

Solution for One-Dimensional Transient Heat Conduction And Thermoelasticity in a Multilayer Hollow Sphere

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Abstract: An analytical solution is obtained for the problem of one dimensional transient heat conduction and thermo elasticity in the multilayered hollow sphere. The sphere has multiple layers in the radial direction and each layer is time dependent and spatially without heat sources are considered. To obtain the temperature distribution The eigen value problem is solved by use of separation of variables method. At $t > 0$ homogenous boundary conditions of the first kind are set on the inner radial surface ($i = 1, r = r_0$) and third kind are set on the outer ($i = n, r = r_n$) radial surfaces (Convection).

Keywords: One dimensional transient heat conduction, Thermal Displacement, Thermal stresses, multilayer sphere.

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1. Introduction

The determination of temperature, displacement and stresses in a one dimensional multilayer hollow sphere it is important both in research and engineering. Earlier literature on these found to be on the homogeneous finite bodies with constant and uniform thermo physical properties. Carslaw and Jeager [8] studied the use of heat sources and sinks in the cases of variable temperature in the hollow and solid sphere, M.N.Ozisk [4,6] discussed homogeneous and non homogeneous heat conduction boundary value problems of solid and hollow sphere. N. Noda [7] discussed heat conduction and thermal stresses for spherical bodies with and without heat generation. In the review of recent literature An integral transform technique was applied to obtained the solution of one dimensional, radial heat conduction problem of a solid sphere with internal heat generation [2] by S.P. Pawar.

The Green's function method has been used to obtained the exact analytical solution of the radial heat conduction problem in a hollow multilayered sphere [3] by Urszula Siedlecka. An analytical method based on separation of variables and finite integral transform has been used to obtain the solution of the heat conduction in the hollow sphere [5] by S. Singh. The aim of present work is to obtain analytical solution for the problem of one dimensional transient heat conduction and thermo elasticity in the multilayered hollow sphere. The sphere has multiple layers in the radial direction and each layer is time dependent and spatially without heat sources are considered. To obtain the temperature distribution eigenvalue problem is solved by use of separation of variables method and results are obtained in the form of series solutions.

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2. Formulation of the Problem

Consider a n -layer composite hollow sphere with radial coordinates $r_0 \leq r \leq r_n$. It is assumed that all the layers are thermally isotropic and make a perfect thermal contact. At $t = 0$, the i^{th} layer has a temperature $f_i(r)$. At $t > 0$, homogeneous boundary conditions of the first kind are set on the inner radial surface ($i = 1, r = r_0$) and third kind are set on the outer ($i = n, r = r_n$) radial surfaces. The thermo physical properties of spherical material are constant. The temperature distribution, displacement and thermal stresses are to be determined and analyze numerically and graphically.

2.1. Heat Conduction Equation

The governing differential equation for the 1-D transient heat conduction in a multilayer composite sphere has the form

$$\frac{\partial^2 T_i}{\partial r^2} + \frac{2}{r} \frac{\partial T_i}{\partial r} = \frac{1}{\alpha_i} \frac{\partial T_i}{\partial t} \quad (1)$$

$T_i = T_i(r, t)$; $r_0 \leq r \leq r_n$; $r_{i-1} \leq r \leq r_i$; $1 \leq i \leq n$. Boundary and initial condition as for Inner surface of the 1st layer ($i = 1$)

$$T_1(r_0, t) = 0 \quad (2)$$

For Outer surface of the n^{th} layer ($i = n$)

$$k_n \frac{\partial T_n(r_n, t)}{\partial r} + h_n T_n(r_n, t) = 0 \quad (3)$$

For Inner interface of the i^{th} layer $i = 2, 3, \dots, n$

$$T_i(r_{i-1}, t) = T_{i-1}(r_{i-1}, t) \quad (4)$$

$$k_i \frac{\partial T_i(r_{i-1}, t)}{\partial r} = k_{i-1} \frac{\partial T_{i-1}(r_{i-1}, t)}{\partial r} \quad (5)$$

For outer interface of the i^{th} layer $i = 1, 2, 3, \dots, n - 1$.

$$T_i(r_i, t) = T_{i+1}(r_i, t) \quad (6)$$

$$k_i \frac{\partial T_i(r_i, t)}{\partial r} = k_{i+1} \frac{\partial T_{i+1}(r_i, t)}{\partial r} \quad (7)$$

The initial condition

$$T_i(r, t = 0) = f_i(r); \quad 1 \leq i \leq n \quad (8)$$

where k is the thermal conductivity, h_n are heat transfer parameters and $\alpha = \frac{k}{c_p \rho}$ is thermal diffusivity of the material, c_p is specific heat and ρ is density of the material of the sphere.

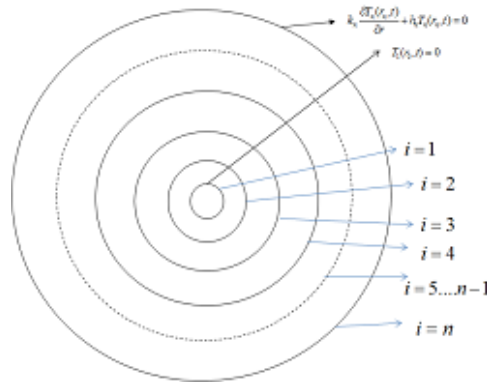


Figure 1. Geometry of multilayer hollow composite sphere

2.2. Thermoelastic Problem

One dimensional problem of thermoelasticity in the spherical coordinates which means spherically symmetric problem, in which the shearing stresses and strains vanish and strain and stress components in spherical coordinates θ and ϕ direction are identical [Noda]. For multilayer composite hollow sphere the equations are as follows.

$$\sigma_{\theta\theta}^{(i)} = \sigma_{\phi\phi}^{(i)}, \varepsilon_{\theta\theta}^{(i)} = \varepsilon_{\phi\phi}^{(i)} \quad (9)$$

$$\sigma_{r\theta}^{(i)} = \sigma_{\theta\phi}^{(i)} = \sigma_{r\phi}^{(i)} = 0 \quad (10)$$

$$\varepsilon_{r\theta}^{(i)} = \varepsilon_{\theta\phi}^{(i)} = \varepsilon_{r\phi}^{(i)} = 0 \quad (11)$$

The equilibrium equation without body force in spherical coordinates reduces to [Noda] as

$$\frac{d\sigma_{rr}^{(i)}}{dr} + \frac{1}{r} (2\sigma_{rr}^{(i)} - \sigma_{\theta\theta}^{(i)} - \sigma_{\phi\phi}^{(i)}) = 0 \quad (12)$$

$$\frac{d\sigma_{rr}^{(i)}}{dr} + \frac{2}{r} (\sigma_{rr}^{(i)} - \sigma_{\theta\theta}^{(i)}) = 0 \quad (13)$$

Stress strain relation or Hooke's relations are

$$\sigma_{rr}^{(i)} = 2\mu^{(i)} \varepsilon_{rr}^{(i)} + \lambda^{(i)} e^{(i)} - \beta^{(i)} \tau^{(i)} \quad (14)$$

$$\sigma_{\theta\theta}^{(i)} = \sigma_{\phi\phi}^{(i)} = 2\mu^{(i)} \varepsilon_{\theta\theta}^{(i)} + \lambda^{(i)} e^{(i)} - \beta^{(i)} \tau^{(i)} \quad (15)$$

where the strain dilatation

$$e^{(i)} = \varepsilon_{rr}^{(i)} + \varepsilon_{\theta\theta}^{(i)} + \varepsilon_{\phi\phi}^{(i)} = \varepsilon_{rr}^{(i)} + 2\varepsilon_{\theta\theta}^{(i)} \quad (16)$$

$\sigma_{rr}^{(i)}$, $\sigma_{\theta\theta}^{(i)}$ and $\sigma_{\phi\phi}^{(i)}$ are the stresses in the radial and tangential direction and $\varepsilon_{rr}^{(i)}$, $\varepsilon_{\theta\theta}^{(i)}$ and $\varepsilon_{\phi\phi}^{(i)}$ are strains in radial and tangential direction, $\tau^{(i)}$ is the temperature change, $e^{(i)}$ is the strain dilatation and $\lambda^{(i)}$ and $\mu^{(i)}$ are the Lamé constants related to the modulus of elasticity $E^{(i)}$ and the Poisson's ratio $\nu^{(i)}$ as,

$$\lambda^{(i)} = \frac{\nu^{(i)} E^{(i)}}{[1 + \nu^{(i)}][1 - 2\nu^{(i)}]} \quad \text{and} \quad \mu^{(i)} = \frac{E^{(i)}}{2[1 + \nu^{(i)}]} \quad (17)$$

The strain component in terms of radial displacement $u_r^{(i)}$ is

$$\varepsilon_{rr}^{(i)} = \frac{du_r^{(i)}}{dr} \quad \text{and} \quad \varepsilon_{\theta\theta}^{(i)} = \varepsilon_{\phi\phi}^{(i)} = \frac{u_r^{(i)}}{r} \quad (18)$$

The boundary and interface conditions as for Inner surface of the 1st layer ($i = 1$)

$$\sigma_{rr}^{(i)}(r_0) = 0 \quad \text{at } r = r_0 \quad (19)$$

For Outer surface of the n^{th} layer ($i = n$)

$$\sigma_{rr}^{(i)}(r_n) = 0 \quad \text{at } r = r_n \quad (20)$$

For Inner interface of the i^{th} layer $i = 1, 2, 3, \dots, n - 1$.

$$\sigma_{rr}^{(i)}(r_i) = \sigma_{rr}^{(i+1)}(r_i) \quad (21)$$

$$k_i \frac{\partial \sigma_{rr}^{(i)}(r_i)}{\partial r} = k_{i+1} \frac{\partial \sigma_{rr}^{(i+1)}(r_i)}{\partial r} \quad (22)$$

Now with equations one can obtain the displacement and thermal stresses as

$$u_r^{(i)} = \left(\frac{1 + \nu^{(i)}}{1 - \nu^{(i)}} \right) a_t^{(i)} \left[\frac{1}{r^2} \int_{r_0}^r \tau^{(i)} r^2 dr + \frac{2[1 - 2\nu^{(i)}]}{[1 + \nu^{(i)}]} \frac{r}{(r_i^3 - r_{i-1}^3)} \int_{r_{i-1}}^{r_i} \tau^{(i)} r^2 dr + \frac{r_{i-1}^3}{(r_i^3 - r_{i-1}^3)r^2} \int_{r_{i-1}}^{r_i} \tau^{(i)} r^2 dr \right] \quad (23)$$

$$\sigma_{rr}^{(i)} = \frac{a_t^{(i)} E^{(i)}}{[1 - \nu^{(i)}]} \left[\frac{2(r^3 - r_{i-1}^3)}{(r_i^3 - r_{i-1}^3)r^3} \int_{r_{i-1}}^{r_i} \tau^{(i)} r^2 dr - \frac{2}{r^3} \int_{r_{i-1}}^r \tau^{(i)} r^2 dr \right] \quad (24)$$

$$\sigma_{\theta\theta}^{(i)} = \sigma_{\phi\phi}^{(i)} = \frac{a_t^{(i)} E^{(i)}}{[1 - \nu^{(i)}]} \left[\frac{2r^3 + r_{i-1}^3}{(r_i^3 - r_{i-1}^3)r^3} \int_{r_{i-1}}^{r_i} \tau^{(i)} r^2 dr + \frac{1}{r^3} \int_{r_{i-1}}^r \tau^{(i)} r^2 dr - \tau^{(i)} \right] \quad (25)$$

where a_t^i is the coefficient of linear thermal expansion. The equation (1)-(25) constitutes the Mathematical formulation of the problem.

3. Solution

We use the procedure adopted by M.N.Ozisk [4], defining a new dependent variable $U_i(r, t)$. The eigenvalue problem is solved by the use of separation of variables method. Defining a new dependent variable $U_i(r, t)$

$$U_i(r, t) = rT_i(r, t) \quad (26)$$

Then the system (1)-(8) becomes

$$\frac{\partial^2 U_i}{\partial r^2} = \frac{1}{\alpha_i} \frac{\partial U_i}{\partial t} \quad (27)$$

where $U_i = U_i(r, t)$ in $r_{i-1} \leq r \leq r_i$; $1 \leq i \leq n$. For Inner surface of the 1st layer ($i = 1$)

$$U_i(r_0, t) = 0 \quad (28)$$

For Outer surface of the n^{th} layer ($i = n$)

$$\frac{\partial U_n(r_n, t)}{\partial r} + \left(\frac{h_n}{k_n} - \frac{1}{r} \right) U_n(r_n, t) = 0, \quad (29)$$

For Inner interface of the i^{th} layer $i = 2, 3, \dots, n$.

$$U_i(r_{i-1}, t) = U_{i-1}(r_{i-1}, t) \quad (30)$$

In the above matrix equation, β_{im} ($i \neq 1$) may be written in terms of β_{1m} using the equation. Subsequently, transverse eigen condition can be obtained by setting the determinant of the coefficient matrix equal to zero. And after that eigen value determined the constants a_{im} and b_{im} . We get

$$\frac{d\Gamma_i(t)}{dt} + \alpha_i \beta_{im}^2 \Gamma_i(t) = 0 \quad (41)$$

We get the solution

$$\Gamma_{im}(t) = c_3 e^{-\alpha_i \beta_{im}^2 t} \quad (42)$$

using Initial condition as

$$\sum_{m=1}^{\infty} \Gamma_{im}(0) R_{im}(r) = r f_i(r)$$

Using orthogonal condition the coefficient is $\Gamma_{im}(0)$

$$\Gamma_{im}(0) = \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} \quad (43)$$

$$\Gamma_{im}(t) = \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} e^{-\alpha_i \beta_{im}^2 t} \quad (44)$$

$$\begin{aligned} U_i(r, t) &= \sum_{m=1}^{\infty} \Gamma_{im}(t) R_{im}(r) \\ U_i(r, t) &= \sum_{m=1}^{\infty} \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} e^{-\alpha_i \beta_{im}^2 t} (a_{im} \cos \beta_{im} r + b_{im} \sin \beta_{im} r) \\ T_i(r, t) &= \sum_{m=1}^{\infty} \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} \left(\frac{a_{im} \cos \beta_{im} r + b_{im} \sin \beta_{im} r}{r} \right) e^{-\alpha_i \beta_{im}^2 t} \end{aligned} \quad (45)$$

4. Displacement and Thermal Stress Function

The temperature change $\tau^{(i)}$ is obtained as

$$\tau^{(i)} = T_i(r, t) - f_i(r)$$

Where $i = 1, 2, 3, \dots, n$.

$$\tau^{(i)} = \sum_{m=1}^{\infty} \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} \left(\frac{a_{im} \cos \beta_{im} r + b_{im} \sin \beta_{im} r}{r} \right) e^{-\alpha_i \beta_{im}^2 t} - f_i(r) \quad (46)$$

Now consider the integral, which is required to obtain displacement and stress functions

$$\begin{aligned} \int r^2 \tau^{(i)} dr &= \sum_{m=1}^{\infty} \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} e^{-\alpha_i \beta_{im}^2 t} \int r (a_{im} \cos \beta_{im} r + b_{im} \sin \beta_{im} r) dr - \int r^2 f_i(r) dr \\ \int r^2 \tau^{(i)} dr &= \sum_{m=1}^{\infty} \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} \\ &\quad e^{-\alpha_i \beta_{im}^2 t} \left[\left(\frac{a_{im} \beta_{im} r + b_{im}}{\beta_{im}} \right) \sin \beta_{im} r - \left(\frac{b_{im} \beta_{im} r - a_{im}}{\beta_{im}} \right) \cos \beta_{im} r \right] - \int r^2 f_i(r) dr \end{aligned} \quad (47)$$

Using this integral in the equations of displacement and thermal stress in i^{th} layers functions as

$$u_r^{(i)} = \left(\frac{1+\nu^{(i)}}{1-\nu^{(i)}} \right) \frac{a_i^{(i)}}{r^2} \left\{ \sum_{m=1}^{\infty} \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} e^{-\alpha_i \beta_{im}^2 t} \left[\left(\frac{a_{im} \beta_{im} r + b_{im}}{\beta_{im}} \right) \sin \beta_{im} r - \left(\frac{b_{im} \beta_{im} r - a_{im}}{\beta_{im}} \right) \cos \beta_{im} r \right] \right\} - \int_{r_{i-1}}^r r^2 f_i(r) dr$$

$$\begin{aligned}
 & + \left(\frac{2(1+2\nu^{(i)})}{1-\nu^{(i)}} \right) \frac{a_i^{(i)} r}{(r_i^3 - r_{i-1}^3)} \left\{ \sum_{m=1}^{\infty} \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} e^{-\alpha_i \beta_{im}^2 t} \left[\left(\frac{a_{im} \beta_{im} r + b_{im}}{\beta_{im}^2} \right) \sin \beta_{im} r - \left(\frac{b_{im} \beta_{im} r - a_{im}}{\beta_{im}^2} \right) \cos \beta_{im} r \right]_{r_{i-1}}^{r_i} - \int_{r_{i-1}}^{r_i} r^2 f_i(r) dr \right\} \\
 & + \left(\frac{1+\nu^{(i)}}{1-\nu^{(i)}} \right) \frac{a_i^{(i)} r_{i-1}^3}{r^2 (r_i^3 - r_{i-1}^3)} \left\{ \sum_{m=1}^{\infty} \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} e^{-\alpha_i \beta_{im}^2 t} \left[\left(\frac{a_{im} \beta_{im} r + b_{im}}{\beta_{im}^2} \right) \sin \beta_{im} r - \left(\frac{b_{im} \beta_{im} r - a_{im}}{\beta_{im}^2} \right) \cos \beta_{im} r \right]_{r_{i-1}}^{r_i} - \int_{r_{i-1}}^{r_i} r^2 f_i(r) dr \right\} \quad (48)
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{rr}^{(i)} &= \frac{a_i^{(i)} E^{(i)} 2(r_i^3 - r_{i-1}^3)}{[1-\nu^{(i)}] r^3 (r_i^3 - r_{i-1}^3)} \left\{ \sum_{m=1}^{\infty} \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} e^{-\alpha_i \beta_{im}^2 t} \left[\left(\frac{a_{im} \beta_{im} r + b_{im}}{\beta_{im}^2} \right) \sin \beta_{im} r - \left(\frac{b_{im} \beta_{im} r - a_{im}}{\beta_{im}^2} \right) \cos \beta_{im} r \right]_{r_{i-1}}^{r_i} - \int_{r_{i-1}}^{r_i} r^2 f_i(r) dr \right\} \\
 & - \frac{2a_i^{(i)} E^{(i)}}{[1-\nu^{(i)}] r^3} \left\{ \sum_{m=1}^{\infty} \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} e^{-\alpha_i \beta_{im}^2 t} \left[\left(\frac{a_{im} \beta_{im} r + b_{im}}{\beta_{im}^2} \right) \sin \beta_{im} r - \left(\frac{b_{im} \beta_{im} r - a_{im}}{\beta_{im}^2} \right) \cos \beta_{im} r \right]_{r_{i-1}}^r - \int_{r_{i-1}}^r r^2 f_i(r) dr \right\} \quad (49)
 \end{aligned}$$

$$\begin{aligned}
 \sigma_{\theta\theta}^{(i)} &= \sigma_{\phi\phi}^{(i)} = \frac{a_i^{(i)} E^{(i)} (2r_i^3 + r_{i-1}^3)}{[1-\nu^{(i)}] r^3 (r_i^3 - r_{i-1}^3)} \left\{ \sum_{m=1}^{\infty} \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} e^{-\alpha_i \beta_{im}^2 t} \left[\left(\frac{a_{im} \beta_{im} r + b_{im}}{\beta_{im}^2} \right) \sin \beta_{im} r - \left(\frac{b_{im} \beta_{im} r - a_{im}}{\beta_{im}^2} \right) \cos \beta_{im} r \right]_{r_{i-1}}^{r_i} - \int_{r_{i-1}}^{r_i} r^2 f_i(r) dr \right\} \\
 & + \frac{a_i^{(i)} E^{(i)}}{[1-\nu^{(i)}] r^3} \left\{ \sum_{m=1}^{\infty} \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} e^{-\alpha_i \beta_{im}^2 t} \left[\left(\frac{a_{im} \beta_{im} r + b_{im}}{\beta_{im}^2} \right) \sin \beta_{im} r - \left(\frac{b_{im} \beta_{im} r - a_{im}}{\beta_{im}^2} \right) \cos \beta_{im} r \right]_{r_{i-1}}^r - \int_{r_{i-1}}^r r^2 f_i(r) dr \right\} \\
 & - \left(\frac{a_i^{(i)} E^{(i)}}{1-\nu^{(i)}} \right) \sum_{m=1}^{\infty} \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} \left(\frac{a_{im} \cos \beta_{im} r + b_{im} \sin \beta_{im} r}{r} \right) e^{-\alpha_i \beta_{im}^2 t} - f_i(r) \quad (50)
 \end{aligned}$$

The equations provides general solution for displacement and thermal stresses subject to the interface conditions as

$$\sigma_{rr}^{(i)}(r_i) = \sigma_{rr}^{(i+1)}(r_i) \quad \text{and} \quad k_i \frac{\partial \sigma_{rr}^{(i)}(r_i)}{\partial r} = k_{i+1} \frac{\partial \sigma_{rr}^{(i+1)}(r_i)}{\partial r} \quad \text{for } i = 1, 2, 3, \dots, n-1$$

5. Special Case Study

We consider a three layer hollow sphere $r_0 \leq r \leq r_3$ which is initially $t = 0$ at uniform temperature. For time $t > 0$ thermal convection occurs from the outer radial surface at $r = r_3$

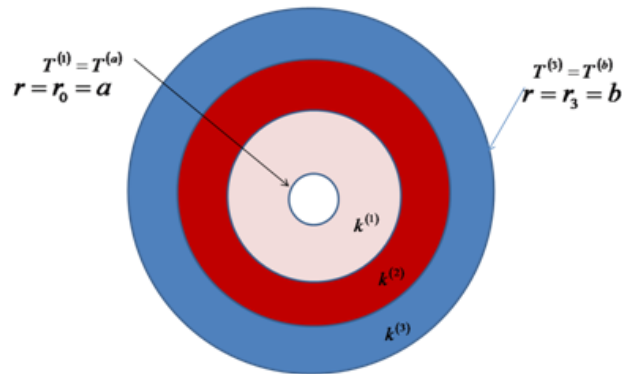


Figure 2. Geometry of three layer hollow composite sphere

The governing differential equation for the 1D transient heat conduction without heat sources in this three layer spherical region is as follows.

$$\frac{\partial^2 T_i}{\partial r^2} + \frac{2}{r} \frac{\partial T_i}{\partial r} = \frac{1}{\alpha_i} \frac{\partial T_i}{\partial t}$$

$T_i = T_i(r, t)$; $r_0 \leq r \leq r_3$; $r_{i-1} \leq r \leq r_i$; $1 \leq i \leq 3$. This equation is used with the following boundary conditions for Inner surface of the 1st layer ($i = 1$):

$$T_1(r_0, t) = 0$$

Outer surface of the 3^{rd} layer ($i=3$)

$$k_3 \frac{\partial T_3(r_3, t)}{\partial r} + h_3 T_3(r_3, t) = 0$$

Inner interface of the i^{th} layer $i=2, 3$

$$T_i(r_{i-1}, t) = T_{i-1}(r_{i-1}, t)$$

Outer interface of the i^{th} layer $i=1, 2$

$$\begin{aligned} T_i(r_i, t) &= T_{i+1}(r_i, t) \\ k_i \frac{\partial T_i(r_i, t)}{\partial r} &= k_{i+1} \frac{\partial T_{i+1}(r_i, t)}{\partial r} \end{aligned}$$

Initial condition for $t=0$ is as follows

$$T_i(r, t=0)' = 1'; \quad 1 \leq i \leq 3$$

$$R_{im}(r) = a_{im} \cos \beta_{im} r + b_{im} \sin \beta_{im} r$$

By using boundary and interface conditions equation

$$\begin{bmatrix} \cos \beta_{1m} r_0 & \sin \beta_{1m} r_0 & 0 & 0 & 0 & 0 \\ \cos \beta_{1m} r_1 & \sin \beta_{1m} r_1 & -\cos \beta_{2m} r_1 & -\sin \beta_{2m} r_1 & 0 & 0 \\ -k_1 r_1 \beta_{1m} \sin \beta_{1m} r_1 - k_1 \cos \beta_{1m} r_1 & k_1 r_1 \beta_{1m} \cos \beta_{1m} r_1 - k_1 \sin \beta_{1m} r_1 & k_2 r_1 \beta_{2m} \sin \beta_{2m} r_1 + k_2 \cos \beta_{2m} r_1 & -k_2 r_1 \beta_{2m} \cos \beta_{2m} r_1 + k_2 \sin \beta_{2m} r_1 & 0 & 0 \\ 0 & 0 & \cos \beta_{2m} r_2 & \sin \beta_{2m} r_2 & -\cos \beta_{3m} r_2 & -\sin \beta_{3m} r_2 \\ 0 & 0 & -k_2 r_2 \beta_{2m} \sin \beta_{2m} r_2 - k_2 \cos \beta_{2m} r_2 & k_2 r_2 \beta_{2m} \cos \beta_{2m} r_2 - k_2 \sin \beta_{2m} r_2 & k_3 r_2 \beta_{3m} \sin \beta_{3m} r_2 + k_3 \cos \beta_{3m} r_2 & -k_3 r_2 \beta_{3m} \cos \beta_{3m} r_2 + k_3 \sin \beta_{3m} r_2 \\ 0 & 0 & 0 & 0 & -\beta_{3m} \cos \beta_{3m} r_3 + M \cos \beta_{3m} r_3 & \beta_{3m} \sin \beta_{3m} r_3 + M \sin \beta_{3m} r_3 \end{bmatrix} \begin{bmatrix} a_{1m} \\ b_{1m} \\ a_{2m} \\ b_{2m} \\ a_{3m} \\ b_{3m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Where $M = \frac{h_3}{k_3} - \frac{1}{r}$. The heat flux continuity conditions at the interfaces imply the following:

$$\beta_{im} = \beta_{1m} \sqrt{\frac{\alpha_1}{\alpha_i}}; \quad i=2, 3.$$

By using initial condition $t=0$

$$U_i(r, t=0) = r$$

$$\sum_{m=1}^{\infty} \Gamma_{im}(0) R_{im}(r) = r$$

By using orthogonal expansion method [8].

$$\begin{aligned} \Gamma_{im}(0) &= \frac{\int_{r_{i-1}}^{r_i} r R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} \\ T_i(r, t) &= \sum_{m=1}^{\infty} \left\{ \left[\frac{a_{im} \left(\frac{r_i \beta_{im} \sin \beta_{im} r_i + \cos \beta_{im} r_i - r_{i-1} \beta_{im} \sin \beta_{im} r_{i-1} - \cos \beta_{im} r_{i-1}}{\beta_{im}^2} \right) - b_{im} \left(\frac{r_i \beta_{im} \cos \beta_{im} r_i - \sin \beta_{im} r_i - r_{i-1} \beta_{im} \cos \beta_{im} r_{i-1} + \sin \beta_{im} r_{i-1}}{\beta_{im}^2} \right)}{\left(\frac{a_{im}^2 + b_{im}^2}{2} \right) (r_i - r_{i-1}) + \left(\frac{a_{im}^2 - b_{im}^2}{4\beta_{im}^2} \right) (\sin 2\beta_{im} r_i - \sin 2\beta_{im} r_{i-1}) - \frac{a_{im} b_{im}}{2\beta_{im}} (\cos 2\beta_{im} r_i - \cos 2\beta_{im} r_{i-1})} \right] \right. \\ &\quad \left. \left(\frac{a_{im} \cos \beta_{im} r + b_{im} \sin \beta_{im} r}{r} \right) e^{-\alpha_i \beta_{im}^2 t} \right\} \end{aligned} \quad (51)$$

For different three layers the Temperature equations as follows

$$\begin{aligned} T_1(r, t) &= \sum_{m=1}^{\infty} \left\{ \left[\frac{a_{1m} \left(\frac{r_1 \beta_{1m} \sin \beta_{1m} r_1 + \cos \beta_{1m} r_1 - r_0 \beta_{1m} \sin \beta_{1m} r_0 - \cos \beta_{1m} r_0}{\beta_{1m}^2} \right) - b_{1m} \left(\frac{r_1 \beta_{1m} \cos \beta_{1m} r_1 - \sin \beta_{1m} r_1 - r_0 \beta_{1m} \cos \beta_{1m} r_0 + \sin \beta_{1m} r_0}{\beta_{1m}^2} \right)}{\left(\frac{a_{1m}^2 + b_{1m}^2}{2} \right) (r_1 - r_0) + \left(\frac{a_{1m}^2 - b_{1m}^2}{4\beta_{1m}^2} \right) (\sin 2\beta_{1m} r_1 - \sin 2\beta_{1m} r_0) - \frac{a_{1m} b_{1m}}{2\beta_{1m}} (\cos 2\beta_{1m} r_1 - \cos 2\beta_{1m} r_0)} \right] \right. \\ &\quad \left. \left(\frac{a_{1m} \cos \beta_{1m} r + b_{1m} \sin \beta_{1m} r}{r} \right) e^{-\alpha_1 \beta_{1m}^2 t} \right\} \\ T_2(r, t) &= \sum_{m=1}^{\infty} \left\{ \left[\frac{a_{2m} \left(\frac{r_2 \beta_{2m} \sin \beta_{2m} r_2 + \cos \beta_{2m} r_2 - r_1 \beta_{2m} \sin \beta_{2m} r_1 - \cos \beta_{2m} r_1}{\beta_{2m}^2} \right) - b_{2m} \left(\frac{r_2 \beta_{2m} \cos \beta_{2m} r_2 - \sin \beta_{2m} r_2 - r_1 \beta_{2m} \cos \beta_{2m} r_1 + \sin \beta_{2m} r_1}{\beta_{2m}^2} \right)}{\left(\frac{a_{2m}^2 + b_{2m}^2}{2} \right) (r_2 - r_1) + \left(\frac{a_{2m}^2 - b_{2m}^2}{4\beta_{2m}^2} \right) (\sin 2\beta_{2m} r_2 - \sin 2\beta_{2m} r_1) - \frac{a_{2m} b_{2m}}{2\beta_{2m}} (\cos 2\beta_{2m} r_2 - \cos 2\beta_{2m} r_1)} \right] \right. \\ &\quad \left. \left(\frac{a_{2m} \cos \beta_{2m} r + b_{2m} \sin \beta_{2m} r}{r} \right) e^{-\alpha_2 \beta_{2m}^2 t} \right\} \\ T_3(r, t) &= \sum_{m=1}^{\infty} \left\{ \left[\frac{a_{3m} \left(\frac{r_3 \beta_{3m} \sin \beta_{3m} r_3 + \cos \beta_{3m} r_3 - r_2 \beta_{3m} \sin \beta_{3m} r_2 - \cos \beta_{3m} r_2}{\beta_{3m}^2} \right) - b_{3m} \left(\frac{r_3 \beta_{3m} \cos \beta_{3m} r_3 - \sin \beta_{3m} r_3 - r_2 \beta_{3m} \cos \beta_{3m} r_2 + \sin \beta_{3m} r_2}{\beta_{3m}^2} \right)}{\left(\frac{a_{3m}^2 + b_{3m}^2}{2} \right) (r_3 - r_2) + \left(\frac{a_{3m}^2 - b_{3m}^2}{4\beta_{3m}^2} \right) (\sin 2\beta_{3m} r_3 - \sin 2\beta_{3m} r_2) - \frac{a_{3m} b_{3m}}{2\beta_{3m}} (\cos 2\beta_{3m} r_3 - \cos 2\beta_{3m} r_2)} \right] \right. \\ &\quad \left. \left(\frac{a_{3m} \cos \beta_{3m} r + b_{3m} \sin \beta_{3m} r}{r} \right) e^{-\alpha_3 \beta_{3m}^2 t} \right\} \end{aligned}$$

$$\begin{aligned}
 & - \left(\frac{a_t^{(i)} E^{(i)}}{1-\nu^{(i)}} \right) \left[\sum_{m=1}^{\infty} \frac{\int_{r_{i-1}}^{r_i} r f_i(r) R_{im}(r) dr}{\int_{r_{i-1}}^{r_i} R_{im}^2(r) dr} \left(\frac{a_{im} \cos \beta_{im} r + b_{im} \sin \beta_{im} r}{r} \right) e^{-\alpha_i \beta_{im}^2 t} - f_i(r) \right] \\
 \sigma_{\theta\theta}^{(i)} = \sigma_{\phi\phi}^{(i)} &= \frac{a_t E_i (2r^3 + r_{i-1}^3)}{(1-\nu_i) r^3 (r_i^3 - r_{i-1}^3)} \times \\
 & \sum_{m=1}^{\infty} \left\{ \left[\frac{a_{im} \left(\frac{r_i \beta_{im} \sin \beta_{im} r_i + \cos \beta_{im} r_i - r_{i-1} \beta_{im} \sin \beta_{im} r_{i-1} - \cos \beta_{im} r_{i-1}}{\beta_{im}^2} \right) - b_{im} \left(\frac{r_i \beta_{im} \cos \beta_{im} r_i - \sin \beta_{im} r_i - r_{i-1} \beta_{im} \cos \beta_{im} r_{i-1} + \sin \beta_{im} r_{i-1}}{\beta_{im}^2} \right)}{\left(\frac{a_{im}^2 + b_{im}^2}{2} \right) (r_i - r_{i-1}) + \left(\frac{a_{im}^2 - b_{im}^2}{4\beta_{im}} \right) (\sin 2\beta_{im} r - \sin 2\beta_{im} r_{i-1}) - \frac{a_{im} b_{im}}{2\beta_{im}} (\cos 2\beta_{im} r - \cos 2\beta_{im} r_{i-1})} \right] \right. \\
 & \left. \left[\left(\frac{a_{im} \beta_{im} r + b_{im}}{\beta_{im}^2} \right) \sin \beta_{im} r - \left(\frac{b_{im} \beta_{im} r - a_{im}}{\beta_{im}^2} \right) \cos \beta_{im} r \right]_{r_{i-1}}^r e^{-\alpha_i \beta_{im}^2 t} \right\} \\
 & + \frac{a_t E_i}{(1-\nu_i) r^3} \times \sum_{m=1}^{\infty} \left\{ \left[\frac{a_{im} \left(\frac{r_i \beta_{im} \sin \beta_{im} r_i + \cos \beta_{im} r_i - r_{i-1} \beta_{im} \sin \beta_{im} r_{i-1} - \cos \beta_{im} r_{i-1}}{\beta_{im}^2} \right) - b_{im} \left(\frac{r_i \beta_{im} \cos \beta_{im} r_i - \sin \beta_{im} r_i - r_{i-1} \beta_{im} \cos \beta_{im} r_{i-1} + \sin \beta_{im} r_{i-1}}{\beta_{im}^2} \right)}{\left(\frac{a_{im}^2 + b_{im}^2}{2} \right) (r_i - r_{i-1}) + \left(\frac{a_{im}^2 - b_{im}^2}{4\beta_{im}} \right) (\sin 2\beta_{im} r - \sin 2\beta_{im} r_{i-1}) - \frac{a_{im} b_{im}}{2\beta_{im}} (\cos 2\beta_{im} r - \cos 2\beta_{im} r_{i-1})} \right] \right. \\
 & \left. \left[\left(\frac{a_{im} \beta_{im} r + b_{im}}{\beta_{im}^2} \right) \sin \beta_{im} r - \left(\frac{b_{im} \beta_{im} r - a_{im}}{\beta_{im}^2} \right) \cos \beta_{im} r \right]_{r_{i-1}}^r e^{-\alpha_i \beta_{im}^2 t} \right\} \\
 & - \left(\frac{a_t E_i}{1-\nu_i} \right) \times \sum_{m=1}^{\infty} \left\{ \left[\frac{a_{im} \left(\frac{r_i \beta_{im} \sin \beta_{im} r_i + \cos \beta_{im} r_i - r_{i-1} \beta_{im} \sin \beta_{im} r_{i-1} - \cos \beta_{im} r_{i-1}}{\beta_{im}^2} \right) - b_{im} \left(\frac{r_i \beta_{im} \cos \beta_{im} r_i - \sin \beta_{im} r_i - r_{i-1} \beta_{im} \cos \beta_{im} r_{i-1} + \sin \beta_{im} r_{i-1}}{\beta_{im}^2} \right)}{\left(\frac{a_{im}^2 + b_{im}^2}{2} \right) (r_i - r_{i-1}) + \left(\frac{a_{im}^2 - b_{im}^2}{4\beta_{im}} \right) (\sin 2\beta_{im} r - \sin 2\beta_{im} r_{i-1}) - \frac{a_{im} b_{im}}{2\beta_{im}} (\cos 2\beta_{im} r - \cos 2\beta_{im} r_{i-1})} \right] \right. \\
 & \left. \left[\frac{a_{im} \cos \beta_{im} r + b_{im} \sin \beta_{im} r}{r} \right] e^{-\alpha_i \beta_{im}^2 t} \right\} \\
 & - \frac{a_t E_i (2r^3 + r_{i-1}^3)}{(1-\nu_i) r^3 (r_i^3 - r_{i-1}^3)} \left(\frac{r_i^3 - r_{i-1}^3}{3} \right) - \frac{a_t E_i}{(1-\nu_i) r^3} \left(\frac{r^3 - r_{i-1}^3}{3} \right) + \left(\frac{a_t E_i}{1-\nu_i} \right)
 \end{aligned}$$

The Above equations provides general solution for displacement and thermal stresses subject to the interface conditions as.

$$\begin{aligned}
 \sigma_{rr}^1(r_1) &= \sigma_{rr}^2(r_1) \quad \text{and} \quad k_1 \frac{\partial \sigma_{rr}^1(r_1)}{\partial r} = k_2 \frac{\partial \sigma_{rr}^2(r_1)}{\partial r} \\
 \sigma_{rr}^2(r_2) &= \sigma_{rr}^3(r_2) \quad \text{and} \quad k_2 \frac{\partial \sigma_{rr}^2(r_2)}{\partial r} = k_3 \frac{\partial \sigma_{rr}^3(r_2)}{\partial r}
 \end{aligned}$$

7. Numerical and Graphical Analysis

Layers	1	2	3
Material	Aluminum (pure)	Copper(pure)	Iron(pure)
k_i (W/mK)	204.2	386	72.7
ρ_i (kg/m ³)	2707	8954	7897
C_i (J/kgK)	896	383	452
$\alpha_i = k_i/\rho_i C_i$ (m ² /s)	$84.18 * 10^{-6}$	$112.34 * 10^{-6}$	$20.34 * 10^{-6}$
r_i (m)	$r_o = 0.02$ to $r_1 = 0.100$	$r_1 = 0.100$ to $r_2 = 0.200$	$r_2 = 0.200$ to $r_3 = 0.300$

Table 1. Three material and their thermal properties

k_i -Thermal conductivity, ρ_i -density, C_i -Specific heat capacity and α_i -Thermal diffusivity. The heat transfer coefficients at the outer surface r_3 are fixed to $h_3 = 90w/m^2k$.

m		1	2	3	4	5
Layer-1	β_{1m}	2.20227	10.5764	21.0745	25.3193	32.1144
	a_{1m}	1	1	1	1	1
	b_{1m}	-22.7045	-4.65571	-2.25	-1.8	-1.14286
Layer-2	β_{2m}	1.906373	9.155355	18.2429	21.9174	31.26207
	a_{2m}	0.7442	0.811374	1.02006	0.9828	0.412357
	b_{2m}	-23.8496	-2.12	-2.219	-1.6129	-1.05286
Layer-3	β_{3m}	4.480223	21.5103	42.87325	51.5087	73.4699
	a_{3m}	2.7502	7.8948	-4.1894	1.4654	1.90348
	b_{3m}	-12.9176	2.1969	-3.42645	-2.5451	1.59108

Table 2. Three Layers numerical values

The numerical calculation and graphs are obtained by using the MATLAB software. The radial temperature variation curve in Figure 2 show jump in derivative at the layer interfaces due to step change in material properties. Due to induced condition the temperature at the inner surface of the first layer are zero. In the first innermost layer and second layer temperature grows and then slowly decays in third layer to satisfy convective boundary condition at the outer surface. Figure 4 shows the radial stress distribution in the layers for different values of t . due to conditions, the radial stresses at the inner and outer surfaces of the hollow sphere are zero.

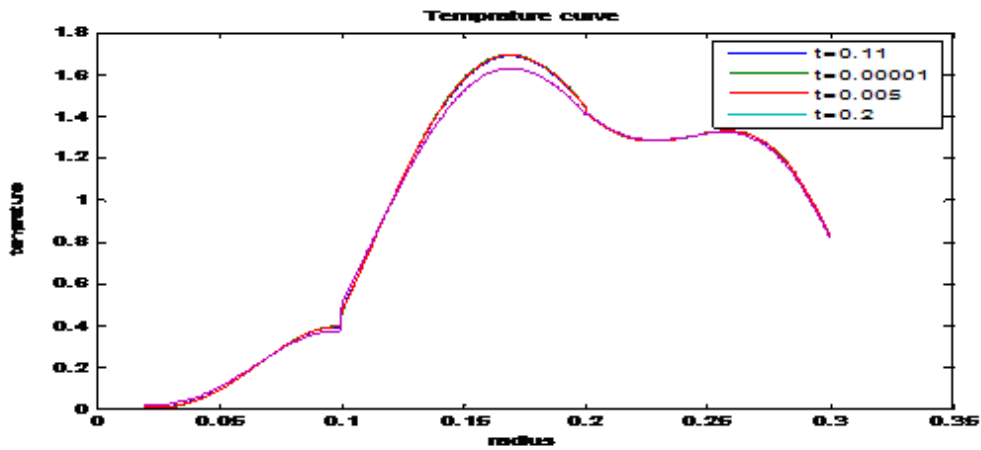


Figure 3. Transient Temperature distribution in radial direction

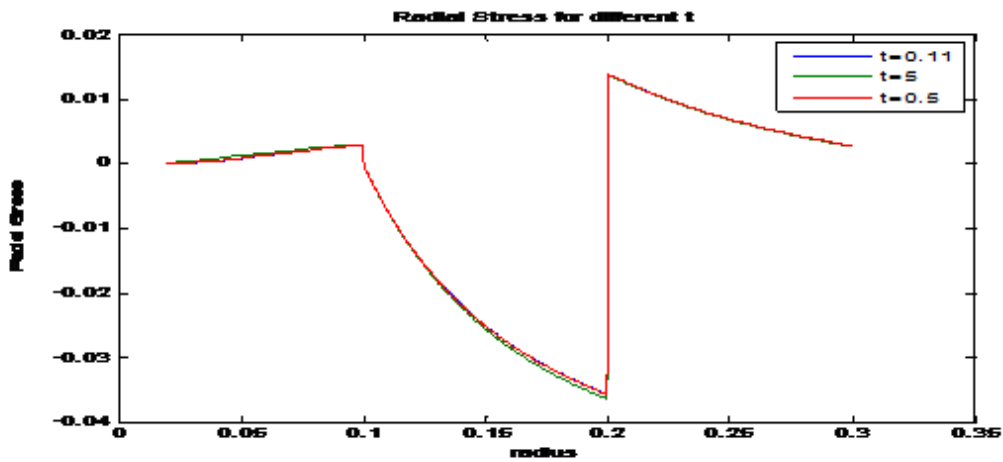


Figure 4. Radial stress variation for different values of t .

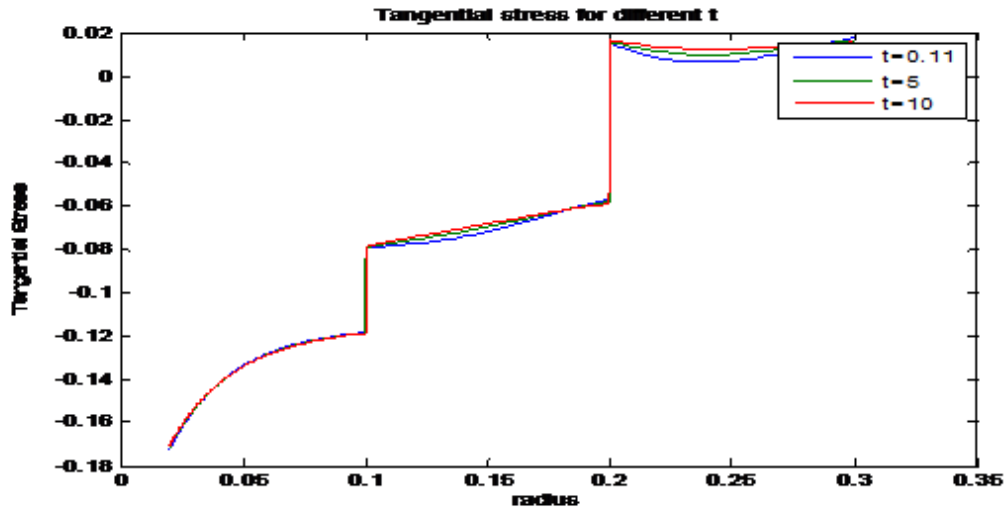


Figure 5. Tangential stress variation for different values of t .

8. Conclusion

This problem deals with determination of temperature, displacement, radial and tangential stress in a multilayer composite hollow sphere under unsteady temperature field. A sphere is considered having $f_i(r)$ initial temperature and heat is dissipated by convection from the boundary at $r = r_n$ into a surrounding temperature of which is varying with time. As a special case mathematical model is constructed for innermost layer contain Aluminum (pure) middle layer contain copper (pure) and outer layer contain Iron (pure) of a multilayer composite hollow sphere with thermophysical properties and functions and parameters specified as above. The transient temperature field and thermal stress profile is obtained in the multilayer composite hollow sphere. The layer wise temperature distribution for inner layer radii are $r \approx 0.02 - 0.1 (m)$, Middle layer radii are $r \approx 0.1 - 0.2 (m)$ outer layer radii are $r \approx 0.2 - 0.3 (m)$ are observed where the temperature is constant irrespective of time. The nature of the thermal stresses varies with either compressive or tensile. This model can be applied to sphere and to design useful structural applications. The proposed method may be readily extended to solve a wide range of physical engineering problems with change in the form of arbitrary initial and surrounding temperature. The results presented here are obtained after an extensive search and this is a new approach.

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