matrix}\right.\right)=G_{p,\,q}^{m,\,n}\left(z\left|\begin{matrix}{}_1(a_j,\,1;\,1)_p\\ {}_1(b_j,1;\,1)_q\end{matrix}\right.\right)$$

Corollary 3.7. *Let the condition of Theorem 2.1 be satisfied, we have*

$$\int_0^\infty \Phi^{-\lambda-1}\,{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,G_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]dx$$
$$=\frac{\sqrt{\pi}}{2\rho\,(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu\,r}}G_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\left|\begin{matrix}\left(\frac{1}{2}-\lambda-\mu\,r,\omega;\,1\right),\,{}_1(a_j,1;\,1)_p\\ {}_1(b_j,1;\,1)_q,\,(-\lambda-\mu\,r,\omega;\,1)\end{matrix}\right.\right].\tag{23}$$

Corollary 3.8. *Let the condition of Theorem 2.2 be satisfied, we obtain*

$$\int_0^\infty \frac{1}{x^2}\,\Phi^{-\lambda-1}\,{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,G_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]dx$$
$$=\frac{\sqrt{\pi}}{2\tau\,(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu\,r}}G_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\left|\begin{matrix}\left(\frac{1}{2}-\lambda-\mu\,r,\omega;\,1\right),\,{}_1(a_j,1;\,1)_p\\ {}_1(b_j,1;\,1)_q,\,(-\lambda-\mu\,r,\omega;\,1)\end{matrix}\right.\right].\tag{24}$$

Corollary 3.9. *Let the condition of Theorem 2.3 be satisfied, we get*

$$\int_0^\infty \left(\rho+\frac{\tau}{x^2}\right)\,\Phi^{-\lambda-1}\,{}_2F_1\left(a,\,b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,{}_2F_1\left(c-a,\,c-b,\,c+\frac{1}{2};\,y\{\Phi\}^{-\mu}\right)\,G_{p,q}^{m,n}\left[z\{\Phi\}^{-\omega}\right]dx$$
$$=\frac{\sqrt{\pi}}{(4\rho\tau+\gamma)^{\lambda+1/2}}\sum_{r=0}^{\infty}\frac{(c)_r}{\left(c+\frac{1}{2}\right)_r}\frac{a_r\,y^r}{(4\rho\tau+\gamma)^{\mu\,r}}G_{p+1,q+1}^{m,n+1}\left[\frac{z}{(4\rho\tau+\gamma)^{\omega}}\left|\begin{matrix}\left(\frac{1}{2}-\lambda-\mu\,r,\omega;\,1\right),\,{}_1(a_j,1;\,1)_p\\ {}_1(b_j,1;\,1)_q,\,(-\lambda-\mu\,r,\omega;\,1)\end{matrix}\right.\right].\tag{25}$$

4. Concluding Remark

We have established three new definite integrals on the product of $\bar{I}$-function with the hypergeometric function. We also derived analogous result in the form of $\bar{H}$-function, H- function and G- function, which have been depicted in corollaries. Further, the results presented in this article are easily converted in terms of a similar type [1, 3, 17, 18] of new interesting integrals with different arguments after some suitable parametric replacements. On account of being general and unified in nature, the results established here yield a large number of known and new results involving simpler functions on suitable specifications of the parameters involved.

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References

[1] P. Agarwal and M. Chand, *New Finite Integrals of Generalized Mellin-Barnes Type of Contour Integrals*, Global Journal of Science Frontier Research Mathematics & Decision Sciences, 12(3)(2012), 32-43.

[2] B. L. J. Braaksma, *Asymptotic expansions and analytic continuations for a class of Barnes integrals*, Composition Mathematica, 15(1964), 239341.

[3] J. Daiya, J. Ram and D. Kumar, *The Multivariable H-Function and the General Class of Srivastava Polynomials Involving the Generalized Mellin-Barnes Contour Integrals*, Filomat, 30(6)(2016), 14571464.

[4] C.Fox, *The G and H -functions as symmetrical Fourier Kernels*, Trans. Amer. Math. Soc., 98(1961), 395-429.

[5] A. A. Inayat-Hussain, *New properties of hypergeometric series derivable from Feynman integrals: II A generalization of the H -function*, J. Phys. A: Math. Gen., 20(1987), 41194128.

[6] Y. L. Luke, *The Special Functions and their Approximations*, Mathematics in Science and Engineering, 53(1969).

[7] A. M. Mathai and R. K. Saxena, *The H -function with applications in Statistics and other Disciplines*, John Wiley, New York, (1978).

[8] M. I. Qureshi, A. Kaleem, Quraishi and Ram Pal, *Some definite integrals of GradshteynRyzhil and other integrals*, Global Journal of Science Frontier Research, 11(4)(2011), 75-80.

[9] A. K. Rathie, *A new generalization of generalized hypergeometric function*, Le-Matematiche, 52(2), 297310.

[10] B. Satyanarayana, L. Prakas and Y. Pragathi Kumar, *Expansion formulas for I-function*, Journal of Progressive Research in Mathematics, 3(2)(2015), 161-166.

[11] V. P. Saxena, *The I-Function*, Anamaya Publishers, New Delhi, (2008).

[12] R. K. Saxena, J. Ram and D. L. Suthar, *Integral formulas for the H-function generalized fractional calculus II*. South East Asian J. Math. Math. Sci., 5(2)(2007), 23-31.

[13] R. K. Saxena, J. Ram and D. L. Suthar, *Unified fractional derivative formulas for the multivariable H-function*, Vijnana Parishad Anusandhan Patrika, 49(2)(2006), 159-175.

[14] R. K. Saxena, J. Ram and D. L. Suthar, *On two-dimensional Saigo-Maeda fractional calculus involving two-dimensional H-transforms*, Acta Cienc. Indica Math., 30(4)(2004), 813-822.

[15] K. Shantha Kumari, T. M. Vasudevan Nambisan and A. K. Rathie, *A study of I-function of two variables*, arXiv: 1212.6717v1 [math.CV], (2012).

[16] L. J. Slater, *Generalized Hypergeometric functions*, Cambridge University Press, (1966).

[17] D. L. Suthar and P. Agarwal, *Generalized Mittag-Leffler function and the multivariable H-function involving the Generalized Mellin-Barnes contour integrals*, Commun. Numer. Anal., 2017(1)(2017), 25-33.

[18] D. L. Suthar, S. D. Purohit and S. Agarwal, *Class of integrals involving generalized hypergeometric function and Srivastavas polynomials*, Int. J. Appl. Comput. Math., 3(1)(2017), 1197-1203.

[19] V. M. Vyas and A. K. Rathie, *A study of I function II*, Vijnana Parishad Anusandhan Patrika, 41(4)(1998), 253-257.