

Serendipity Fixed Point Using Dual Quasi Contraction

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Abstract: The concept of Serendipity fixed point was initiated by Powar [1]. Further, Powar [2] established the result of existence of Serendipity Fixed Point using dual F-Contraction. Recently, Poom Kuman [3] extended the result of Ciric and established uniqueness of Fixed Point. Considering the lighter concept of completeness, in the present paper, authors have established the existence and uniqueness of Serendipity Fixed Point under dual quasi and dual generalized quasi contraction condition.

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1. Introduction

The idea of weak complete metric space involving the weak convergence of weak Cauchy sequence has been described in [4]. In 2013 [1] Powar and Sahu applied it on the Fixed Point theory and defined Serendipity Fixed Point. They also established the result of existence and uniqueness of Serendipity Fixed Point. Using the concept of dual F-Contraction, the existence and uniqueness of Serendipity Fixed Point has been established in [2]. In the present paper, the idea of T-Orbitally complete space initiated. By Poom in [3] has been extended over the weak complete space (viz. weak T-Orbitally complete space). The existence and uniqueness of Serendipity Fixed Point has been also established in weak T-Orbitally complete space. In 2015 [3], Poom Kuman et al. defined generalized quasi contraction and proved existence and uniqueness of Fixed Point. In this paper, considering the lighter concept of completeness, viz. weak T-Orbitally complete, established the result of uniqueness of Serendipity Fixed Point in weak T-Orbitally complete space.

2. Preliminaries

In order to establish our results, we require the following definitions.

Definition 2.1 ([4]). Let $\{x_n\}$ be a sequence in a normed linear space X . $\{x_n\}$ is said to converge weakly to x in X if for every linear functional $f \in X^*$ (dual space of X)

$$f(x_n) \rightarrow f(x), \text{ as } n \rightarrow \infty$$

In this case, we write $x_n \xrightarrow{w} x$ and x is called the weak limit of the sequence $\{x_n\}$.

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Example 2.2. Consider the Hilbert space $L_2 [0, 2\pi]$ which is the space of the square-integrable functions on the interval $[0, 2\pi]$. The sequence of functions r_1, r_2, r_3, \dots defined by $r_n(x) = \cos(nx)$, where $x \in [0, 2\pi]$. Let $f : L_2 [0, 2\pi] \rightarrow R$ defined by $f(r(x)) = \int_0^{2\pi} x^2 \cdot r(x) dx$. Then $f(r_n(x)) = \int_0^{2\pi} \cos nx \cdot x^2 dx$ tends to zero when n tends to infinity. However, $\cos(nx)$ does not converge, as $n \rightarrow \infty$.

Definition 2.3 ([4]). A Fixed Point of a mapping $T : X \rightarrow X$ of a set X into itself is a point $x \in X$ which is mapped onto itself, i.e. $Tx = x$.

Example 2.4. Let $T : R \rightarrow R, T(x) = x^2$. Clearly T has two fixed points 0 and 1.

Definition 2.5 ([2]). A Serendipity Fixed Point of a mapping $T : X \rightarrow X$ of a set X into itself is a point $x \in X$ such that there exists real or complex valued function f on X satisfying the condition $f(Tx) = f(x)$.

Example 2.6. Let $T : R \rightarrow R, T(x) = x + 2$ and $f : R \rightarrow R, f(x) = x^2$. Then T has one Serendipity Fixed Point $x = -1$ and it is clear that T has no fixed point.

Definition 2.7 ([3]). Let $T : X \rightarrow X$ be a map on metric space (X, d) . For each $x \in X$ and for any positive integer n , denote $O_T(x, n) = \{x, Tx, \dots, T^n x\}$ and $O_T(x, +\infty) = \{x, Tx, \dots, T^n x, \dots\}$. The set $O_T(x, +\infty)$ is called the orbit of T at x and the metric space X is called T -Orbitally complete if every Cauchy sequence in $O_T(x, +\infty)$ is convergent in X .

Example 2.8. Let (R, d) be metric space with respect to usual metric d and $T : R \rightarrow R$ defined by $T(x) = \frac{x}{4}$. Then orbit of T is $O_T(x, +\infty) = \{x, \frac{x}{4}, \dots, \frac{x}{4^n}, \dots\}$ and it may be verified easily that R is T -Orbitally complete.

Referring the Definition of 2.7, we introduce the dual orbit of T due to Poom Kuman.

Definition 2.9. Let $T : X \rightarrow X$ be a map on normed linear space X and a real or complex valued function f defined on X . For each $x \in X$ and for any positive integer n , denote

$$O_{fT}(x, n) = \{fx, fTx, \dots, fT^n x\} \text{ and } O_{fT}(x, +\infty) = \{fx, fTx, \dots, fT^n x, \dots\}.$$

The set $O_{fT}(x, +\infty)$ is called the dual orbit of T at x and the normed space X is called weakly T -Orbitally complete if every weakly Cauchy sequence in $O_{fT}(x, +\infty)$ is weakly convergent in X .

Example 2.10. Let $T : X \rightarrow X, T(x) = x^2$ and $f : X \rightarrow R, f(x) = \frac{x}{3} + \frac{1}{2}$ where $X = (0, 1)$. Then orbit of T is $O_T(x, +\infty) = \{x, x^2, \dots, x^{2^n}, \dots\}$. It is clear that all Cauchy sequence of orbit converges to 0 but $0 \notin X$, so T is not T -Orbitally complete. But dual orbit of T is $O_{fT}(x, +\infty) = \left\{ \frac{x}{3} + \frac{1}{2}, \frac{x^2}{3} + \frac{1}{2}, \frac{x^4}{3} + \frac{1}{2}, \dots, \frac{x^{2^n}}{3} + \frac{1}{2}, \dots \right\}$ and it may be verified easily that R is weakly T -Orbitally complete.

Definition 2.11 ([5]). Let $T : X \rightarrow X$ be a mapping on metric space (X, d) . The mapping T is said to be a quasi-contraction if there exists $q \in [0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq q \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\}.$$

Example 2.12. Let (X, d) be the metric space with respect to usual metric d , where $X = [0, \infty)$ and $T : X \rightarrow X, T(x) = \frac{x}{3}$. Then T satisfies the quasi-contraction condition.

Now, we introduce dual quasi-contraction and dual generalized quasi-contraction by referring the Definition 2.6. and Definition 2.8 respectively.

Definition 2.13. Let X be normed linear space and T be a selfmap on X . Let ‘ d ’ be the metric $d : R \times R \rightarrow R$. The mapping T is said to be dual quasi-contraction if there exists $q \in [0, 1)$ such that for all x, y in X and $f \in X^*$.

$$d(fTx, fTy) \leq q \cdot \max\{d(fx, fy), d(fx, fTx), d(fy, fTy), d(fy, fTx), d(fx, fTy)\}.$$

Example 2.14. Let $T : [0, \infty) \rightarrow [0, \infty)$ defined by $T(x) = 2x$, $f : [0, \infty) \rightarrow R$ defined by $f(x) = \frac{x}{6}$ and d be the metric defined $d(x, y) = |x - y|$, $\forall x, y \in R$. It is clear that T satisfies dual generalized quasi-contraction condition.

Definition 2.15 ([3]). Let $T : X \rightarrow X$ be a mapping on metric space (X, d) . The mapping T is said to be a generalized quasi-contraction if there exists $q \in [0, 1)$ such that for all $x, y \in X$,

$$d(Tx, Ty) \leq q \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), (y, Tx), d(x, Ty), d(T^2x, x), (T^2x, Tx), d(T^2x, y), d(T^2x, Ty)\}$$

Example 2.16. Let (X, d) be the metric space with respect to usual metric d , where $X = [0, \infty)$ and $T : X \rightarrow X$ defined by $T(x) = \frac{x}{3}$. Then T satisfies generalized quasi-contraction condition.

Definition 2.17. Let X be normed linear space and T be a selfmap on X . Let d' be the metric $d : R \times R \rightarrow R$. The mapping T is said to be dual generalized quasi-contraction if there exists $q \in [0, 1)$ such that for all x, y in X and $\in X^*$,

$$d(fTx, fTy) \leq q \cdot \max\{d(fx, fy), d(fx, fTx), d(fy, fTy), d(fy, fTx), d(fx, fTy), d(fT^2x, fx), d(fT^2x, fTx), d(fT^2x, fy), d(fT^2x, fTy)\}.$$

Example 2.18. Let $T : [0, \infty) \rightarrow [0, \infty)$ defined by $T(x) = 3x$, $f : [0, \infty) \rightarrow R$ defined by $f(x) = \frac{x}{9}$ and d be the usual metric defined by $d(x, y) = |x - y|$, $\forall x, y \in R$. It is clear that T satisfies dual generalized quasi-contraction condition.

3. Main Result

In this section, we state two main results of this paper. Considering the concept of dual quasi contraction map, in the following result, we have established the existence and uniqueness of Serendipity Fixed Point in the following result.

Theorem 3.1. Let X be a normed linear and weak T -Orbitaly complete space. Consider a metric $d : R \times R \rightarrow R$ and a one-to-one onto function $f \in X^*$ (dual of X). If f satisfies the condition of dual quasi contraction (c.f. Definition 2.7), then T has a unique Serendipity Fixed Point.

Proof. Consider a self map T defined on X and $f \in X^*$. Applying the condition of dual quasi contraction, for each $x \in X$ and $1 \leq i \leq n$ and $1 \leq j \leq n$, we have

$$\begin{aligned} d(fT^i(x), fT^j(x)) &= d(fT(T^{i-1}x), fT(T^{j-1}x)) \\ &\leq q \cdot \max\{d(f(T^{i-1}x), f(T^{j-1}x)), d(f(T^{i-1}x), fT(T^{i-1}x)), \\ &\quad \cdot d(f(T^{i-1}x), fT(T^{j-1}x)), d(f(T^{j-1}x), fT(T^{j-1}x)), d(f(T^{j-1}x), fT(T^{i-1}x))\} \\ &\leq q \cdot \delta[O_{fT}(x, n)] \end{aligned} \tag{1}$$

Where $\delta[O_{fT}(x, n)] = \max\{d(f(T^i x), f(T^j x)) : 0 \leq i, j \leq n\}$. Since $0 \leq q < 1$, $\exists k_n(x) \leq n$ such that

$$d(f(x), f(T^{k_n(x)}x)) = \delta[O_{fT}(x, n)] \tag{2}$$

Now, for any non-negative integer $k_n(x)$, we have

$$\begin{aligned}
 d(f(x), f(T^{k_n(x)}x)) &\leq d(f(x), f(Tx)) + d(f(Tx), f(T^{k_n(x)}x)) \\
 &\leq d(f(x), f(Tx)) + q.\delta[O_{fT}(x, n)] \quad (\text{by using equation (1)}) \\
 &\leq d(f(x), f(Tx)) + q.d(f(x), f(T^{k_n(x)}x)) \quad (\text{by using equation (2)}) \\
 \delta[O_{fT}(x, n)] = d(f(x), f(T^{k_n(x)}x)) &\leq \frac{1}{1-q}d(f(x), f(Tx)) \tag{3}
 \end{aligned}$$

For all $n, m \geq 1$ and $n < m$, consider

$$\begin{aligned}
 d(fT^n(x), fT^m(x)) &= d(fT(T^{n-1}x), fT^{m-n+1}(T^{n-1}x)) \\
 &\leq q.\delta[O_{fT}(T^{n-1}x, m-n+1)] \quad (\text{In view of equation (1)}) \\
 &\leq q.d(f(T^{n-1}x), fT^{k_{m-n+1}(T^{n-1}x)}(T^{n-1}x)) \quad (\text{cf. equation (2)}) \\
 &\leq q.d(fT(T^{n-2}x), fT^{k_{m-n+1}(T^{n-1}x)+1}(T^{n-2}x)) \\
 &\leq q^2.d[O_{fT}(T^{n-2}x, m-n+2)] \quad (\text{Appeal to equation (1)}) \\
 &\leq \dots
 \end{aligned}$$

The process is continued n-times and finally, we get

$$\begin{aligned}
 d(fT^n(x), fT^m(x)) &\leq q^n.\delta[O_{fT}(x, m)] \\
 d(fT^n(x), fT^m(x)) &\leq \frac{q^n}{1-q}d(f(x), f(Tx)) \quad (\text{by using equation (3)})
 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} q^n = 0 \Rightarrow d(fT^n(x), fT^m(x)) = 0$. Hence, the sequence $\{T^n x\}$ is weakly Cauchy in X. Since X is weak T-Orbitally complete, $\exists f(x) \in R$ such that $\lim_{n \rightarrow \infty} f(T^n x) = f(x)$. By using dual quasi contraction condition of T (cf. Definition 2.7), we have

$$\begin{aligned}
 d(f(x), f(Tx)) &\leq d(f(x), f(T^{n+1}x)) + d(f(T^{n+1}x), f(Tx)) \\
 &\leq d(f(x), f(T^{n+1}x)) + d(fT(T^n x), f(Tx)) \\
 &\leq d(f(x), f(T^{n+1}x)) + q.\max\{d(f(T^n x), f(x)), d(f(T^n x), f(T^{n+1}x)), \\
 &\quad d(f(T^n x), f(Tx)), d(f(x), f(Tx)), d(f(T^{n+1}x), f(x))\} \tag{4}
 \end{aligned}$$

Letting $n \rightarrow \infty$ in Equation (4), we get

$$\begin{aligned}
 d(f(x), f(Tx)) &\leq q.d(f(x), f(Tx)) \\
 (1-q)d(f(x), f(Tx)) &\leq 0
 \end{aligned}$$

$\Rightarrow d(f(x), f(Tx)) = 0 \Rightarrow f(Tx) = f(x)$. We therefore conclude that T has a Serendipity Fixed Point $x \in X$.

Claim: x is unique. Let T has two serendipity fixed points viz. x and y (say). Consider

$$\begin{aligned}
 d(f(x), f(y)) &= d(f(Tx), f(Ty)) \\
 &\leq q.\max\{d(f(x), f(y)), d(f(x), f(Tx)), d(f(x), f(Ty)), d(f(y), f(Tx)), d(f(y), f(Ty))\} \text{(cf. Definition 2.7)} \\
 &\leq q.d(f(x), f(y)) \\
 d(f(x), f(y)) = 0 &\Rightarrow f(x) = f(y).
 \end{aligned}$$

Thus, we finally conclude that T has a unique Serendipity Fixed Point. □

In the following theorem, we establish the existence and uniqueness of Serendipity Fixed Point by considering dual generalized quasi contraction.

Theorem 3.2. *Let X be a normed linear and weak T -Orbitally complete space. Consider a metric $d : R \times R \rightarrow R$ and a one-to-one, onto function $f \in f^*$ (dual of X). If f satisfies the condition of dual generalized quasi contraction (c.f. Definition 2.9), then T has a unique Serendipity Fixed Point.*

Proof. Consider a self map T defined on X and $f \in X^*$. Applying the condition of dual generalized quasi contraction, for each $x \in X$ and $1 \leq i \leq n - 1$ and $1 \leq j \leq n$, we have

$$\begin{aligned}
 d(fT^i(x), fT^j(x)) &= d(fT(T^{i-1}x), fT(T^{j-1}x)) \\
 &\leq q \cdot \max\{d(f(T^{i-1}x), f(T^{j-1}x)), d(f(T^{i-1}x), fT(T^{i-1}x)), d(f(T^{i-1}x), fT(T^{j-1}x)), \\
 &d(f(T^{j-1}x), fT(T^{j-1}x)), d(f(T^{j-1}x), fT(T^{i-1}x)), d(fT^2(T^{i-1}x), f(T^{i-1}x)), \\
 &d(fT^2(T^{i-1}x), fT(T^{i-1}x)), d(fT^2(T^{i-1}x), f(T^{j-1}x)), d(fT^2x, fT(T^{j-1}x))\} \\
 &\leq q \cdot \delta[O_{fT}(x, n)]
 \end{aligned} \tag{5}$$

Where $d[O_{fT}(x, n)] = \max\{d(f(T^i x), f(T^j x)) : 0 \leq i, j \leq n\}$. Since $0 \leq q \leq 1$, $\exists k_n(x) \leq n$ such that

$$d(f(x), f(T^{k_n(x)}x)) = \delta[O_{fT}(x, n)] \tag{6}$$

Now, for any non-negative integer $k_n(x)$, we have

$$\begin{aligned}
 d(f(x), f(T^{k_n(x)}x)) &\leq d(f(x), f(Tx)) + d(f(Tx), f(T^{k_n(x)}x)) \\
 &\leq d(f(x), f(Tx)) + q \cdot \delta[O_{fT}(x, n)] \quad (\text{by using Equation (5)}) \\
 &\leq d(f(x), f(Tx)) + q \cdot d(f(x), f(T^{k_n(x)}x)) \quad (\text{cf. Equation (6)}) \\
 \delta[O_{fT}(x, n)] &= d(f(x), f(T^{k_n(x)}x)) \leq \frac{1}{1-q} d(f(x), f(Tx))
 \end{aligned} \tag{7}$$

For all $n, m \geq 1$ and $n < m$, consider

$$\begin{aligned}
 d(fT^n(x), fT^m(x)) &= d(fT(T^{n-1}x), fT^{m-n+1}(T^{n-1}x)) \\
 &\leq q \cdot \delta[O_{fT}(T^{n-1}x, m - n + 1)] \quad (\text{In view of Equation (5)}) \\
 &\leq q \cdot d(f(T^{n-1}x), fT^{k_{m-n+1}(T^{n-1}x)}(T^{n-1}x)) \quad (\text{by using Equation (6)}) \\
 &\leq q \cdot d(fT(T^{n-2}x), fT^{k_{m-n+1}(T^{n-1}x)+1}(T^{n-2}x)) \\
 &\leq q^2 \cdot \delta[O_{fT}(T^{n-2}x, m - n + 2)] \quad (\text{cf. Equation (5)}) \\
 &\leq \dots
 \end{aligned}$$

The process is continued n -times and finally, we get

$$\begin{aligned}
 d(fT^n(x), fT^m(x)) &\leq q^n \cdot \delta[O_{fT}(x, m)] \\
 d(fT^n(x), fT^m(x)) &\leq \frac{q^n}{1-q} d(f(x), f(Tx)) \quad (\text{by using Equation (7)})
 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} q^n = 0 \Rightarrow d(fT^n(x), fT^m(x)) = 0$. Hence, the sequence $\{T^n x\}$ is weakly Cauchy in X . Since X is weak T-Orbitally complete, $\exists f(x) \in R$ such that $\lim_{n \rightarrow \infty} f(T^n x) = f(x)$. By using dual generalized quasi contraction condition of T (cf. Definition 2.9), we have

$$\begin{aligned}
 d(f(x), f(Tx)) &\leq d(f(x), f(T^{n+1}x)) + d(f(T^{n+1}x), f(Tx)) \\
 &\leq d(f(x), f(T^{n+1}x)) + d(fT(T^n x), f(Tx)) \\
 &\leq d(f(x), f(T^{n+1}x)) + q \cdot \max\{d(f(T^n x), f(x)), d(f(T^n x), f(T^{n+1}x)), d(f(T^n x), f(Tx)) \\
 &\quad d(f(x), f(Tx)), d(f(T^{n+1}x), f(x)), d(fT^{n+2}x, fT^n x), \\
 &\quad d(fT^{n+2}x, fT^{n+1}x), d(fT^{n+2}x, f(x)), d(fT^{n+2}x, fTx)\} \tag{8}
 \end{aligned}$$

Letting $n \rightarrow \infty$ in Equation (8), we get

$$\begin{aligned}
 d(f(x), f(Tx)) &\leq q \cdot d(f(x), f(Tx)) \\
 (1 - q) d(f(x), f(Tx)) &\leq 0
 \end{aligned}$$

$\Rightarrow d(f(x), f(Tx)) = 0 \Rightarrow f(Tx) = f(x)$. Therefore, we conclude that T has a Serendipity Fixed Point $x \in X$.

Claim: x is unique. Let if possible T has two serendipity fixed points viz. x and y. Consider

$$\begin{aligned}
 d(f(x), f(y)) &= d(f(Tx), f(Ty)) \\
 &\leq q \cdot \max\{d(f(x), f(y)), d(f(x), f(Tx)), d(f(x), f(Ty)), d(f(y), f(Tx)), d(f(y), f(Ty)), \\
 &\quad d(fT^2x, f(x)), d(fT^2x, fTx), d(fT^2x, f(y)), d(fT^2x, fTy)\} \text{ (cf. Definition 2.9)} \\
 &\leq q \cdot d(f(x), f(y)) \\
 d(f(x), f(y)) &= 0 \Rightarrow f(x) = f(y).
 \end{aligned}$$

Thus, we finally conclude that T has a unique Serendipity Fixed Point. □

4. Conclusion

This concept of Serendipity Fixed Point may be used to find the solution of equations for which the approximate solution does not converge to any unique limit. The existence and uniqueness of a Serendipity Fixed Point in dual T-Orbitally complete space has been established with sufficiently light form of contraction map viz. dual quasi contraction and dual generalized quasi-contraction which is notable.

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