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Modelling of Synthetic Drugs Transmission with Psychological Addicts

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Abstract: The synthetic drugs that are making the rounds of drug-using markets are some of the most destructive drugs available today, which have given rise to serious social issues in recent years. In this paper, a synthetic drugs transmission model with psychological addicts and Holling Type II functional responses is proposed. The basic reproduction number R_0 is obtained by the method of next generation matrix. Then we have discussed the local and global stabilities analysis of the model using the basic reproduction number R_0. By analyzing the sensitivity of parameters, we obtain that controlling psychological addiction is better than drugs treatment. Computer simulation are carried out to validate our analytical findings. The biological implications of analytical and numerical findings are discussed critically.

MSC: 34D20, 34D23.

Keywords: Synthetic drugs, Basic reproduction number, Local and Global stability, Sensitivity analysis.

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1. Introduction

Some drugs are natural, meaning that the plants from which they are derived exist in nature except any help from humankind: opium poppies (heroin, morphine, codeine), coca leaves (cocaine), psilocybin mushrooms (shrooms), and marijuana. Another drugs are synthetic, which means they are created using man-made chemicals, not natural ingredients. New synthetics appear on the market constantly, therefore it is not possible to compile list. Chemists who choice to evade arrest can simply shift the formula slightly and come up with something new that might not be listed in the text of laws that ban drugs. For illustration, K2 also known as Spice, Ecstasy also known as Molly, and bath salts are all types of synthetic drugs. Whereas synthetic drugs are created in illegal labs to bypass regulations prohibiting controlled substances, their strength, composition, and ingredients are unknown to the consumer. Fun names and colorful appearances can sometimes mask their extreme potential for harm, however make no mistake about it: synthetic drugs are extremely dangerous and can cause addiction, severe health issues, and even death. Synthetic drugs began appearing in the United States around 2009, and their popularity has surged in the eight years since then. They are especially popular amongst teenagers due to their high level of accessibility (source). Actually, according to a study conducted in 2013 by the Center for Substance Abuse Research (CESAR) at the University of Maryland at College Park, about 12 % of high school students said they used synthetic drugs regularly (source). Many of these teen-ager arent aware of how dangerous these substances can be.

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In [18], drug-users are more infected with many infectious diseases such as hepatitis and AIDS. The data from South African police statistics showed that only drug-related crimes increased from 621 to more than 3,000 between 2002 and 2006 in Cape Town. In addition, the HIV infections increased from 13.1% in 2003 to 15.7% in 2005 in the Western Cape Province of South Africa from the 2005 antenatal survey [17]. At recent, a survey reflects the number of the synthetic drug-users appears an upward trend. For instance, by the end of 2015, there were 2.354 million drugusers in China, including 1.34 million synthetic drug abusers which accounted for 57.1% and increased by 7.7% over 2014 [33]. we know that, there is no mature treatment of drugs addiction around the world. Like whole addiction diseases, the major treatment measures for drug addicts are medications and psychotherapy.

The principal objective of modeling epidemics is to simulate and reveal the rule of epidemics and provide theoretical basis for preventing and controlling diseases, see [6, 11, 12, 16, 21, 22, 26, 27, 29–34] and references cited therein. The papers [24, 31] studied a general heroin model. In this paper, authors divided people into susceptible, heroin addicts not in treatment and heroin addicts in treatment, and took into account repeated use. Also we have considered Holling type-II functional response for the species susceptible (S), psychological addicts (P_1) and susceptible (S), physiological addicts (P_2). Mulone et al. [26] also discussed a similar heroin model which has treated users and standard incidence rate, the result shows that the steady states of their model are stable. In [30], Samanta et al. discussed a non-autonomous drugs model and obtained the sufficient conditions of global asymptotic stability of the model. Fang et al. [11] established a model with two distribute delays and proved the global asymptotic stabilities of steady states by Lyapunov function. In the above models, all authors discussed traditional drugs. However, for the synthetic drugs, the relevant research is less [24, 27, 29]. Mushanyu et al. [27] considered a mathematical model to explain the methamphetamine transmission of Western Cape province of South Africa. Whereas of eating by mistake, fascinated by friends, the accidental drug-users may fill with guilt after first taking little drug. By the psychological effect, many of them will avoid contacting with synthetic drugs consciously once again, and usually cannot be addictive. So, it is essential to distinguish the effect of synthetic drugs transmission caused by psychology and physiology. We have considered the factor, we attempt to imitate the transmission of synthetic drugs in this paper, and study the dynamics of the model to seek some workable measures for controlling the drugs epidemic.

The rest of the paper is organized as follows. we first establish a synthetic drugs model with psychological and physiological addicts. In section 2 and 3, the basic mathematical model is introduced together with basic considerations. Boundedness and positivity of the solutions of the proposed model are established in section 4. In Section 5, the basic reproduction number (R_0) of the system is calculated using next generation matrix method. Then the deterministic dynamical behaviors of the system are studied. Our study includes the existence and stability analysis of equilibrium points of the system. Section 7 contains the sensitivity analysis of our system which helps us to find out the parameters of greater interest. Computer simulations are carried out to validate our analytical findings numerically in section 8. Section 9 contains the general discussion and biological significance of our analytical findings. In the last section, we end our paper with some reference.

2. The Mathematical Model

Synthetic drugs, also referred to as designer or club drugs, are chemically-created in a lab to mimic another drug such as marijuana, cocaine or morphine. The synthetic drugs show a stronger psychological dependence and spread mainly in youths. In this section, we divide the human population (N) into four classes at time t: susceptible (S), psychological addicts (P_1), physiological addicts (P_2) and addicts in treatment (T). Also we have considered Holling type-II functional response for the species (S, P_1) and (S, P_2). There are data to show that drug users age mainly focus on $15 - 64$ years old [19], therefore the rate of entering this age group every year is roughly equivalent to the recruitment rate of susceptible

denoting Λ which is assumed a constant. After contacting with a drug addict, the susceptible one will first move into the psychological addict class, while after taking plenty of drugs, the psychological ad- dict will become the physiological addict. In generally, a susceptible one is more likely to initiate drug abuse when he contacts with a physiological addict compared to a psychological addict. Here we denote the corresponding contact rates are $\beta_1(N)$ and $\beta_2(N)$. Obviously, the contact rate $\beta_1(N)$ is greater than or equal to $\beta_(N)$. For simplification and analysis sakes, here we make an assumption that there are same probabilities to make susceptible become drug-user when he contacts with psychological and physiological addict. We bring $\beta(N) = \beta_(N) = \beta_2(N)$ into our model as the effective contact rate, where $\beta(N)$ is a function of total population N. Once psychological and physiological addicts accept treatment and rehabilitation, they will enter into treatment compartment. The treatment rates are denoted by γ and ρ respectively. Next we consider, some drug-users in treatment may escape and reenter physiological with rate η. Then The model can be represented by the following set of ordinary differential equations:

$$\begin{aligned}
\frac{dS}{dt} &= \Lambda - \delta S - \frac{\beta(N)S(P_1 + bP_2)}{A + S} \\
\frac{dP_1}{dt} &= \frac{\beta(N)S(P_1 + bP_2)}{A + S} - \alpha P_1 - (\delta + \gamma)P_1 \\
\frac{dP_2}{dt} &= \alpha P_1 + \eta T - \rho P_2 - \delta P_2 \\
\frac{dT}{dt} &= \gamma P_1 + \rho P_2 - \eta T - \delta T
\end{aligned} \tag{1}$$

where the meanings of S, P_1, P_2, T and N are predented concisely in Table 1. Also A is the half saturation constant and all other parameters $\alpha, \delta, \gamma, \eta, \rho$ are positive constants, and their definitions are given in Table 2. In this paper, we enumerate different contact rates related to the total population N in Table 3, which interpretations and meanings can be found in relevant references. Here, we list them no longer.

Variable	Description
$S(t)$	Number of susceptible humans at time t.
$P_1(t)$	Number of psychological addicts at time t.
$P_2(t)$	Number of physiological addicts at time t.
$T(t)$	Number of drug-users in treatment at time t.
$N(t)$	Total human population at time t.

Table 1. **The state variables for this model** (1)

3. Analysis

From the system (1), by adding all equations, we get $\frac{dN}{dt} = \Lambda - dN$. Whereas $N \to \frac{\Lambda}{\delta}$ when $t \to +\infty$. We assume that the total number $N = \frac{\Lambda}{\delta}$. Let us non-dimensionalize the system (2.1) with the following scaling: $s = \frac{S}{N}$, $p_1 = \frac{P_1}{N}$, $p_2 = \frac{P_2}{N}$ and $t_1 = \frac{T}{N}$. Then the system (1) takes the form (after some simplification):

$$\begin{aligned}
\frac{ds}{dt} &= \delta - \delta s - \frac{\beta(N)s(p_1 + bp_2)}{a + s} \\
\frac{dp_1}{dt} &= \frac{\beta(N)s(p_1 + bp_2)}{a + s} - \alpha p_1 - (\delta + \gamma)p_1 \\
\frac{dp_2}{dt} &= \alpha p_1 + \eta t_1 - \rho p_2 - \delta p_2 \\
\frac{dt_1}{dt} &= \gamma p_1 + \rho p_2 - \eta t_1 - \delta t_1
\end{aligned} \tag{2}$$

with the initial conditions $s(0) = s_0 \geq 0$, $p_1(0) = p_{1_0} \geq 0$, $p_2(0) = p_{2_0} \geq 0$, $t_1(0) = t_{1_0} \geq 0$, where $a = AN$.

Parameter	Description and dimensions	Range	Reference
Λ	Immigration rate of susceptible. $Humans \times year^1$	[1-1.2]	[29]
β	Probability of transmission from susceptible to drug addicts. Dimensionless	[0-0.9399]	[17]
δ	Natural death rate of humans. $year^1$	[0-0.025]	[17]
α	Escalation rate from psychological addicts to physiological addicts. $year^1$	[0.0015-0.5]	[17]
γ	Per capita treatment rate for psychological addicts. $year^1$	[0-0.3]	[17]
ρ	Per capita treatment rate for physiological addicts. $year^1$	[0-0.99]	[17]
η	Relapse rate. $year^1$	[0.00002-0.9]	[17]

Table 2. **Description of parameters and their dimensions.**

In the remaining part, we will analysis model (2) with initial conditions to establish the necessary and sufficient conditions of global asymptotic stabilities of synthetic drug-free and synthetic drug addiction equilibria. Before beginning, we must give the following theorem about positivity and boundedness of solutions for the system (2).

4. Basic Properties

4.1. Non-negativity of solutions

Theorem 4.1. *All solutions of the system (2) that start in $\mathbb{R}^4_+$ are positive forever.*

Proof. From the first equation of system (2), we get

$$s(t) = s(0)\exp\left[\int_0^t -\{\delta + \frac{\beta(N)(p_1(\omega)+bp_2(\omega))}{a+s(\omega)}\}d\omega\right] + \delta\int_0^t \exp\left[\int_\theta^t -\{\delta + \frac{\beta(N)(p_1(\omega)+bp_2(\omega))}{a+s(\omega)}\}d\omega\right]d\theta \Rightarrow s(t) > 0.$$

From the second equation of system (2), we get

$$p_1(t) = e^{-(\alpha+\delta+\gamma-\frac{\beta(N)s}{a+s})t}\left[p_1(0) + \int_0^t e^{(\alpha+\delta+\gamma-\frac{\beta(N)s}{a+s})\omega}\frac{\beta(N)s(\omega)bp_2(\omega)}{a+s(\omega)}d\omega\right] \Rightarrow p_1(t) > 0.$$

From the third equation of system (2), we get

$$p_2(t) = e^{-(\rho+\delta)t}\left[p_2(0) + \int_0^t e^{(\rho+\delta)\omega}\{\alpha p_1(\omega) + \eta t_1(\omega)\}d\omega\right] \Rightarrow p_2(t) > 0.$$

From the fourth equation of system (2), we get

$$t_1(t) = e^{-(\eta+\delta)t}\left[t_1(0) + \int_0^t e^{(\eta+\delta)\omega}\{\gamma p_1(\omega) + \rho p_2(\omega)\}d\omega\right] \Rightarrow t_1(t) > 0.$$

This proves the theorem. □

4.2. Invariant region

Theorem 4.2. *The feasible region Γ defined by*

$$\Gamma = \{(s(t), p_1(t), p_2(t), t_1(t)) \in \mathbb{R}^4_+ : s(t) + p_1(t) + p_2(t) + t_1(t) = 1\}$$

with initial conditions $s(0) \geq 0$, $p_1(0) \geq 0$, $p_2(0) \geq 0$, $t_1(0) \geq 0$ is positively invariant.

Proof. Adding all the equations of system (2) and denoting $\hat{N} = s(t) + p_1(t) + p_2(t) + t_1(t)$, then we obtain

$$\frac{d\hat{N}}{dt} = \delta - \delta\hat{N}_1.$$

Since $\hat{N}_1(0) = s(0) + p_1(0) + p_2(0) + t_1(0) > 0$. By after some calculation $\hat{N}(t) = 1(1 - \hat{N}(0))e^{-\delta t}$ is the solution of the system. Obviously, $\lim_{t\to\infty}\hat{N}(t) = 1$ which implies the result. This completes the proof. □

5. The Basic Reproduction Number

Basic reproduction number [1, 4, 5, 7, 10, 14, 15, 28] is defined as the number of new number of psychological addicts individuals produced by a single psychological addicts individual during his or her effective psychological addicts period when introduced into susceptible population. In the following the basic reproduction number R_0 of system (2) will be obtained by the next generation matrix method formulation [10]. Rewrite system (2), denote $x = (p_1, p_2, t_1, s), a_0 = \alpha + \delta + \gamma, a_1 = \rho + \delta, a_2 = \eta + \delta$, then

$$\frac{dx}{dt} = F - V = \begin{pmatrix} \frac{\beta(N)s(p_1+bp_2)}{a+s} \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} a_0 p_1 \\ -\alpha p_1 - \eta t_1 + a_1 p_2 \\ -\gamma p_1 - \rho p_2 + a_2 t_1 \\ -\delta + \delta s + \frac{\beta(N)s(p_1+bp_2)}{a+s} \end{pmatrix}$$

where F is the new infection terms, V is the other terms. The corresponding linearized matrixes of F and V evaluated at E_0 are

$$F = \begin{pmatrix} \frac{\beta(N)}{a+1} & \frac{\beta(N)b}{a+1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, V = \begin{pmatrix} a_0 & 0 & 0 & 0 \\ -\alpha & a_1 & -\eta & 0 \\ -\gamma & -\rho & a_2 & 0 \\ \frac{\beta_1(N_1)}{a+1} & \frac{\beta(N)b}{a+1} & 0 & \delta \end{pmatrix}$$

Now, FV^{-1} is the next generation matrix of system (2). It follows that the spectral radius of matrix FV^{-1} denoted and defined by [10] which is the basic reproduction number of system (2) is given as follows:

$$R_0 = r(FV^{-1}) = \frac{\beta(N)\{a_1a_2 - \eta\rho + b(\alpha a_2 + \eta\gamma)\}}{(a+1)a_0(a_1a_2 - \eta\rho)} = \beta(N)R_0^*,$$

where $R_0^* = \frac{\{a_1a_2-\eta\rho+b(\alpha a_2+\eta\gamma)\}}{(a+1)a_0(a_1a_2-\eta\rho)}$. If taking several contact rates, we can get different basic reproduction numbers of model (2). In Table 3, we list some basic reproduction numbers for several effective contact rates $\beta(N)$ of synthetic drugs model (2).

Effective contact rate	Expression	Reference	Basic reproduction number
Bilinear	β	[2, 25]	$\beta N R_0^*$
Standard	$\frac{\beta}{N}$	[2, 25]	βR_0^*
Michaelis-Menten (M-M)	$\frac{AN}{1+\beta N}$	[8]	$\frac{AN^2}{1+\beta N} R_0^*$
Heesterbeek-Metz (H-M)	$\frac{AN}{1+\beta N+\sqrt{1+2\beta N}}$	[13]	$\frac{AN^2}{1+\beta N+\sqrt{1+2\beta N}} R_0^*$
Mena-Lorca-Metz (M-L-M)	λN^A	[23]	$\lambda N^{A+1} R_0^*$

Table 3. **Basic reproduction numbers of several effective contact rates of synthetic drug-users model**

By Theorem 4.2 in [10], the following result is obtained directly.

Theorem 5.1. *The synthetic drug-free equilibrium E_0 is locally asymptotically stable (unstable) if $R_0 < 1 (R_0 > 1)$.*

6. Equilibrium Points: Their Existence and Stability

In this section, we will study the existence and stability behavior of the system (2) at several equilibrium points. The equilibrium points of the system (1) are:

(1). Drug-free equilibrium (DFE): $E_0(1, 0, 0, 0)$.

(2). Drug addiction equilibrium: $E_1(s^*, p_1^*, p_2^*, t_1^*)$.

6.1. Existence of unique drug addiction equilibrium $E_1(s^*, p_1^*, p_2^*, t_1^*)$

In this section, we will analyze the existence of non-trivial drug addiction equilibrium $E_1(s^*, p_1^*, p_2^*, t_1^*)$ of the system (2). To find the unique drug addiction equilibrium of system (1), we consider the following:

$$s > 0, p_1 > 0, p_2 > 0, t_1 > 0 \text{ and } \frac{ds}{dt} = \frac{dp_1}{dt} = \frac{dp_2}{dt} = \frac{dt_1}{dt} = 0. \tag{3}$$

Solving the equations of system (3), we get:

$$\begin{aligned} s^* &= \frac{a_0 a(a_1a_2 - \eta\rho)}{\beta(N)\{a_1a_2 - \eta\rho + ba_2\alpha + b\eta\gamma - a_0(a_1a_2 - \eta\rho)\}}, p_1^* = \frac{(a_1a_2 - \eta\rho)p_2^*}{a_2\alpha + \eta\gamma}, \\ p_2^* &= \frac{(\delta - \delta s^*)(a + s^*)(a_2\alpha + \eta\gamma)}{\beta(N)s^*(a_1a_2 - \eta\rho + ba_2\alpha + b\eta\gamma)} \quad \text{and} \\ t_1^* &= \frac{(a_1\gamma + \alpha\rho)p_2^*}{\eta\gamma + a_2\alpha}. \end{aligned}$$

Next we will prove the local and global asymptotic stabilities of equilibria E_0 and E^*.

6.2. Local stability analysis

In this section we will study the local stability of the system (2) at drug-free equilibrium $E_0(1, 0, 0, 0)$. The variational matrix of system (2) at $E_0(1, 0, 0, 0)$ is given by

$$V(E_0) = \begin{bmatrix} -\delta & 0 & 0 & 0 \\ 0 & -a_0 & 0 & 0 \\ 0 & \alpha & -a_1 & \eta \\ 0 & \gamma & \rho & -a_2 \end{bmatrix}$$

Therefore, eigenvalues of the characteristic equation of $V(E_0)$ are

$$\lambda_1 = -\delta, \lambda_2 = -(\alpha + \delta + \gamma) \text{ and } \lambda_{3,4} = \frac{-(a_1 + a_2) \pm \sqrt{(a_1 + a_2)^2 - 4(a_1a_2 - \eta\rho)}}{2}.$$

Here, λ_1 and λ_2 are clearly real and negative. Now, E_0 is stable if λ_3 and $\lambda_4 < 0$ i.e. $a_1a_2 - \eta\rho > 0$ when, $R_0 < 1$. Therefore the system (2) is local asymptotically stability at $E_0(1, 0, 0, 0)$ if $R_0 < 1$. So, we arrive to the following result:

Theorem 6.1. *The synthetic drug-free equilibrium E_0 of system (2) is locally asymptotically stable if $R_0 < 1$ and $a_1a_2 - \eta\rho > 0$ are satisfied.*

Now, the variational matrix of system (2) at $E_1(s^*, p_1^*, p_2^*, t_1^*)$ is given by

$$V(E_1) = \begin{bmatrix} -\delta - \frac{\beta(N)a(p_1 + bp_2)}{(a+s)^2} & -\frac{\beta(N)s}{a+s} & -\frac{\beta(N)sb}{a+s} & 0 \\ \frac{\beta(N)a(p_1 + bp_2)}{(a+s)^2} & \frac{\beta(N)s}{a+s} - a_0 & \frac{\beta(N)sb}{a+s} & 0 \\ 0 & \alpha & -a_1 & \eta \\ 0 & \gamma & \rho & -a_2 \end{bmatrix}.$$

Therefore, the characteristic equation of $V(E_1)$ is given by

$$\lambda^4 + B_1\lambda^3 + B_2\lambda^2 + B_3\lambda + B_4 = 0 \tag{4}$$

where

$$u = \frac{\beta(N)a(p_1 + bp_2)}{(a+s)^2},\ v = \frac{\beta(N)s}{a+s},\ w = a_1a_2 - \eta\gamma,\ B_1 = a_0 + a_1 + a_2 - v - u - \delta,$$

$$B_2 = (\delta - u)(v - a_0 - a_1 - a_2) + a_0(a_1 + a_2) + w - b\alpha v + uv,$$

$$B_3 = (\delta - u)(a_1y + a_2y - a_0a_1 - a_0a_2 - w + b\alpha v) - vw + a_0w - a_2b\alpha v - b\eta\gamma v + a_1uv + a_2uv + buv\alpha \text{ and}$$

$$B_4 = (\delta - u)(wv + a_2b\alpha v + b\eta\gamma v) + wuv + buva_2\alpha + buv\eta\gamma.$$

By the Routh-Hurwitz criterion [20] it follows that all eigenvalues of the characteristic equation (4) have negative real part if and only if:

$$\begin{aligned} &(i) \quad B_1 > 0 \\ &(ii) \quad B_3 > 0 \\ &(iii) \quad B_4 > 0 \text{ and} \\ &(iv) \quad B_1B_2B_3 > B_3^2 + B_1^2B_4. \end{aligned} \tag{5}$$

Therefore the system (2) shows local asymptotic stability at E_1 when $R_0 > 1$ which guarantees the existence of E_1 and conditions (5) are satisfied. So, we arrive to the following result.

Theorem 6.2. *The synthetic drug addiction equilibrium E_1 of system (2) is locally asymptotically stable if $R_0 > 1$ and conditions (5) are satisfied.*

6.3. Global stability analysis

Theorem 6.3. *The synthetic drug-free equilibrium E_0 of system (5) is globally asymptotically stable if $R_0 \leq 1$.*

Proof. Let (s, p_1, p_2, t_1) be a positive solution of the system (2) and let us consider the following positive definite function about E_0 :

$$L = \frac{1}{a+1}p_1 + \frac{\beta(N)a_2b}{(a+1)(a_1a_2 - \eta\rho)}p_2 + \frac{\beta(N)\eta b}{(a+1)(a_1a_2 - \eta\rho)}t_1.$$

Obviously, $L(E_0) = 0$ and $L(s, p_1, p_2, t_1) > 0$ in Γ. Because of $0 < s < 1$ in Γ and $R_0 \leq 1$, by simplifying the time derivative of L along the solutions of the system (2), we obtain

$$\begin{aligned} \frac{dL}{dt} &= \frac{1}{a+1}\frac{dp_1}{dt} + \frac{\beta(N)a_2b}{(a+1)(a_1a_2 - \eta\rho)}\frac{dp_2}{dt} + \frac{\beta(N)\eta b}{(a+1)(a_1a_2 - \eta\rho)}\frac{dt_1}{dt} \\ &= \frac{\beta(N)s(p_1 + bp_2)}{(a+s)(a+1)} - \frac{a_0p_1}{a+1} + \frac{\beta(N)a_2b}{(a+1)(a_1a_2 - \eta\rho)}(\alpha p_1 + \eta t_1 - a_1p_2) + \frac{\beta(N)\eta b}{(a+1)(a_1a_2 - \eta\rho)}(\gamma p_1 + \rho p_2 - a_2t_1) \\ &\leq \frac{\beta(N)(p_1 + bp_2)}{(a+1)} - \frac{a_0p_1}{a+1} + \frac{\beta_1(N_1)a_2b}{(a+1)(a_1a_2 - \eta\rho)}(\alpha p_1 + \eta t_1 - a_1p_2) \\ &+ \frac{\beta(N)\eta b}{(a+1)(a_1a_2 - \eta\rho)}(\gamma p_1 + \rho p_2 - a_2t_1) \\ &= a_0\{\frac{\beta(N)}{a+1} - \frac{1}{a+1} + \frac{\beta(N)b(a_2\alpha + \eta\gamma)}{(a+1)(a_1a_2 - \eta\rho)}\}p_1 \\ &= a_0(R_0 - 1)p_1 \\ &\leq 0, \end{aligned}$$

and $\frac{dL}{dt}$ if and only if $R_0 = 1$. Consequently, by the Lyapunov Theorem [9], the drug-free equilibrium E_0 is globally asymptotically stable if $R_0 \leq 1$. □

Theorem 6.4. *The synthetic drug addiction equilibrium $E_1(s^*, p_1^*, p_2^*, t_1^*)$ of system (2) is globally asymptotically stable if $R_0 > 1$.*

Proof. Let us consider the positive solution of the system (2) about $E_1(s^*, p_1^*, p_2^*, t_1^*)$ and construct the following Lyapunov function:

$$L = (s - s^* - s^* ln\frac{s}{s^*}) + (p_1 - p_1^* - p_1^* ln\frac{p_1}{p_1^*}) + \frac{\beta(N)a_2 s^*}{a_1a_2 - \eta\rho}\left(p_2 - p_2^* - p_2^* ln\frac{p_2}{p_2^*}\right) + \frac{\beta(N)\eta s^*}{a_1a_2 - \eta\rho}(t_1 - t_1^* - t_1^* ln\frac{t_1}{t_1^*}).$$

Differentiating L with respect to t along the solution of (2), a little algebraic manipulation yields

$$\begin{aligned}\frac{dL}{dt} &= (1 - \frac{s^*}{s})\frac{ds}{dt} + (1 - \frac{p_1^*}{p_1})\frac{dp_1}{dt} + \frac{\beta(N)a_2 s^*}{a_1a_2 - \eta\rho}(1 - \frac{p_2^*}{p_2})\frac{dp_2}{dt} + \frac{\beta(N)\eta s^*}{a_1a_2 - \eta\rho}(1 - \frac{t_1^*}{t_1})\\ &= (1 - \frac{s^*}{s})(\delta - \delta s - \frac{\beta(N)s(p_1 + bp_2)}{a + s}) + (1 - \frac{p_1^*}{p_1})(\frac{\beta(N)s(p_1 + bp_2)}{a + s} - a_0p_1)\\ &+ \frac{\beta(N)a_2 s^*}{a_1a_2 - \eta\rho}(\alpha p_1 + \eta t_1 - a_1p_2) + \frac{\beta(N)\eta s^*}{a_1a_2 - \eta\rho}(\gamma p_1 + \rho p_2 - a_2t_1).\end{aligned}$$

Here,

$$\begin{aligned}\delta &= \delta s^* + \frac{\beta(N)s^*(p_1^* + bp_2^*)}{a + s^*}, \quad & a_0 &= \frac{\beta(N)s^*(p_1^* + bp_2^*)}{a + s^*}\frac{1}{p_1^*},\\ a_1 &= \frac{\alpha p_1^* + \eta t_1^*}{p_2^*} \text{ and} & a_2 &= \frac{\gamma p_1^* + \rho p_2^*}{t_1^*}.\end{aligned}$$

and we assume

$$m_1 = \frac{s}{s^*}, \quad m_2 = \frac{p_1}{p_1^*}, \quad m_3 = \frac{p_2}{p_2^*} \text{ and } m_4 = \frac{t_1}{t_1^*}.$$

Substituting these values, we get

$$\begin{aligned}
&= -\tfrac{\delta}{s}(s - s^*)^2 + \beta(N)(1 - \tfrac{1}{m_1})\left[\tfrac{s^*(p_1^* + bp_2^*)}{a + s^*} - \tfrac{s(p_1 + bp_2)}{a + s}\right]\\
&+ \beta(N)(1 - \tfrac{1}{m_1})\left[\tfrac{s(p_1 + bp_2)}{a + s} - p_1\tfrac{s^*(p_1^* + bp_2^*)}{a + s^*}\right] + \tfrac{\beta(N)a_2 s^*}{a_1a_2 - \eta\rho}(1 - \tfrac{1}{m_3})\left[\alpha p_1^*(m_2 - m_3) + \eta t_1^*(m_4 - m_3)\right]\\
&+ \tfrac{\beta(N)\eta s^*}{a_1a_2 - \eta\rho}(1 - \tfrac{1}{m_4})\left[\gamma p_1^*(m_2 - m_4) + \rho p_2^*(m_3 - m_4)\right]\\
&= -\tfrac{\delta}{s}(s - s^*)^2 + \beta(N)(1 - \tfrac{1}{m_1})\left[\{\tfrac{s^*p_1^*}{a + s^*} - \tfrac{sp_1}{a + s}\} + b\{\tfrac{p_2^*s^*}{a + s^*} - \tfrac{sp_2}{a + s}\}\right]\\
&+ \beta(N)(1 - \tfrac{1}{m_2})\left[\{\tfrac{sp_1}{a + s} - \tfrac{s^*p_1^*}{a + s^*}p_1\} + b\{\tfrac{sp_2}{a + s} - \tfrac{p_2^*s^*}{a + s^*}p_1\}\right] + \tfrac{\beta(N)a_2 s^*}{a_1a_2 - \eta\rho}(1 - \tfrac{1}{m_3})[\alpha p_1^*(m_2 - m_3)\\
&+ \eta t_1^*(m_4 - m_3)] + \tfrac{\beta(N)\eta s^*}{a_1a_2 - \eta\rho}(1 - \tfrac{1}{m_4})\left[\gamma p_1^*(m_2 - m_4) + \rho p_2^*(m_3 - m_4)\right]\\
&= -\tfrac{\delta}{s}(s - s^*)^2 + \tfrac{\beta(N)}{(a+s)(a+s^*)}(1 - \tfrac{1}{m_1})[s^*p_1^*\{a(1 - m_1m_2) + s(1 - m_2)\} + s^*p_2^*\{a(1 - m_1m_3)\\
&+ s(1 - m_3)\}] + \tfrac{\beta(N)}{(a+s)(a+s^*)}(1 - \tfrac{1}{m_2})[s^*p_1^*\{a(m_1m_2 - p_1) + s(m_2 - p_1)\} + s^*p_2^*\{a(m_1m_3 - p_1)\\
&+ \tfrac{\beta(N)a_2 s^*}{a_1a_2 - \eta\rho}[\alpha p_1^*(1 - \tfrac{m_2}{m_3} + m_2 - m_3) + \eta t_1^*(m_4 - m_3 + 1 - \tfrac{m_4}{m_3})]\\
&+ s(m_3 - p_1)\}] + \tfrac{\beta(N)\eta s^*}{a_1a_2 - \eta\rho}[\gamma p_1^*(1 - \tfrac{m_2}{m_4} + m_2 - m_4) + \rho p_2^*(m_3 - m_4 + 1 - \tfrac{m_3}{m_4})]\\
&= \tfrac{\beta(N)s^*p_1^*}{a + s^*}[1 - p_1 - \tfrac{1}{m_1} - a(m_1 - m_2 - p_1) - s(1 - \tfrac{m_2}{m_1} - \tfrac{p_1}{m_2})] + \tfrac{\beta(N)s^*p_2^*}{a + s^*}[1 - p_1 - \tfrac{1}{m_1}\\
&- a(\tfrac{m_1m_3}{m_2} - m_3 - \tfrac{p_1}{m_2}) - s(\tfrac{m_3}{m_2} - \tfrac{p_1}{m_2}) - \tfrac{m_3}{m_1}] + \tfrac{\beta(N)a_2 s^*}{a_1a_2 - \eta\rho}[\alpha p_1^*(1 - \tfrac{m_2}{m_3} + m_2 - m_3)\\
&+ \eta t_1^*(m_4 - m_3 + 1 - \tfrac{m_4}{m_3})] + \tfrac{\beta(N)\eta s^*}{a_1a_2 - \eta\rho}[\gamma p_1^*(1 - \tfrac{m_2}{m_4} + m_2 - m_4) + \rho p_2^*(m_3 - m_4 + 1 - \tfrac{m_3}{m_4})].
\end{aligned}$$

Since $R_0 > 1$, then $s^*, p_1^*, p_2^*, t_1^* > 0$. Let us choose $1 + m_2 < m_4 < m_3 - 1$, $\frac{p_1}{m_2} < Min\{(\frac{m_1}{m_2} - 1), (1 - \frac{m_2}{m_1})\}$ and $m_1(1 + m_2^2) > m_1m_3 > m_2m_3 + p_1$. Then using the condition, we see that $\frac{dL}{dt}$ is negative definite. Consequently, L is a Lyapunov function and the theorem is established. □

7. Sensitivity Analysis

In this section, we shall discuss the sensitivity analysis of the system with respect to some important parameters, especially the treatment rates γ, ρ and the effective contact rate $\beta(N)$, taking $\beta(N) = \beta_1$ for example, we calculate the following

partial derivatives.

$$\frac{\partial R_0}{\partial \gamma} = -\frac{\beta_1 b\delta(\alpha+\rho+\delta)}{(a+1)(a_1a_2-\eta\rho)a_0^2} < 0,$$
$$\frac{\partial R_0}{\partial \rho} = -\frac{\beta_1 b\delta(\alpha a_2+\eta\gamma)}{a_0(a+1)(a_1a_2-\eta\rho)^2} < 0,$$
$$\frac{\partial R_0}{\partial \beta} = R_0^* > 0.$$

From the above discussion it is clear that increasing the treatment rates γ and ρ can decrease the basic reproduction number R_0, and decreasing or increasing the contact rate also can obtain the basic reproduction number R_0 are same effect. It is concluded that reducing contact with drug-users and improving the treatment effect can serve as control strategies. In addition, from the method of Arriola and Hyman [3], we also examine the sensitivity of R_0 to the parameters γ, ρ and β, normalized forward sensitivity index with respect to each of those parameters are computed as follows:

$$A_\gamma = |\frac{\frac{\partial R_0}{R_0}}{\frac{\partial \gamma}{\gamma}}| = |\frac{\gamma}{R_0}\frac{\partial R_0}{\partial \gamma}| < \frac{\delta(\alpha+\rho+\delta)}{\delta(\alpha+\rho+\delta)+\eta(\alpha+\gamma+\delta)} < 1$$
$$A_\rho = |\frac{\frac{\partial R_0}{R_0}}{\frac{\partial \rho}{\rho}}| = |\frac{\rho}{R_0}\frac{\partial R_0}{\partial \rho}| < \frac{b(\alpha a_2+\eta\gamma)}{\delta(\alpha a_2+\eta\gamma)+\delta(\rho+\eta+\delta)} < 1$$
$$A_\beta = |\frac{\frac{\partial R_0}{R_0}}{\frac{\partial \beta}{\beta}}| = |\frac{\beta}{R_0}\frac{\partial R_0}{\partial \beta}| = 1.$$

From the above discussion it is clear that the basic reproduction number (R_0) is more sensitive to changes in γ,ρ and β. If β will increase R_0 will increase in same proportion and if β will decrease R_0 will also decrease in same proportion. On the other hand, the size of the increase in γ or ρ will lead to a smaller size of the increase in R_0. This sensitivity analysis tells us that the controlling psychological addiction is better than drugs treatment.

8. Numerical Simulations

Numerical simulations are equally important beside the analytical findings to verify them. In this section, we present computer simulations of different solutions of the system (2) using MATLAB.

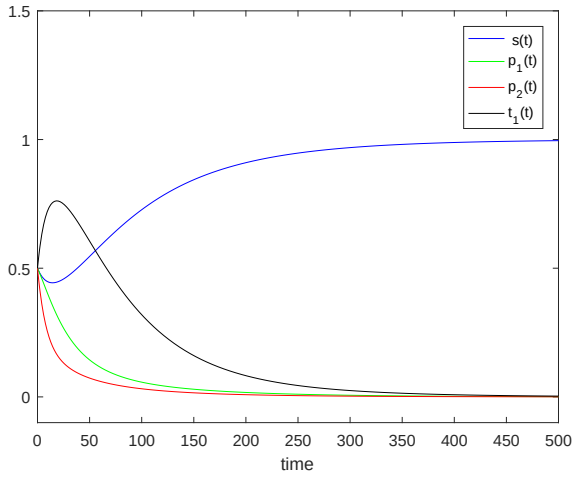

Figure 1. Here $\delta = 0.02, \beta = 0.08, b = 0.5, a = 1.0, \alpha = 0.03, \gamma = 0.0008, \eta = 0.005, \rho = 0.1, R_0 = 0.7986 < 1$. So E_0 is locally asymptotically stable.

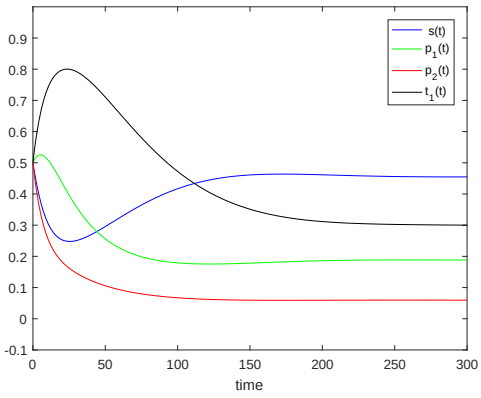

Figure 2. Here $\delta = 0.02, \beta = 0.16, b = 0.5, a = 1.0, \alpha = 0.03, \gamma = 0.0008, \eta = 0.005, \rho = 0.1, R_0 = 1.5972 > 1$. So E^* is locally asymptotically stable.

From Table 2, we fix $\delta = 0.02$, $\beta = 0.0007$, $b = 0.01$, $a = 1.0$, $\alpha = 0.03$, $\gamma = 0.0008$, $\eta = 0.005$, $\rho = 0.1$ in their ranges and take the total population $N = 54$ million in South African. If $\beta(N) = \beta$, let $\beta = 0.0007$, then $R_0 = 0.0061 < 1$, the E_0 is globally asymptotically stable (see figure 3).

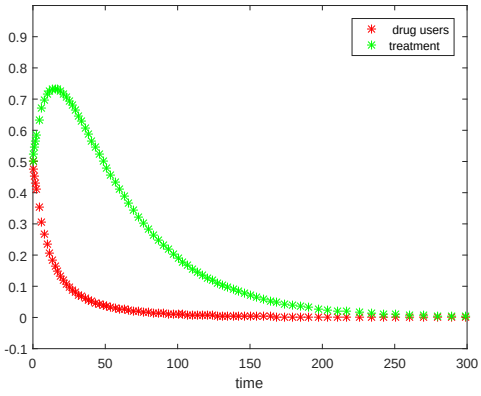

Figure 3. Here $\delta = 0.02, \beta = 0.0007, b = 0.01, a = 1.0, \alpha = 0.03, \gamma = 0.0008, \eta = 0.005, \rho = 0.1, R_0 = 0.0061$. So E_0 is globally asymptotically stable.

If $\beta(N) = \beta$, let $\beta = 0.7$, then $R_0 = 6.0536 > 1$, the E^* is globally asymptotically stable (see figure 4).

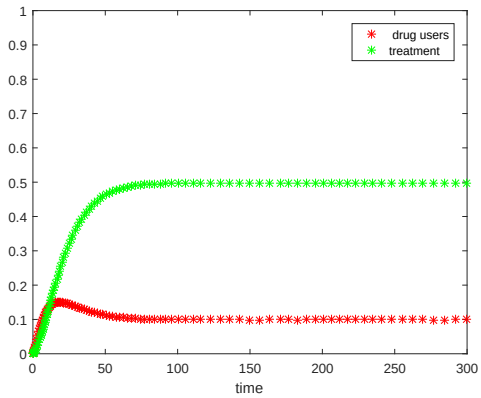

Figure 4. Here $\delta = 0.02, \beta = 0.7, b = 0.01, a = 1.0, \alpha = 0.03, \gamma = 0.0008, \eta = 0.005, \rho = 0.1, R_0 = 6.0536$. So E^* is globally asymptotically stable.

If $\beta(N) = \frac{\beta}{N}$, let $\beta = 0.0007$, then $R_0 = 0.00086 < 1$, the E_0 is globally asymptotically stable (see figure 5).

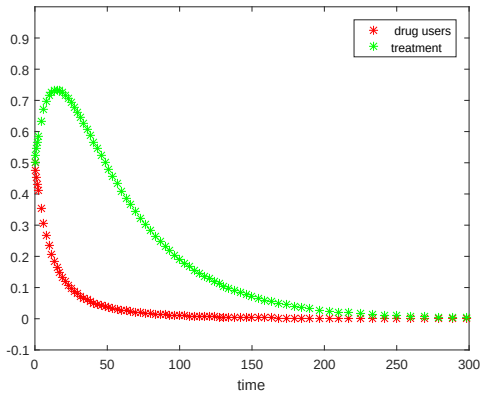

Figure 5. Here $\delta = 0.02, \beta = 0.0007, b = 0.01, a = 1.0, \alpha = 0.03, \gamma = 0.0008, \eta = 0.005, \rho = 0.1, R_0 = 0.00086$. So E_0 is globally asymptotically stable.

If $\beta(N) = \frac{\beta}{N}$, let $\beta = 0.9$, then $R_0 = 2.0331 > 1$, the E^* is globally asymptotically stable (see figure 6).

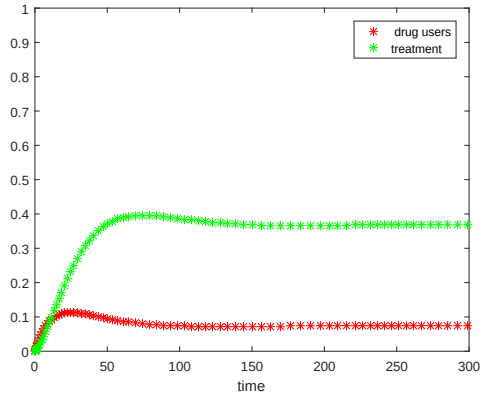

Figure 6. Here $\delta = 0.02, \beta = 0.9, b = 1.5, a = 1.0, \alpha = 0.03, \gamma = 0.0008, \eta = 0.005, \rho = 0.1, R_0 = 2.0331$. So E^* is globally asymptotically stable.

9. Conclusion

Synthetic drugs are created using man-made chemicals, not natural ingredients. For instance, K2 (also known as Spice), Ecstasy (also known as Molly), and bath salts are all types of synthetic drugs. Compared with the traditional ones, the spread of synthetic drugs has its common characteristics. For instance, it is widely believed that the synthetic drugs toxicity is not greater than the traditional ones, so if someone sucks it only a little, he will not become addict. Depend on this wrong psychology, many people neglect precautions and be captured by synthetic drugs.

In this paper, we have considered a model with psychological and physiological addicts. The basic reproduction number R_0 is obtained by the method of next generation matrix. Then we have discussed the local and global stabilities analysis of the model using the basic reproduction number. The system (2) is locally as well as globally asymptotically stable at drug-free equilibrium E_0 when $R_0 < 1$ under some conditions. when $R_0 > 1$, the drug addiction equilibrium is locally as well as globally asymptotically stable. It is understood that if changing the value of basic reproduction number R_0 from more than 1 to less than 1, the drug addiction could be eliminated. The next focus of this paper, we analyze the sensitivity of parameters γ, ρ and β about R_0. If β will increase R_0 will increase in same proportion and if β will decrease R_0 will also decrease in same proportion. On the other hand, the size of the increase in γ or ρ will lead to a smaller size of the

increase in R_0. This sensitivity analysis tells us that the controlling psychological addiction is better than drugs treatment. Consequently, how to design the rational measures to control the drugs transmission is urgent. By the meanings of the parameters, we point out the following model. For example, the government departments could introduce the severe harm to the public. Schools and families should reinforce the health awareness for youths, et al. According to a series of measures, people will realize the danger of synthetic drugs and decrease the interest, which could lead to a lower contact rate. Already, medical staffs and medical researchers also try their best to help addicts keeping away from drugs.

The next study of this paper is to set up an optimal control problem relative to the drug abuse epidemic model so as to minimize the drug addiction as well as to minimize the cost of treatment. Also we can study this model by using distributed time delays. Then the model will be more realistic and significant biologically. We leave this for future investigation.

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References

[1] R. M. Anderson and R. M. May, *Population biology of infectious diseases Part I*, Nature, 280(1979), 361-367.

[2] R. M. Anderson, R. M. May and B. Anderson, *Infectious diseases of humans: dynamics and control*, Oxford, Oxford university press, (1992).

[3] L. Arriola and J. Hyman, *forward and adjoint sensitivity analysis: with applications in Dynamical Systems*, Linear Algebra and Optimisation Mathematical and Theoretical Biology Institute, Summer, (2005).

[4] F. Brauer and C. Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology*, Springer, Berlin, (2001).

[5] C. Castillo-Chavez and B. Song, *Dynamical models of tuberculosis and their applications*, Math. Biosci. Engrg., 1(2004), 361-404.

[6] X. Cen, Z. Feng and Y. Zhao, *Emerging disease dynamics in a model coupling within-host and between-host systems*, J. Theor. Biol., 361(2014), 141-151.

[7] O. Diekmann, J. A. P. Heesterbeek and J. A. J. Metz, *On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations*, J. Math. Biol., 28(1990), 365-382.

[8] K. Dietz, *Overall population patterns in the transmission cycle of infectious disease agents, In Population biology of infectious diseases*, Springer Berlin Heidelberg, (1982).

[9] F. Dumortier, J. Llibre and J. C. Arts, *Qualitative Theory of Planar Differential Systems*, Springer, Berlin, Heidelberg, New York, (2006).

[10] P. Van den Driessche and J. Watmough, *Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission*, Math. Biosci., 180(2002), 29-48.

[11] B. Fang, X. Li, M. Martcheva and L. Cai, *Global stability for a heroin model with two distributed delays*, Discrete Contin. Dynam. Syst. Ser. B, 19(2014), 715-733.

[12] Z. Feng, X. Cen, Y. Zhao and J. X. Velasco-Hernandez, *Coupled within-host and between-host dynamics and evolution of virulence*, Math. Biosci., 270(2015), 204-212.

[13] J. A. P. Heesterbeek and J. A. Metz, *The saturating contact rate in marriage-and epidemic models*, J. Math. Biol.,

31(1993), 529-539.

[14] H. Hethcote, *The mathematics of infectious diseases*, SIAM Rev., 42(2000), 599-653.

[15] H. W. Hethcote and P. Van den Driessche, *Some epidemiological models with nonlinear incidence*, J. Math. Biol., 29(1991), 271-287.

[16] H. Huo, R. Chen and X. Wang, *Modelling and stability of HIV/AIDS epidemic model with treatment*, Appl. Math. Model, 40(2016), 6550-6559.

[17] A. S. Kalula and F. Nyabadza, *A theoretical model for substance abuse in the presence of treatment*, S. Afr. J. Sci., 108(2011), 96-107.

[18] C. Kapp, *Crystal meth boom adds to South Africas health challenges*, Lancet, 37(2008), 193-194.

[19] A. Kelly, M. Carvalho and C. Teljeur, *A 3- Source Capture Recapture Study of the Prevalence of Opiate Use in Ireland 2000 to 2001: key findings summary tables*, Dublin, (2003).

[20] M. Kot, *Elements of Mathematical Ecology*, Cambridge University Press, Cambridge, (2001).

[21] J. Li, Y. Zhao and S. Li, *Fast and Slow dynamics of Malaria model with relapse*, Math. Biosci., 246(2013), 94-104.

[22] J. Li, Y. Zhao and H. Zhu, *Bifurcation of an SIS model with nonlinear contact rate*, J. Math. Anal. Appl., 432(2015), 1119-1138.

[23] J. Mena-Lorcat, H. W. Hethcote, *Dynamic models of infectious diseases as regulators of population sizes*, J. Math. Biol., 30(1992), 693-716.

[24] Mingju Ma, Sanyang Liu, Hong Xiang and Jun Li, *Dynamics of synthetic drugs transmission model with psychological addicts and general incidence rate*, Physica A, (2017), (Accepted Manuscript).

[25] R. M. May and R. M. Anderson, *Population biology of infectious diseases: Part II*, Nature, 280(1979), 455-461.

[26] G. Mulone and B. Straughan, *A note on heroin epidemics*, Math. Biosci., 218(2009), 138-141.

[27] J. Mushanyu, F. Nyabadza and A. G. R. Stewart, *Modelling the trends of inpatient and outpatient rehabilitation for methamphetamine in the Western Cape province of South Africa*, BMC research notes, 8(2015), 1-13.

[28] F. Nyabadza and S. D. Hove-Musekwa, *From heroin epidemics to methamphetamine epidemics: Modeling substance abuse in a South African province*, Math. Biosci., 225(2010), 132-140.

[29] F. Nyabadza, J. B. H. Njagarah and R. J. Smith, *Modelling the dynamics of crystal meth (tik) abuse in the presence of drug-supply chain in South Africa*, B. Math. Biol., 75(2013), 24-48.

[30] G. P. Samanta, *Dynamic behaviour for a nonautonomous heroin epidemic model with time delay*, J. Appl. Math. Comput., 35(2011), 161-178.

[31] E. White and C. Comiskey, *Heroin epidemics, treatment and ODE modelling*, Math. Biosci., 208(2007), 312-324.

[32] H. Xiang, Y. L. Tang and H. F. Huo, *A viral model with intracellular delay and humoral immunity*, B. Malays. Math. Sci. So., (2016), doi:10.1007/s40840-016-0326-2.

[33] www.nncc626.com/2016-02/18/c 128731173 2.htm.

[34] www.nncc626.com.