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Dynamical Analysis of a Fractional Order Delayed Prey-Predator Model With Stage Structure

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Abstract: This paper is mainly connected with the investigation of fractional order stage-structured predator-prey system with time delay. By analyzing the corresponding characteristic equations, the local stability of the equilibria is investigated and conditions at which the existence of Hopf bifurcation are derived at positive equilibrium by employing Routh Hurwitz criterion. Both fractional order and time delay are chosen as bifurcation parameters. Further, it is concluded that, if the values of fractional order and time delay exceeds the derived critical value then the solutions of addressed system exhibits oscillatory behavior. Moreover, Lyapunov global stability, complex dynamics of the predator-prey systems are also investigated with or without delay for the incommensurate fractional order. Finally, numerical illustrations are provided to validate the effectiveness of derived analytical results.

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1. Introduction

Fractional calculus is considered as one of old mathematical concepts, it has been always thought of as a pure mathematical problem for nearly three centuries [1, 2, 3]. The investigation of fractional order integral and derivative operators over real or complex domains is known as fractional calculus. In the recent years, the studies on fractional calculus (includes fractional differential equations) have drawn much attentions due to its more precise descriptions for real world problems. In the fields of continuous-time modeling, the fractional calculus plays a vital role in describing linear viscoelasticity, acoustics, rheology, polymeric chemistry, and so forth. Fractional calculus is nonlocal in nature. Hence it lies in the fact that it has a memory. It had been proved to be a very suitable tool for the description of memory and hereditary properties of various materials and processes.

An increasing interest has currently turned towards fractional order differential equations(FODEs). Recently, many researchers, have demonstrated that FODEs are used as an effective tool to describe complex dynamics in the field of physical, biological, and engineering problems. Material and energy cannot be instantaneously transmitted to almost all the natural systems hence the existence of time delays cannot be ignored. Delay differential equation (DDE) is a differential equation in delay in the model will enrich its dynamics and provide exact definition of real life phenomena. In DDE, one has to provide

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history of the system over the delay interval $[-\tau, 0]$ as the initial condition. Dynamical analysis of systems with time delay is more complex due to the non-deterministic polynomial time hard (NP-hard) nature of the stability problem [4]. Fractional order systems describe behavior of real physical systems more truthfully than the differential equations which are involved only integer order derivatives [5, 6]. Stability of linear fractional order systems has been exhaustively studied [7, 8]. The time delay in the fractional order systems causes some difficulties for stability analysis because their characteristics equation involves exponential type transcendental terms as well as non-integer orders.

Fractional delay systems of related types are the bounded-input bounded-output (BIBO) stable if and only if all the roots of the characteristics equation lie in the left of the imaginary axis in the complex plane. For more details sec [9, 13, 15]. By employing a generalized form of Hassard's theorem, an analytical criterion is derived to determine the number of unstable roots of the characteristics equation for each given constant value of the delay [14].

In [10], authors addressed the location of characteristics roots of the scalar system that can be determined by using the Lambert W function. In [11], the authors investigated the delay margin of fractional delay system of retarded type by using the Orlando formula. In [12], the authors proposed a numerical algorithm based on Cauchy's integral theorem to investigate the stability of fractional delay systems with a constant delay. The combination of fractional calculus and delay was successfully applied into many areas of engineering as well as science, especially when one can model the phenomena to describe the complex systems with memory effects. The bifurcations can be analyzed entirely through changes in the local stability properties of equilibrium, periodic orbits as parameters cross through critical thresholds. There are different types of bifurcations which occurs in nature.

In the work, we consider Hopf type bifurcation causes the appearance or the disappearance of a periodic orbit when there is a small change in the stability properties of equilibrium. Hopf bifurcation analysis plays a vital role in modeling the effects of real world situations. Mathematical modeling through differential equations and simulation via computers play a significant role in the study of multi species populations interactions. Lot of works had been already done on these species interaction (predator-prey) based on both ordinary differential equations (ODEs) and delay differential equations 9DDEs. Based on the literature review, there may be few works available on multi-species interactions by considering the fractional differential equations (FDEs) with time delay. FDE models for interactions between species are the more important of the classical applications of mathematics biology.

In [16] have discussed the stability and Hopf bifurcation in a ratio-dependent predator-prey system with stage structure. While the global stability and Hopf bifurcation of a predator-prey model with stage structure and discrete type delayed predator response was investigated in [17]. The authors have analyzed the Hopf bifurcation for a ratio-dependent predator-prey system along with two different types of delay and considered the stage structure for the predator. Improvements of above models will provide a new modeling which describes the significance of fractional differential equations in the interaction of multiple species. When comparing the integer order differential equations, FODEs have ability to provide precise description of the modeled mathematical problems. Predator-prey models are significant in the modeling of multi-species populations interactions and these interactions through integer order models have been studied by many authors. In this work, we oppose a fractional order multi-species interaction model along with time delay. The authors of [17], have studied the global stability and entire Hopf bifurcation by considering the stage structure for predator. Also, the time delay in the predator response is consider in the system the integer order prey-predator model is given as

$$\begin{aligned}
\frac{dx}{dt} &= x(t)\left(r - ax(t) - \frac{a_1 y_2(t)}{1 + mx(t)}\right) \\
\frac{dy_1}{dt} &= \frac{a_2 x(t-\tau)\, y_2(t-\tau)}{1 + mx(t-\tau)} - r_1 y_1(t) - D y_1(t)
\end{aligned}$$

$$\frac{dy_2}{dt} = Dy_1(t) - r_2 y_2(t)$$

In the above integer order system $x(t)$ represents the density of the prey with respect to time t. y_1 and y_2 denote the densities of the immature and mature predator with respect to time respectively. The parameters $a, a_1, a_2, m, r, r_1, r_2$ and D are positive constants in which a is the intra-specific constant of the prey, r is intrinsic growth rate of the prey r_1 and r_2 are the death rates of the immature and the mature predator respectively. The response function of the mature predator is denoted by $\frac{a_1 x}{1+mx}$. The capturing rate of mature predator is represented by. $a_1.\frac{a_1}{a_2}$ is the rate of conversing prey into a new immature predator. The rate of immature predator becomes mature predator is denoted by non-negative parameter D and this rate is proportional to the density of the immature predator. The time delay due to the gestation of mature predator is considered by the constant $\tau \geq 0$. It is assumed that mature adult predator can only contribute to the reproduction of predator biomass. In this work, we investigate a fractional order prey-predator interaction along with time delay is described by

$$\begin{aligned}
\frac{d^{\alpha_1} x}{dt} &= x(t)\left(r - ap_1(t) - \frac{a_1 y_2(t)}{1+mx(t)}\right) \\
\frac{d^{\alpha_2} y_1}{dt} &= \frac{a_2 x(t)\, y_2(t-\tau)}{1+mx(t-\tau)} - r_1 y_1(t) - \beta y_1(t) \qquad (1) \\
\frac{d^{\alpha_3} y_2}{dt} &= \beta y_1(t) - r_2 y_2(t)
\end{aligned}$$

With the following initial conditions $x(0)$, $y_1(0) > 0$ and $y_2(t) = \varphi(t)$, $\varphi \in [-\tau,\ 0)$, $\alpha \in (0,\ 1]$, $\phi(t)$ is a smooth function. The parameter descriptions are same as in the integer order system. The organization of this paper is as follows: in the following section, we represent the basic definitions that could utilized in the analysis of the proposed system (1). In subsection 3.1, we investigate the stability and existence of Hopf bifurcation of fractional order predator prey system by choosing the fractional orders to be commensurate. The Lyapunov global stability of the fractional order system with incommensurate fractional orders is presented in the subsection 3.2. Finally, the numerical simulations are provided in the section 4 validate the derived theoretical predictions. Main results and discussion are presented in the section 5.

2. Basic Tools

In this section, we present three oftenly used definitions of fractional derivatives, that is, Riemann–Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald Letnikov fractional derivative. Among these Caputo fractional derivative definition is most commonly used definition because of its accuracy in solutions of real world problems.

Definition 2.1. *The Riemann-Liouville (R-L) fractional integral operator of order $\alpha > 0$, of function $f \in (R_+)$ is defined as*

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-a)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau$$

where $\Gamma(.)$ is the Euler gamma function.

Definition 2.2. *An equilibrium point is a saddle point at which the linearized fractional model has atleast one eigenvalue in the stable region and one in the unstable region.*

Definition 2.3. *A saddle point is called a saddle point of index 1 if one of the eigenvalues is unstable and the others are stable. A saddle point of index 2 is a saddle point with one stable eigenvalue and two unstable ones.*

Now, we provide the stability theorem on FODEs and their results.

Theorem 2.4. *Let the following autonomous system*

$$\frac{d^{\alpha}x}{dt^{\alpha}} = Ax, \quad x(0) = x_0$$

With $0 < \alpha \leq 1$, $x \in R^n$ *and* $A \in R^{nxn}$ *is asymptotically, stable if and only if* $|\arg(\lambda)| > \frac{\alpha\pi}{2}$ *is satisfied or all eigenvalues of matrix* A*. Also, this system is stable if and only* $|\arg(\lambda)| > \alpha\pi$ *is satisfied for all eigenvalues of matrix* A *with these critical eigenvalues satisfying* $|\arg(\lambda)| \geq \alpha\pi$ *having geometric multiplicity of one. The geometric multiplicity of an eigenvalue* λ *of the matrix* A *is the dimension of the subspace of vectors* v *for which* $Av = \lambda v$*.*

3. Dynamics in a Fractional Order Predator-Prey System

3.1. Stability Analysis

In this section we proceed with stability analysis of commensurate fractional order system $\alpha = \alpha_1 = \alpha_2 = \alpha_3$.

Stability of the Equilibrium Points

In this section, we analyzed the local stability of each of feasible equilibria of the system (1) and the existence of Hopf bifurcation at the coexistence equilibrium. Equating the derivatives to the zeros and solving the system (1), one can get three distinct types of equilibria.

- $E_0\,(0,\ 0,\ 0)$ represents the trivial equilibrium of the system.
- $E_1\left(\frac{r}{a},\ 0,\ 0\right)$ the predator extinction equilibrium.
- Further, if the condition, (H_1) $a_2 r\beta - r_2\,(a + mr)\,(\beta + r) > 0$, holds then the system (1) has a unique coexistence equilibrium $E_2\,(\overline{x},\ \overline{y_1},\ \overline{y_2})$ where

$$\begin{aligned}
\overline{x} &= \frac{r_2\,(\beta + r_1)}{\alpha_2\beta - mr_2\,(\beta + r_1)} \\
\overline{y_1} &= \frac{r_2}{\beta}\overline{y_2} \\
\overline{y_2} &= \frac{a_2\beta[a_2 r\beta - r_2\,(a + mr)\,(\beta + r_1)]}{a_1[a_2\beta - mr_2\,(\beta + r_1)]^2}
\end{aligned}$$

The local stability analysis of the system (1) can be done based upon the standard linearization technique and using Jacobian matrix of the system (1).

$$\begin{vmatrix}
\lambda^{\alpha} - \left(\left[-ax + \frac{a_1 y_2 xm}{(1+mx)^2}\right] + r - ax - \frac{a_1 y_2}{(1+mx)}\right) & 0 & \frac{a_1 x e^{-\lambda\tau}}{(1+mx)} \\
-\frac{a_2 y_2}{(1+mx)^2} & \lambda^{\alpha} + r_1 + \beta & \frac{a_2 x e^{-\lambda\tau}}{(1+mx)} \\
0 & -\beta & \lambda^{\alpha} + r_2
\end{vmatrix} = 0$$

The characteristic polynomial of the system (1) at trivial equilibrium is

$$P\,(\lambda^{\alpha}) = (\lambda^{\alpha} + r_2)\,(\lambda^{\alpha} - r)\,(\lambda^{\alpha} + \beta + r) = 0$$

Hence, from the definition 4, trivial equilibrium point is saddle point of index 1 because it contains one positive root $\lambda\alpha = r$ which is unstable. Let the characteristic polynomial of the system at predator extinction equilibrium is of the form

$$P\,(\lambda^{\alpha}) = (\lambda^{\alpha} + r)\left(\lambda^{2\alpha} + \lambda^{\alpha}(r_1 + \beta + r_2) + r_2(r_1 + \beta) - \frac{a_2 r\beta}{a + mr}\right) = 0$$

It is obvious that the characteristic polynomial (2) has one negative real root $\lambda^{\alpha} = -r$. And the remaining roots are determined by the following equation

$$\lambda^{2\alpha} + \lambda^{\alpha} P_1 + P_0 = 0 \tag{2}$$

where $P_1 = r_1 + \beta + r_2$, $P_0 = r_2(r_1+\beta) - \frac{a_2 r\beta}{a+mr}$. Let $f(\lambda^{\alpha}) = \lambda^{2\alpha} + \lambda^{\alpha} P_1 + P_0$. If the condition (H_1) holds, it is clear that, for λ_{α} real,

$$f(0) = \frac{-a_2 r\beta - r_2 (a+mr)(r_1+\beta)}{(a+mr)} < 0$$
$$\lim_{n\to\infty} f(\lambda^{\alpha}) = +\infty,$$

Hence, $f(\lambda^{\alpha}) = 0$ has atleast one positive real root. Therefore, if (H_1) holds the equilibrium is E_1 unstable. If $a_2 r\beta < r_2 (a+mr)(r_1+\beta)$, then from equation (2), we arrive that E_1 is locally asymptotically stable.

Proposition 3.1. *Consider the following three dimensional commensurate fractional order system $D^{\alpha}u = f(u,\mu^*)$ where $\alpha \in (0,1)$, $u \in R^3$ and let u^* is an equilibrium point of the above system, then its characteristic polynomial is given as*

$$P(\lambda) = \lambda^{3\alpha} + P_1\lambda^{2\alpha} + P_2\lambda^{\alpha} + P_3 = 0$$

And its discriminant is defined as

$$K(P) = 18p_1p_2p_3 + (p_1p_2)^2 - 4p_3(p_1)^3 - 4(p_2)^3 - 27(p_3)^3$$
$$P_1{}^2 - 2P_0 = (r_1+\beta)^2 + r^2 > 0 \quad and \quad P_0{}^2 > 0.$$

We know that if $a_2 r\beta < r_2(a+mr)(\beta+r_1)$, then E_1 is globally asymptotically stable. The characteristic polynomial of systems (1) at the coexistence equilibrium E_2 is of the form

$$\lambda^{3\alpha} + p_2\lambda^{2\alpha} + p_1\lambda^{\alpha} + p_0 = 0 \tag{3}$$

where

$$p_2 = \frac{r_1 + r_2 + \beta - r + 2a\overline{x} + a\overline{y_2}}{(1+m\overline{x})^2}$$
$$p_1 = (r_1 + r_2 + \beta)\frac{(2a\overline{x} - r) + a_1\overline{y_2}}{(1+m\overline{x})^2} + (r_1+\beta)r_2 - \frac{a_2\beta\overline{x}}{1+m\overline{x}}$$
$$p_0 = \frac{(r_1+\beta)r_2(2a\overline{x} - r) + ((r_1+\beta)r_2 + a_1\overline{y_2})}{(1+m\overline{x})^2} - \frac{a_2\beta\overline{x}(2a\overline{x} - r)}{1+m\overline{x}}$$

Theorem 3.2. *The equilibrium E_2 of the fractional order predator prey system (1) is locally asymptotically stable if and only if $\min_{1\le i\le 3} |\arg(\lambda_i)| > \frac{\alpha\pi}{2}$.*

Remark 3.3. *Based on Theorem 3.4., the equilibrium E of fractional order system (1) unstable, if the following condition holds*

$$\min_{1\le i\le 3} |\arg(\lambda_i)| > \frac{\alpha\pi}{2}$$

Hence, it is clear that min $|\arg(\lambda_i)|$, $i = 1,2,3$ depends on the roots of the characteristics polynomial (3). Thus the local stability of the equilibrium E depends on the characteristic polynomial (3) with the discriminate $K(P)$ and condition of fractional order Routh-Hurwitz criterion are defined as above.

3.2. Condition for Hopf Bifurication in Fractional Order System

In the section, we present the general condition to be satisfied for the existence of Hopf bifurcation in the fractional-order predator-prey interaction. Consider the following three dimensional fractional-order predator prey system

$$D^{\alpha}u = f(u, \mu^{*})$$

where $\alpha \in (0,1)$, $u \in R^3$ the critical value of the bifurcation parameter μ is $\mu-$ and E_2 is an equilibrium of the system. In integer order case (when $\alpha = 1$), the stability of E_2 is related to the sign of $\mathrm{Re}\,(\lambda_i)$, $i = 1, 2, 3\ldots$ where λ_i are the eigenvalues of the Jacobian matrix $\frac{\partial f}{\partial u}|E_2$ if $\mathrm{Re}\,(\lambda_i) < 0$, $i = 1, 2, 3\ldots$, then E_2 is locally asymptotically stable. If there exists i such that $\mathrm{Re}\,(\lambda_i) > 0$, then E_2 is unstable. The conditions to be satisfied by the system (1) to undergo a Hopf bifurcation at $\mu = \mu^{*}$ and $\alpha = 1$ are given as the Jacobian matrix has two complex-conjugate eigenvalues $\lambda^{\alpha_{12}} = \theta\,(\mu) + i\omega\,(\mu)$ one real $\lambda^{\alpha_{12}}\,(\mu)$ that is $K(P_E\,(\mu^{*}))$.

- $\theta\,(\mu^{*}) = 0$ and $\lambda^{\alpha_3}\,(\mu^{*}) = 0$
- $\omega\,(\mu^{*}) = 0$

But in the fractional case, the stability of E_2 is related to the sign of $F\,(\alpha) = \frac{\alpha\pi}{2} - \min\limits_{1\leq i\leq 3} |\arg(\lambda_i)|$. If $F_i\theta\,(\alpha, \mu) < 0$ for all $i = 1, 2, 3, \ldots$ then E_2 is locally asymptotically stable. If there exist such that $F_i\theta\,(\alpha, \mu) > 0$ then E_2 is unstable. So the function $F_i(\alpha, \mu)$ has a similar effect as the real part of eigenvalue in integer systems. Therefore, we extend the Hopf bifurcation conditions to the fractional order systems by replacing $Re(\lambda_i)$ instead of $F_i\theta\,(\alpha, \mu)$ as given below,

- $K(P_E\,(\mu^{*})) < 0$
- $-F_{1,2}\,(\alpha, \mu) = 0$ and $\lambda^{\alpha_3}\,(\mu^{*}) = 0$
- $\frac{\partial f}{\partial u}|\mu = \mu^{*} = 0$

Hopf bifurcation analysis verses fractional order α

Based on the above discussions, it is found clear that the fractional order α has an effect on stability of fractional order system. Hence, the perturbing parameter α can be chosen as a bifurcation parameter. In this section, we will analyse the existence of Hopf Bifurcation by fixing the fractional order α as the bifurcation parameter. Let us define the function with respect to a.

$$F\,(\alpha) = \frac{\alpha\pi}{2} \min_{1\leq i\leq 3} |\arg(\lambda_i)| \tag{4}$$

From Theorem 3.1 and the discussions above, if $F\,(\alpha) < 0$, then the equilibrium point is locally asymptotically stable, otherwise unstable. Now, the function $F\,(a)$ is used to analyse the existence of Hopf bifurcation in the fractional order in the non-delayed predator-prey system (1) verses the fractional order α.

Hopf bifurcation analysis verses the parameter

The characteristic polynomial of delayed fractional order predator-prey system (1) at the co-existence equilibrium E_2 is of the form

$$\lambda^{3\alpha} + p_2\lambda^{2\alpha} + p_1\lambda^{\alpha} + p_0 + (q_1\lambda^{\alpha} + q_0)\,e^{-\lambda\tau} = 0 \tag{5}$$

where

$$p_2 = \frac{r_1 + r_2 + \beta - r + 2a\overline{x} + a\overline{y_2}}{(1 + m\overline{x})^2},$$

$$p_1 = \frac{(2a\overline{x} - r)(r_1 + r_2 + \beta) + (r_1 + \beta)r_2 + a_1\overline{y_2(r_1 + r_2 + \beta)}}{(1 + m\overline{x})^2},$$
$$p_0 = \frac{(r_1 + \beta)r_2(2a\overline{x} - r) + ((r_1 + \beta)r_2 + a_1\overline{y_2})}{(1 + m\overline{x})^2},$$
$$q_1 = -\frac{a_2\beta\overline{x}}{1 + m\overline{x}} \quad and$$
$$q_0 = -\frac{a_2\beta\overline{x}(2a\overline{x} - r)}{1 + m\overline{x}}$$

The equilibrium E_2 of the fractional order predator-prey system (1) at $\tau = 0$ is locally asymptotically stable is already proved in the Section 4.1.

Suppose $\lambda^\alpha = i\omega\ \omega > 0$ is a solution of (5). Separating real and imaginary parts by substituting $\lambda^\alpha = i\omega$ we have

$$\begin{aligned} -\omega^{3\alpha} + p_1\omega^\alpha &= q_0 \sin\omega\tau - q_1\omega^\alpha \cos\omega\tau, \\ p_2\omega^{2\alpha} - p_0 &= q_0 \cos\omega\tau - q_1\omega^\alpha \sin\omega\tau. \end{aligned} \tag{6}$$

Squaring and adding the two equations of the (6), it follows that

$$\omega^{6\alpha} + (p_2 - 2p_1)\omega^{4\alpha} + (p_2 - 2p_0p_2 - q_2)\omega^{2\alpha} - p_2 - q_2 = 0. \tag{7}$$

Hence, if $p_0 > q_0$, (7) has no positive real roots. If $p_0 < q_0$, then (7) has a unique positive root denoted by ω_0 and the characteristic polynomial (6) has a pair of conjugate roots for $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$. From (6), we have

$$\tau^* = \frac{1}{\omega_0} arccos(((\frac{(\omega_0 - p_2\omega_0)q_2\omega_0 + q_0(q_2p_2\omega_0 - p_0)}{q^2\omega^2\alpha} + 2j\alpha\pi, \quad where \ \ j = 1, 2, 3 \dots. \tag{8}$$

Nothing that the condition (i) of Theorem 3.2 holds, E_2 is locally stable when $\tau = 0$ according to the general theory on delay differential, E_2 will remains stable for $\tau < \tau^*$. Also, E_2 will undergoes a Hopf bifurcation when $\tau = \tau^*$ and becomes unstable when $\tau > \tau^*$. Finally, we summarize the above findings as

Theorem 3.4. *For system (1) we have the following:*

(1). The equilibrium $E_0\,(0,\ 0,\ 0)$ is always unstable.

(2). if $a_2r\beta - r_2\,(a + mr)\,(\beta + r) < 0$ then the predator extinction equilibrium $E_1\left(\frac{r}{a},\ 0,\ 0\right)$ is locally asymptotically stable, if $a_2r\beta - r_2\,(a + mr)\,(\beta + r) > 0$ then E_1 is unstable.

(3). Let (H_1) and condition (i) of Theorem 3.2 hold. If $p_0 - q_0 < 0$ then there exists a positive roots and critical value τ^* such that E_2 is locally asymptotically stable if $0 < \tau < \tau^*$ and is unstable if $\tau > \tau^*$. Further, system undergoes a Hopf bifurcation for a fixed α at E_2 when $\tau = \tau^*$.

4. Numerical Simulation

In this section, we provide the stability and existence of Hopf bifurcation for commensurate fractional order predator prey systems through Section 4.1 and Section 4.2 the Lyapunov global stability of incommensurate fractional order system is provided in the Section 4.3. The parameter values chosen for the numerical simulations are $a = 16$, $a_1 = 5$, $a_2 = 3$, $m = 0.1$, $\beta = 1$, $r = \frac{1}{8}$, $r_1 = \frac{1}{8}$, $r_2 = \frac{1}{8}$ and the initial conditions of the populations are $x(0) = 0.2$, $y_1\,(0) = 0.2$, $y_2\,(0) = 0.2$.

Equilibrium	Eigenvalues	Discriminant	Nature	Index
$E_0\,(0,\ 0,\ 0)$	8, -0.125, -1.125	5496.8	Saddle	1
$E_1\,(0.812,\ 0,\ 0)$	-8, 0.6706, -1.92	18656	Saddle	1
$E_2\,(0.047,\ 0.18,\ 1.45)$	$-1.78, -0.092 \pm 0.748i$	-26.202	A. Saddle	-

Table 1. **Stable nature for $r = 8$ and commensurate fractional order $\alpha = 1$**

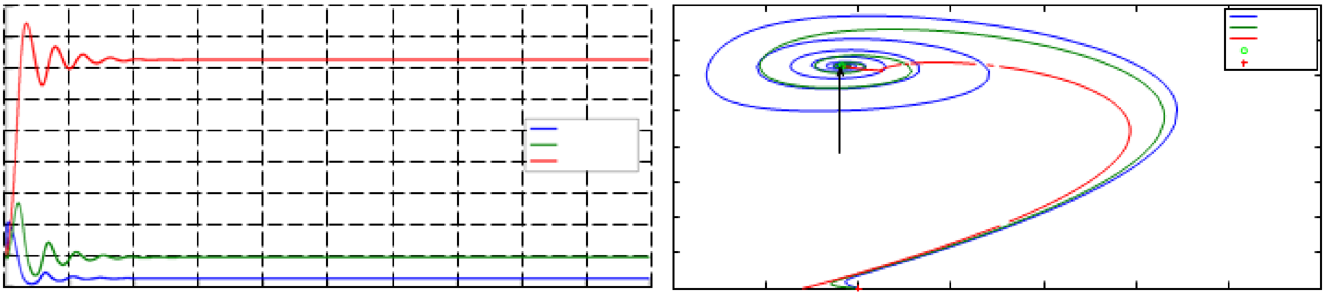

Figure 1. **Solutions of the system (1) converges to the equilibrium E_2.**

Hence, the equilibrium E_2 is asymptotically stable for $\tau = 0$.

4.1. Commensurate fractional order $(\alpha_1 = \alpha_2 = \alpha_3 = \alpha)$ and $\tau = 0$

In this subsection, we analyse the non-delayed predator-prey system by considering the commensurate fractional order $(\alpha_1 = \alpha_2 = \alpha_3 = \alpha)$ and varying the value of the growth rate r of prey. The equilibrium points of the system (1) and the eigenvalues of the corresponding Jacobian matrix are give in Table 1 and Table 2. From Table 1, it can seen that E_0, E_1 are saddle points of index 1 and E_2 is asymptotically stable for $r = 8$. The theoretical results are proved through pictorial representation of Figure 1. If the growth rate of prey is increased that $r = 15$ then the corresponding equilibrium E_0, E_1, E_2 and their respective eigenvalues are given in Table 2. It can be seen that E_0, E_1 are saddle points of index 1 and E_2 is saddle point of index 2. For the equilibrium point E_2 of Table 2, we have the $\omega_0 = 0.6828$, and two pair of complex conjugates 0.2733.

Equilibrium	Eigenvalues	Discriminant	Nature	Index
$E_0(0, 0, 0)$	15, -0.125, -1.125	59483	Saddle	1
$E_1(0.812, 0, 0)$	-15, 1.0547, -2.304	468830	Saddle	1
$E_2(0.047, 0.18, 1.45)$	$-2.005, -0.034 \pm 0.996i$	-105.39	Saddle	2

Table 2.

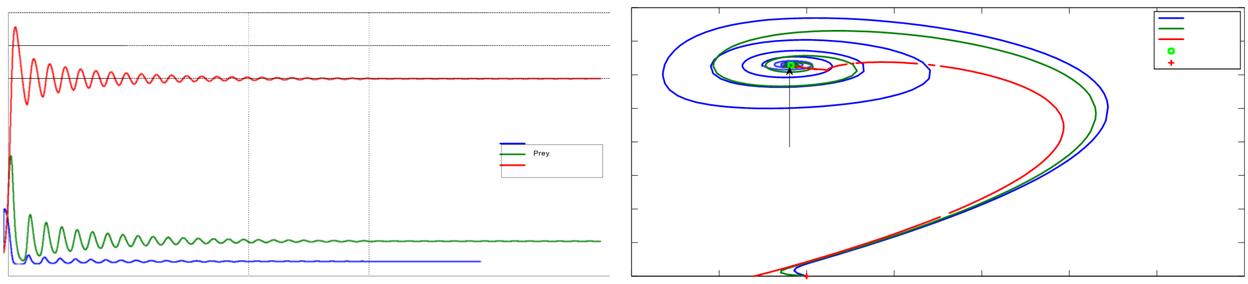

Figure 2. Equilibrium E_2 is asymptotically stable when $\alpha = 0.96$, $\tau = 0$ and growth rate $r = 15$. If the commensurate fractional order is increased that is $\alpha = 0.98$ then solutions of the system (1) shows oscillatory.

Hence it is clear that for the given parameter values and the fixed r the system (1) will converge to a fixed point for $\alpha < \overline{\alpha}$ and the system shows oscillatory behaviour when $\alpha > \overline{\alpha}$ which is shown in the Figure 2. For a given set of parameter values the stability of the commensurate fractional order system can be perturbed by α.

4.2. Dynamics for different τ and fixed commensurate α

In this section, we analyse the effect of time delay in the stability of the delayed predator prey system (1) by choosing that α to be fixed. In numerical simulations, the parameters and the initial values are considered as $a = 16$, $a_1 = 5$, $a_2 = 3$, $m = 0.1$, $r = 8$, $\beta = 1$, $r_1 = \frac{1}{8}$, $r_2 = \frac{1}{8}$ and $x(0) = 0.2$, $y_1(0) = 0.2$, $y_2(0) = 0.2$. For $\alpha = 0.98$, the critical value of time delay is calculated as $\tau^* = 0.4882$ with unique positive root $0.2733 \pm 1.1187i$, $-0.3444 \pm 1.0989i$ with one satisfying the condition for existence of Hopf bifurcation that is $0.2733 > 0$. If the value of time delay τ exceeds the critical value τ^* then the system undergoes Hopf bifurcation at $\tau = \tau^*$ which is shown in the Figure 3 and 4. Figure 5 shows that decrease in the fractional order derivative $\alpha = 0.96$ will increase the value of critical time delay $\tau^* = 1.1$.

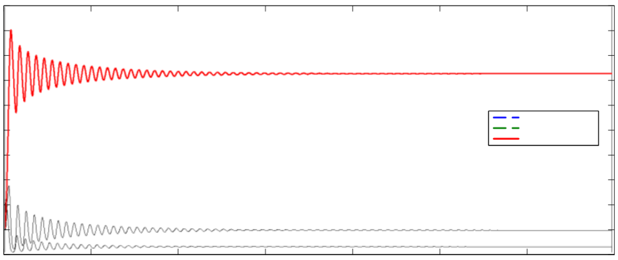
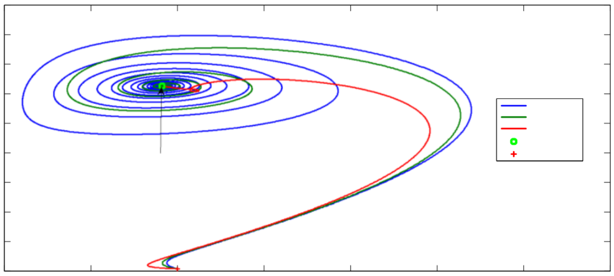

Figure 3. **Equilibrium E_2 is asymptotically stable for fixed τ and various α.**

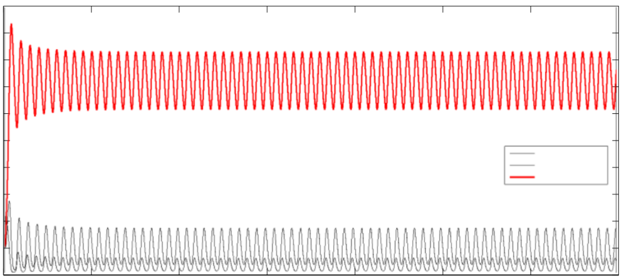
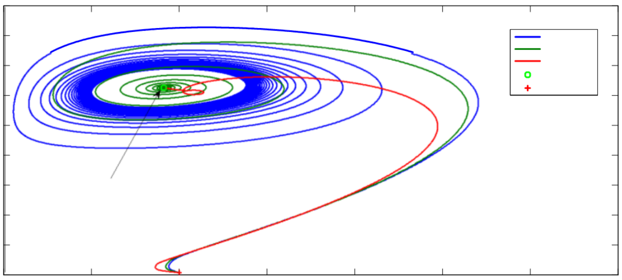

Figure 4. Solutions of the system (1) undergoes Hopf bifurcation when the value of time delay exceeds its critical value that is, $\tau = 0.7 > 0.4882(\tau^*)$ and become stable for decreasing the fractional order $\alpha = 0.95, 0.8$.

4.3. Dynamics for Incommensurate Fractional Order with Fixed τ

In this section, we do not provide explicit expression for critical magnitude of τ for stability but we provide information about the existence of such value. Figure 7 depicts the stability and periodic solutions of incommensurate fractional order by considering the time delay to be constant.

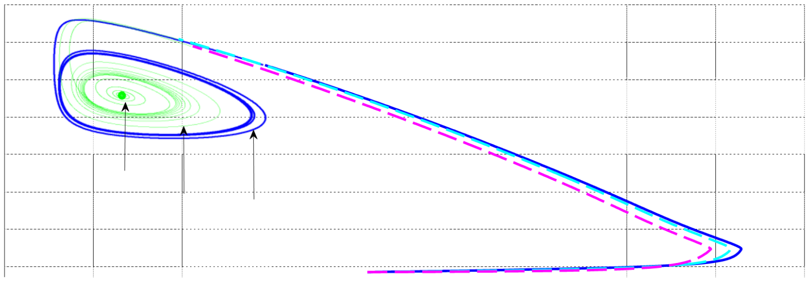

Figure 5. Solution of the incommensurate fractional order system is globally asymptotically stable for $\alpha_1 = 0.6$, and becomes unstable for $\alpha_2 = \alpha_3 = 1$. Also the solutions will exhibit oscillatory behavior when $\alpha_1 = 0.94$, $\alpha_2 = \alpha_3 = 1$.

5. Results and Discussions

In this work, we have developed a theoretical framework that includes sufficient biological; complexity to accurately describe the dynamics of multi-species interaction. The complex dynamics of a proposed fractional order stage structured predator-prey system with and without time delay have been investigated in detail via numerical simulations. We studied the stability of trivial equilibrium predator extinction equilibrium, co-existence equilibrium through the roots of the characteristics polynomial and fractional order Routh-Hurwitz criterion. The significance of incorporating the delay into the system has been explored, and it has been shown that an appropriate time delay will destabilize the system when it exceeds its derived critical value. In the present work we discussed the fractional order predator-prey system with stage structure for the predator and time delay. This work may be extended to further investigation by considering the stage structure for both prey and predator. The Crowley-Martin functional response will be considered in the interaction of predator-prey species along with multiple time delays.

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