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On the Limitations of Complex Growth Rate in Triply Diffusive Convection in Porous Medium: Darcy-Brinkman Model

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Abstract: The paper mathematically establishes that the complex growth rate (p_r, p_i) of an arbitrary, neutral or unstable oscillatory perturbation of growing amplitude in triply diffusive convection analogous to stern type fluid layer heated from below, must lie inside a semicircle in the right half of the (p_r, p_i)-plane whose centre is at the origin and radius equals $\frac{1}{E\sigma}\left(\sqrt{|R|\,E\sigma - \frac{27}{4}p^4}\right)$, where R is the thermal Rayleigh number, s is the Prandtl number,E is a constant. Further this result is uniformly valid for quite general nature of bounding surfaces.

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1. Introduction

Research on Thermohaline instability in porous medium or more generally known as double diffusive convection in porous medium has been an area of great activity due to its importance in the predication of ground water movement in aquifers, in assessing the effectiveness of fibrous materials, in engineering geology and in nuclear engineering. Double diffusive convection is now well known. For a broad view of the subject one may be referred to Nield and Bezan [12], Murray and Chen [10], Nield [11], Taunton [32], Kuznetsov and Nield [8], Lombardo and Mulone [9], Basu and Layek [2].

All these researchers have considered two component systems. However, it has been recognized later that there are many fluid systems, in which more than two components are present. For example, Degens [3] reported that the saline waters of geothermally heated Lake kivu are strongly stratified by heat and a salinity which is the sum of comparable concentrations of many salts. Similarly the oceans contain many salts having concentrations less than a few percent of the sodium chloride concentration. Multi-component concentrations can also be found in magmas and substratum of water reservoirs. The subject with more than two components (in porous and non porous medium) has attracted the attention of many researchers. For the detailed study of multi-diffusive convection problem one may be referred to Grifiths [4, 5], Poulikakos [15], Pearlstein [14], Terrones and Pearlstein [29], Rudraiah and Vortmeyer [23], Lopez [7], Tracey [30, 31], Rionero ([20,21,22]), Straughan and Tracey [27], Terrones [28], Ryzhkov and Shevtsova [24]. The summary of the investigations of these researchers is that

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small salinity of a third component with a smaller mass diffusivity can have a significant effect upon the nature of convection; and 'oscillatory' and direct 'salt finger' modes are simultaneous possible under a wide range of conditions, when the density gradients due to components with greatest and smallest diffusivity are of same signs. Further, the problem of obtaining the upper limits for the linear growth rate of an arbitrary oscillatory motion of growing amplitude in various hydrodynamic stability problems is an important problem especially when atleast one boundary is rigid, so that exact solutions in the closed form are not obtainable and one has to depend on numerical solutions which are rather laborious.

Banerjee [1] derived such bounds for double diffusive convection problems. Later Gupta [6] extended their results to rotatory/magnetorotatory hydrodynamic double diffusive convection problems. Recently, Prakash et al. [16, 17, 18, 19] further extended these results to Triply diffusive, hydromagnetic, rotatory hydrodynamic and magneto rotatory hydrodynamic Triply diffusive convection problems respectively. In the present paper, we use their technique to derive upper bounds for the complex growth rate of an arbitrary, oscillatory perturbation of growing amplitude for Triply diffusive convection in porous medium, analogous to Stern (1960) type, using Darcy-Brinkman model. Further, these results are uniformly valid for any combination of rigid and free boundaries.

2. Mathematical Formulation and Analysis

A viscous finitely heat conducting Boussinesq fluid layer, saturating a porous medium, of infinite horizontal extension is statically confined between two horizontal boundaries $z = 0$ and $z = d$ which are respectively maintained at uniform temperatures T_0 and T_1 $(> T_0)$ and uniform concentrations S_{10}, S_{20} and S_{11} $(> S_{10})$, S_{21} $(> S_{20})$ (as shown in Figure 1). It is assumed that the saturating fluid and the porous layer are incompressible and that the porous medium is a constant porosity medium. It is further assumed that the cross-diffusion effects of the stratifying agencies can be neglected. The Darcy- Brinkman model has been used to investigate the triple diffusive convection in porous medium.

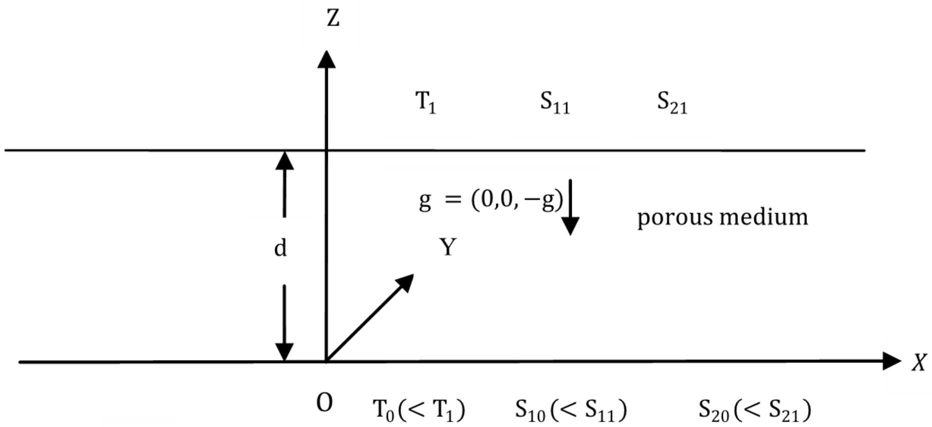

Figure 1. **Physical Configuration**

Non-dimensional hydrodynamical equations that govern the problem are given by (Vafai [30], Prakash [16])

$$\Lambda \left(D^2 - a^2\right)^2 w - \left(p + D_a^{-1}\right)\left(D^2 - a^2\right) w = -|R|\, a^2\theta + |R_1|\, a^2\phi_1 + |R_2|\, a^2\phi_2. \tag{1}$$

$$\left(D^2 - a^2 - E\sigma p\right)\theta = -w, \tag{2}$$

$$\left(D^2 - a^2 - \frac{E_1\sigma p}{\tau_1}\right)\phi_1 = -\frac{w}{\tau_1}, \tag{3}$$

$$\left(D^2 - a^2 - \frac{E_2\sigma p}{\tau_2}\right)\phi_2 = -\frac{w}{\tau_2}. \tag{4}$$

The equations (1)-(4) are to be solved by using the following boundary conditions:

$$w=\theta=\phi_1=\phi_2=Dw=0 \quad at \quad z=0 \quad and\ at \quad z=1, \quad \text{(when both the boundaries are rigid)} \tag{5}$$

$$or \quad w=\theta=\phi_1=\phi_2=D^2w=0 \quad at \quad z=0 \quad and\ at \quad z=1, \quad \text{(when both the boundaries are free)} \tag{6}$$

$$\left.\begin{array}{l} or \quad w=\theta=\phi_1=\phi_2=Dw=0 \quad at \quad z=0, \quad \text{(when lower boundary is rigid)} \\ and \quad w=\theta=\phi_1=\phi_2=D^2w=0 \quad at \quad z=1, \quad \text{(when upper boundary is free)} \end{array}\right\} \tag{7}$$

$$\left.\begin{array}{l} or \quad w=\theta=\phi_1=\phi_2=D^2w=0 \quad at \quad z=0, \quad \text{(when lower boundary is free)} \\ and \quad w=\theta=\phi_1=\phi_2=Dw=0 \quad at \quad z=1, \quad \text{(when upper boundary is rigid)} \end{array}\right\} \tag{8}$$

where z is the real independent such that $0 \le z \le 1$, D is the differentiation w.r.t. z, a^2 is square of the wave number, $\sigma = \frac{\nu\epsilon}{\kappa}$ is the Prandtl number, $\tau_1 = \frac{\kappa_1}{\kappa}$ and $\tau_2 = \frac{\kappa_2}{\kappa}$ are the Lewis numbers, $R = \frac{g\alpha\beta d^4}{\kappa\nu}$ is the thermal Rayleigh number, $R_1 = \frac{g\alpha_1\beta_1 d^4}{\kappa\nu}$ and $R_2 = \frac{g\alpha_2\beta_2 d^4}{\kappa\nu}$ are the two concentration Rayleigh numbers, $p = p_r + ip_i$ is the complex growth rate where p_r and p_i are the real constants, w is the vertical velocity, θ, is the temperature, ϕ_1 and ϕ_2 are the two concentrations. It may further be noted that in Equations (1)-(4) together with the boundary conditions (5)-(8) describe an eigenvalue problem for p and govern triply diffusive convection in porous medium for any combination of dynamically free and rigid boundaries.

3. Main Result

Theorem 3.1. *If* $(w, \theta, \phi_1, \phi_2, p)$, $p = p_r + ip_i$, $p_r \ge 0$ *is a solution of Equations* (1)-(8) *with* $R < 0$, $R_1 < 0$, $R_2 < 0$ *and* $\frac{|R|E\sigma}{2p^4} = 1$ *then* $p_i = 0$. *In particular* $p_r = 0$ *implies* $p_i = 0$, *if* $\frac{|R|E\sigma}{2p^4} = 1$.

Proof. Multiplying equation (1) by w^* (the superscript * henceforth denotes complex conjugation) on both sides and integrating over vertical range of z, we obtain

$$\Lambda \int_0^1 w^* \left(D^2 - a^2\right)^2 w\, dz - \left(p + D_a^{-1}\right) \int_0^1 w^* \left(D^2 - a^2\right) w dz = -|R|\, a^2 \int_0^1 w^*\theta dz + |R_1|\, a^2 \int_0^1 w^*\phi_1 dz + |R_2|\, a^2 \int_0^1 w^*\phi_2 dz. \tag{9}$$

Making use of Equations (2)-(4) and the fact that $w(0) = 0 = w(1)$, we can write

$$|R|\, a^2 \int_0^1 w^*\theta dz = |R|\, a^2 \int_0^1 \theta \left(D^2 - a^2 - E\sigma p^*\right) \theta^* dz, \tag{10}$$

$$|R_1|\, a^2 \int_0^1 w^*\phi_1 dz = -|R_1|\, a^2\tau_1 \int_0^1 \phi_1 \left(D^2 - a^2 - \frac{E_1\sigma p^*}{\tau_1}\right) \phi_1^* dz, \tag{11}$$

$$|R_2|\, a^2 \int_0^1 w^*\phi_2 dz = -|R_2|\, a^2\tau_2 \int_0^1 \phi_2 \left(D^2 - a^2 - \frac{E_2\sigma p^*}{\tau_2}\right) \phi_2^* dz. \tag{12}$$

Combining Equations (9)-(12), we obtain

$$\Lambda \int_0^1 w^* \left(D^2 - a^2\right)^2 w dz - \left(p + D_a^{-1}\right) \int_0^1 w^* \left(D^2 - a^2\right) w dz = |R|\, a^2 \int_0^1 \theta \left(D^2 - a^2 - E\sigma p^*\right) \theta^* dz$$
$$- |R_1|\, a^2\tau_1 \int_0^1 \phi_1 \left(D^2 - a^2 - \frac{E_1\sigma p^*}{\tau_1}\right) \phi_1^* dz\, |R_{21}|\, a^2\tau_2 \int_0^1 \phi_2 \left(D^2 - a^2 - \frac{E_2\sigma p^*}{\tau_2}\right) \phi_2^* dz. \tag{13}$$

Integrating various terms of equation (13), by parts, for an appropriate number of times and making use of either of the boundary conditions (5)-(8), it follows that

$$\Lambda \int_0^1 \left(\left|D^2 w\right|^2 + 2a^2 |Dw|^2 + a^4 |w|^2\right) dz + \left(p + D_a^{-1}\right) \int_0^1 \left(|Dw|^2 + a^2 |w|^2\right) dz$$
$$= -|R|\, a^2 \int_0^1 \left(|D\theta|^2 + a^2 |\theta|^2 + E\sigma p^* |\theta|^2\right) dz + |R_1|\, a^2\tau_1 \int_0^1 \left(|D\phi_1|^2 + a^2 |\phi_1|^2 + \frac{E_1\sigma p^*}{\tau_1} |\phi_1|^2\right) dz \tag{14}$$
$$+ |R_2|\, a^2\tau_2 \int_0^1 \left(|D\phi_2|^2 + a^2 |\phi_2|^2 + \frac{E_2\sigma p^*}{\tau_2} |\phi_2|^2\right) dz$$

Equating imaginary parts of both sides of equation (14) and cancelling $p_i(\neq 0)$ throughout, we have

$$\int_0^1 \left(|Dw|^2 + a^2 |w|^2\right) dz = |R| a^2 E\sigma \int_0^1 |\theta|^2 dz - |R_1| a^2 E_1\sigma \int_0^1 |\phi_1|^2 dz - |R_2| a^2 E_2\sigma \int_0^1 |\phi_2|^2 dz. \quad (15)$$

We first note that since w, θ, ϕ_1 and ϕ_2 satisfy $w(0) = 0 = w(1)$, $\theta(0) = 0 = \theta(1)$, $\phi_1(0) = 0 = \phi_1(1)$, $\phi_2(0) = 0 = \phi_2(1)$, we have by the Rayleigh-Ritz inequality [22].

$$\int_0^1 |Dw|^2 dz = \pi^2 \int_0^1 |w|^2 dz \quad (16)$$

$$\int_0^1 |D\theta|^2 dz = \pi^2 \int_0^1 |\theta|^2 dz \quad (17)$$

$$\int_0^1 |D\phi_1|^2 dz = \pi^2 \int_0^1 |\phi_1|^2 dz \quad (18)$$

$$\int_0^1 |D\phi_2|^2 dz = \pi^2 \int_0^1 |\phi_2|^2 dz \quad (19)$$

Using inequality (16), we have

$$\int_0^1 \left(|Dw|^2 + a^2 |w|^2\right) dz \geq \left(\pi^2 + a^2\right) \int_0^1 |w|^2 dz. \quad (20)$$

Combining Equation (15) and inequality (20), we get

$$|R| a^2 Es \int_0^1 |\theta|^2 dz \geq \left(\pi^2 + a^2\right) \int_0^1 |w|^2 dz. \quad (21)$$

Now, multiplying equation (2) by its complex conjugate and integrating the resulting equation for a suitable number of times and use the boundary condition on θ namely, $\theta(0) = 0 = \theta(1)$, we obtain

$$\int_0^1 \left(\left|D^2\theta\right|^2 + 2a^2 |D\theta|^2 + a^4 |\theta|^2\right) dz + 2E\sigma p_R \int_0^1 \left(|D\theta|^2 + a^2 |\theta|^2\right) dz + E^2\sigma^2 |p|^2 \int_0^1 |\theta|^2 dz = \int_0^1 |w|^2 dz. \quad (22)$$

Now,

$$\begin{aligned} \int_0^1 |D\theta|^2 dz &= -\int_0^1 \theta^* D^2\theta dz & (\because\ \theta(0) = 0 = \theta(1)) \\ &\leq \left|-\int_0^1 \theta^* D^2\theta dz\right|, \\ &\leq \int_0^1 \left|\theta^* D^2\theta\right| dz, \\ &\leq \left[\int_0^1 |\theta|^2\ dz\right]^{\frac{1}{2}} \left[\int_0^1 \left|D^2\theta\right|^2 dz\right]^{\frac{1}{2}}, & \text{(using Schwartz inequality)} \\ &\leq \frac{1}{\pi} \left[\int_0^1 |D\theta|^2\ dz\right]^{\frac{1}{2}} \left[\int_0^1 \left|D^2\theta\right|^2\ dz\right]^{\frac{1}{2}}, & \text{(using inequality (17))} \end{aligned}$$

so that we have,

$$\int_0^1 \left|D^2\theta\right|^2 dz \geq \pi^2 \int_0^1 |D\theta|^2 dz. \quad (23)$$

Combining inequalities (17) and (23), we get

$$\int_0^1 \left|D^2\theta\right|^2 dz \geq \pi^4 \int_0^1 |\theta|^2 dz. \quad (24)$$

From inequalities (17) and (24) it follows that

$$\int_0^1 \left(\left|D^2\theta\right|^2 + 2a^2 |D\theta|^2 + a^4 |\theta|^2\right) dz \geq \left(\pi^2 + a^2\right)^2 \int_0^1 |\theta|^2 dz. \quad (25)$$

Since $p_r = 0$, we get from Equation (22) and inequality (25), that

$$\left(\pi^2 + a^2\right)^2 \int_0^1 |\theta|^2 \, dz + E^2\sigma^2 |p|^2 \int_0^1 |\theta|^2 \, dz < \int_0^1 |w|^2 \, dz, \tag{26}$$

which can be rearranged as

$$\int_0^1 |\theta|^2 \, dz < \frac{1}{E^2\sigma^2 |p|^2} \int_0^1 |w|^2 \, dz - \frac{\left(p^2 + a^2\right)^2}{E^2\sigma^2 |p|^2} \int_0^1 |\theta|^2 \, dz. \tag{27}$$

Combining inequalities (21) and (27), we obtain

$$|R| \, a^2 \int_0^1 |\theta|^2 \, dz < \frac{a^2}{E^3\sigma^3 |p|^2} \left[|R| \, E\sigma - \frac{\left(p^2 + a^2\right)^3}{a^2} \right] \int_0^1 |w|^2 \, dz \tag{28}$$

Use inequality (28) in Equation (15), we get

$$\int_0^1 |Dw|^2 \, dz + a^2 \left[1 - \frac{1}{E^2\sigma^2 |p|^2} \left\{ |R| \, E\sigma - \frac{27}{4}\pi^4 \right\} \right] \int_0^1 |w|^2 \, dz + |R_1| \, a^2 E_1\sigma \int_0^1 |\phi_1|^2 \, dz + |R_2| \, a^2 E_2\sigma \int_0^1 |\phi_2|^2 \, dz < 0, \tag{29}$$

since the minimum value of $\frac{\left(\pi^2+a^2\right)^3}{a^2}$ is $\frac{27}{4}\pi^4 \left(for \ a^2 = \frac{\pi^2}{2} \right)$. Inequality (29) clearly implies that

$$|p|^2 < \frac{1}{E^2\sigma^2} \left(|R| \, E\sigma - \frac{27}{4}\pi^4 \right).$$

This establishes the desired result. □

The above theorem may be stated in an equivalent form as: the complex growth rate of an arbitrary, neutral or unstable oscillatory perturbation of growing amplitude in triply diffusive convection in porous medium analogous to Stern (1960) type, must lie inside a semicircle in the right half of the (p_r, p_i)-plane whose centre is at the origin and radius equals $\frac{1}{E\sigma} \left(\sqrt{|R| \, E\sigma - \frac{27}{4}\pi^4} \right)$. Further this result is uniformly valid for quite general nature of bounding surfaces.

4. Conclusion

Linear stability theory is used to derive the upper limits for complex growth rate in triply diffusive convection problem in porous medium analogous to Stern (1960) type using Darcy-Brinkman model. These limits are important especially when both the boundaries are not dynamically free so that exact solutions in the closed form are not obtainable. Further, the results so obtained are uniformly for all the combinations of rigid and free boundaries.

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