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Domination in Some Image Graphs

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Abstract: Let $G=(V,E)$ be a simple connected and undirected graph. A subset D of V is called a dominating set of G if every vertex not in D is adjacent to some vertex in D. The domination number of G denoted by $\gamma(G)$ is the minimal cardinality taken over all dominating sets of G. The shadow graph of G, denoted $D_2(G)$ is the graph constructed from G by taking two copies of G, say G itself and $G^{'}$ and joining each vertex u in G to the neighbors of the corresponding vertex $u^{'}$ in $G^{'}$. Let D be the set of all distinct pairs of vertices in G and let D_s (called the distance set) be a subset of D. The distance graph of G, denoted by $D(G,D_s)$ is the graph having the same vertex set as that of G and two vertices u and v are adjacent in $D(G,D_s)$ whenever $d(u,v)\in D_s$. The image graph of a connected graph G, denoted by $I_{mg}(G)$ is the graph obtained by joining the vertices of the original graph G to the corresponding vertices of a copy of G. In this paper, we determine the domination number of the image graph of path graph and cycle graph. We also determine the domination number of the shadow distance graph of the image graph of path graph and cycle graph with specified distance sets.

MSC: 05C69.

Keywords: Dominating set, Minimal edge dominating set, Minimal Domination number, Shadow distance graph, Image graph.

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1. Introduction

By a graph $G=(V,E)$ we mean a finite undirected graph without loops and multiple edges. A subset D of V is called a dominating set of G if every vertex not in D is adjacent to some vertex in D. The domination number of G denoted by $\gamma(G)$ is the minimal cardinality taken over all dominating sets of G. A vertex v in a graph G dominates the vertices in its closed neighbourhood $N[v]$, that is, v is said to dominating itself and each of its neighbours. The shadow graph of G, denoted by $D_2(G)$ is the graph constructed from G by taking two copies of G, namely G itself and $G^{'}$ and by joining each vertex u in G to the neighbors of the corresponding vertex $u^{'}$ in $G^{'}$. Let D be the set of all distances between distinct pairs of vertices in G and let D_s (called the distance set) be a subset of D. The distance graph of G denoted by $D(G,D_s)$ is the graph having the same vertex set as that of G and two vertices u and v are adjacent in $D(G,D_s)$ whenever $d(u,v)\in D_s$. The shadow distance graph [2] of G, denoted by $D_{sd}(G,D_s)$ is constructed from G with the following conditions:

(1). consider two copies of G say G itself and $G^{'}$

(2). if $u\in V(G)$ (first copy) then we denote the corresponding vertex as $u^{'}\in V(G^{'})$ (second copy)

(3). the vertex set of $D_{sd}(G,D_s)$ is $V(G)\cup V(G^{'})$

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(4). the edge set of $D_{sd}(G, D_s)$ is $E(G) \cup E(G^{'}) \cup E_{ds}$ where E_{ds} is the set of all edges (called the shadow distance edges) between two distinct vertices $u \in V(G)$ and $v^{'} \in V(G^{'})$ that satisfy the condition $d(u, v) \in D_s$ in G.

By P_n and C_n respectively we mean the path graph and cycle graph on n vertices. The image graph of a connected graph G [1], denoted by $I_{mg}(G)$, is the graph obtained by joining the vertices of the original graph G to the corresponding vertices of copy of G. By definition, the image graph of the path graph for $n \geq 2$ is the ladder graph L_n on $2n$ vertices. Also, $\gamma(I_{mg}(K_{1,n})) = 2$ for $n \geq 2$ and $\gamma(I_{mg}(K_n)) = 2$ for $n \geq 2$.

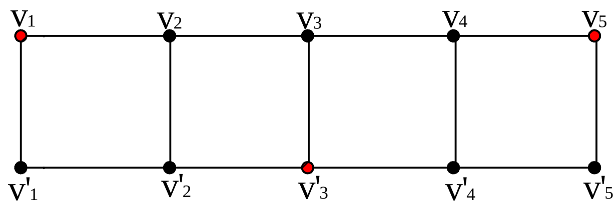

Figure 1. **The image graph of path graph** $P_5(\cong L_5)$

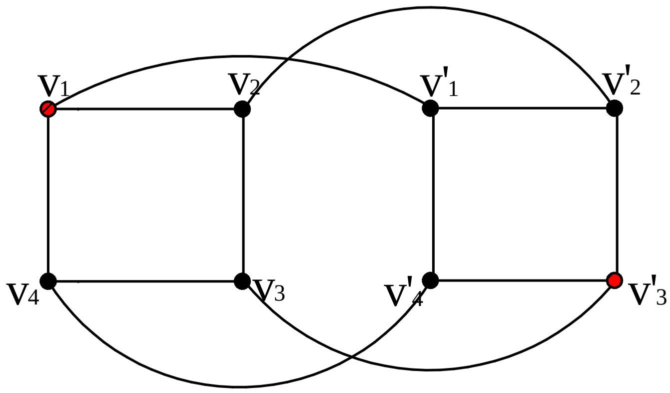

Figure 2. **The image graph of cycle graph** C_4

2. Main Results

We recall the following result related to the vertex domination number of a graph.

Theorem 2.1. *[6] A dominating set D is a minimal dominating set if and only if for each vertex v in D, one of the following condition holds:*

(1). v is an isolated vertex of D

(2). there exists a vertex $u \in V - D$ such that $N(u) \cap D = \{v\}$.

We begin our results with the domination number of the image graph of path graph and cycle graph. In [8] Mostaghim and Sayed Khalkhali proved the following result related to the domination number of the ladder graph L_n.

Theorem 2.2. *Let L_n be a ladder graph of order $2n$. Then*

$$\gamma(L_n) = \begin{cases} \frac{n}{2}, & 4 \mid n \\ \frac{n}{2} + 1, & \textit{if } 2 \mid n,\ 4 \nmid n \\ \lceil \frac{n}{2} \rceil + 1, & 2 \nmid n \end{cases}$$

From this theorem, $\gamma(L_4) = 2$. As a couter example we observe that the graph L_4 illustrated in figure 3 has domination number is 3.

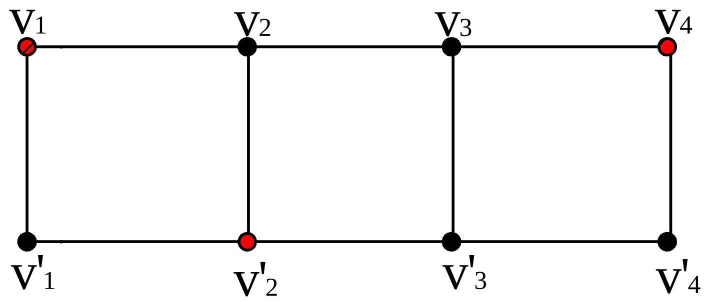

Figure 3. $\gamma(I_{mg}P_4) = 3$ **where** $D = \{v_1, v_4, v_2^{'}\}$

We now provide a correct value of $\gamma(I_{mg}(P_n))$ in Theorem 2.3.

Theorem 2.3. *Let* $n \geq 2$*. Then* $\gamma(I_{mg}(P_n)) = \lfloor \frac{n}{2} \rfloor + 1$.

Proof. Let $G = I_{mg}(P_n)$. We consider the vertex set of G as $V(G) = \{v_i, v_j^{'}\}$, where $i, j = 1, 2, 3, \ldots, n$ and the edge set of G as $E(G) = \{E_1 \cup E_2 \cup E_3\}$, where $E_1 = \{e_1, e_2, e_3, ...e_{n-1}\}$ such that $e_i = (v_i, v_{i+1})$, where $i = 1, 2, \ldots, n-1$, $E_2 = \{e_1^{'}, e_2^{'}, e_3^{'}, \ldots, e_{n-1}^{'}\}$ such that $e_i^{'} = (v_i^{'}, v_{i+1}^{'})$ where $i = 1, 2, \ldots, n-1$, $E_3 = (v_i, v_i^{'})$, where $i = 1, 2, 3, \ldots, n$.
For $n = 2$, the set $D = \{v_1, v_2^{'}\}$ is a minimal edge dominating set with minimum cardinality and hence $\gamma(G) = 2$. For $n = 3$, the set $D = \{v_1, v_3^{'}\}$ is a minimal edge dominating set with minimum cardinality and hence $\gamma(G) = 2$. For $n = 4$, the set $D = \{v_1, v_4, v_2^{'}\}$ is a minimal edge dominating set with minimum cardinality and hence $\gamma(G) = 3$. Let $n \geq 5$. Consider the set

$$D = \begin{cases} \{v_{4j-3}\} \cup \{v_n\} \cup \{v_{4k-1}^{'}\}, & \text{where } n \equiv 0, 1 \ (mod\ 4) \\ \{v_{4p-3}\} \cup \{v_n^{'}\} \cup \{v_{4q-1}^{'}\}, & \text{where } n \equiv 2, 3 \ (mod\ 4) \end{cases}$$

where $1 \leq j, k \leq \lfloor \frac{n}{4} \rfloor$, $1 \leq p, q \leq \lfloor \frac{n}{4} \rfloor$. This set D is a minimal dominating set with minimum cardinality since for any vertex $v \in D$, $D - \{v\}$ is not a dominating set. Thus, some vertex u in $V - D \cup \{v\}$ is not dominated by any vertex in $D \cup \{v\}$. If $u \in V - D$ and u is not dominated by $D - \{v\}$, but is dominated by D, then u is adjacent only to vertex v in D, i.e $N(v) \cap D = \{v\}$. This implies that the set D described above is of minimum cardinality and since $|D| = \lfloor \frac{n}{2} \rfloor + 1$, it follows that $\gamma(I_{mg}(P_n)) = \lfloor \frac{n}{2} \rfloor + 1$. □

Theorem 2.4. *Let* $n \geq 4$*. Then*

$$\gamma(I_{mg}(C_n)) = \begin{cases} \frac{n}{2}, & \textit{where } n \equiv 0 \ (mod\ 4) \\ \lceil \frac{n}{2} \rceil, & \textit{where } n \equiv 1, 3 \ (mod\ 4) \\ \frac{n}{2} + 1, & \textit{where } n \equiv 2 \ (mod\ 4) \end{cases}$$

Proof. Let $G = I_{mg}(C_n)$. We consider the vertex set of G as $V(G) = \{v_i, v_j^{'}\}$, where $i, j = 1, 2, 3, \ldots, n$ and the edge set of G as $E(G) = \{E_1 \cup E_2 \cup E_3\}$, where $E_1 = \{e_1, e_2, e_3, \ldots, e_n\}$such that $e_i = (v_i, v_{i+1})$ and $e_n = (v_n, v_1)$, where $1 \leq i \leq n-1$, $E_2 = \{e_1^{'}, e_2^{'}, e_3^{'}, \ldots, e_n^{'}\}$ such that $e_i^{'} = (v_i^{'}, v_{i+1}^{'})$ and $e_n^{'} = (v_n, v_1)$ for $i = 1, 2, \ldots, n-1$, $E_3 = (v_i, v_i^{'})$, where $i = 1, 2, 3, \ldots, n$. For $n = 4$, the set $D = \{v_1, v_3^{'}\}$ is a minimal edge dominating set with minimum cardinality and hence $\gamma(G) = 2$. Let $n \geq 5$. Consider the set $D = V_1 \cup V_2$, where $V_1 = v_{4j-3}$ and

$$V_2 = \begin{cases} \{v_{4k-1}^{'}\}, & \text{where } n \equiv 0, 1, 3 \ (mod\ 4) \\ \{v_{4p-1}^{'}\} \cup \{v_n^{'}\}, & \text{where } n \equiv 2 \ (mod\ 4) \end{cases}$$

where $1 \leq j \leq \lceil \frac{n}{4} \rceil$, $1 \leq k \leq \lceil \frac{n}{4} \rceil - 1$, $1 \leq p \leq \lfloor \frac{n}{4} \rfloor$. This set D is a minimal dominating set with minimum cardinality since for any vertex $v \in D$, $D - \{v\}$ is not a dominating set. Thus, some vertex u in $V - D \cup \{v\}$ is not dominated by any vertex in $D \cup \{v\}$. If $u \in V - D$ and u is not dominated by $D - \{v\}$, but is dominated by D, then u is adjacent only to vertex v in D, i.e $N(v) \cap D = \{v\}$. This implies that the set D described above is of minimum cardinality and since

$$|D| = \begin{cases} \frac{n}{2}, & \text{where } n \equiv 0 \ (mod\ 4) \\ \lceil \frac{n}{2} \rceil, & \text{where } n \equiv 1,3 \ (mod\ 4) \\ \frac{n}{2} + 1, & \text{where } n \equiv 2 \ (mod\ 4) \end{cases}$$

the result follows. Hence the proof. □

We now provide results related to distance graphs.

Theorem 2.5. *For $n \geq 3$, $\gamma(D_{sd}\{I_{mg}(P_n), \{2\}\}) = 2(\lfloor \frac{n}{2} \rfloor + 1)$.*

Proof. Consider two copies of $G = I_{mg}(P_n)$, one G itself and the other denoted by $G^{'}$. Let $V(G) = \{v_i, v_j^{'}\}$, where $i, j = 1, 2, 3, \ldots, n$ and $E(G) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{e_1, e_2, e_3, \ldots, e_{n-1}\}$ such that $e_i = (v_i, v_{i+1})$, where $i = 1, 2, \ldots, n-1$, $E_2 = \{e_1^{'}, e_2^{'}, e_3^{'}, \ldots, e_{n-1}^{'}\}$ such that $e_i^{'} = (v_i^{'}, v_{i+1}^{'})$ where $i = 1, 2, \ldots, n-1$, $E_3 = (v_i, v_j^{'})$, where $i, j = 1, 2, 3 \ldots, n$. Let $V(G^{'}) = v_i^{''}, v_j^{'''}$, where $i, j = 1, 2, 3, \ldots, n$ and $E(G) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{e_1^{''}, e_2^{''}, e_3^{''}, \ldots, e_{n-1}^{''}\}$ such that $e_i = (v_i^{''}, v_{i+1}^{''})$, where $i = 1, 2, \ldots, n-1$, $E_2 = \{e_1^{'''}, e_2^{'''}, e_3^{'''}, \ldots, e_{n-1}^{'''}\}$ such that $e_i^{'} = (v_i^{'''}, v_{i+1}^{'''})$ where $i = 1, 2, \ldots, n-1$, $E_3 = (v_i^{''}, v_i^{'''})$, where $i = 1, 2, 3, \ldots, n$.

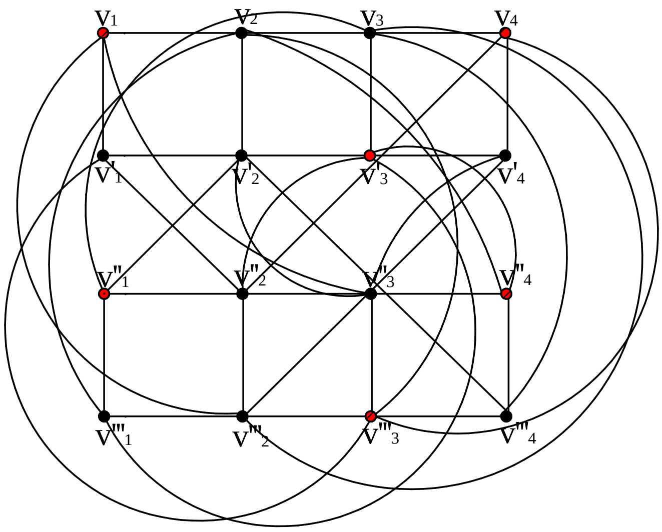

Figure 4. $\gamma(D_{sd}\{I_{mg}(P_n), \{2\}\}) = 6$ **where** $D = \{v_1, v_4, v_3^{'}, v_1^{''}, v_4^{''}, v_3^{'''}\}$

Let $G^{''} = D_{sd}\{I_{mg}(P_n), \{2\}\}$. For $n = 3$, the set $D_{sd} = \{v_1, v_3^{'}, v_1^{''}, v_3^{'''}\}$ is a minimal edge dominating set with minimum cardinality and hence $\gamma(G^{''}) = 4$. Let $n \geq 4$. Consider the set

$$D_{sd} = \begin{cases} \{v_{4j-3}\} \cup \{v_{4j-3}^{''}\} \cup \{v_n\} \cup \{v_n^{''}\} \cup \{v_{4k-1}^{'}\} \cup \{v_{4k-1}^{'''}\}, & \text{where } n \equiv 0,1 \ (mod\ 4) \\ \{v_{4p-3}\} \cup \{v_{4p-3}^{''}\} \cup \{v_n^{'}\} \cup \{v_n^{'''}\} \cup \{v_{4q-1}^{'}\}, \cup \{v_{4q-1}^{'''}\}, & \text{where } n \equiv 2,3 \ (mod\ 4) \end{cases}$$

where $1 \leq j, k \leq \lfloor \frac{n}{4} \rfloor$, $1 \leq p, q \leq \lfloor \frac{n}{4} \rfloor$. This set D_{sd} is a minimal dominating set with minimum cardinality since for any vertex $v \in D_{sd}$, $D_{sd} - \{v\}$ is not a dominating set. Thus, some vertex u in $V - D_{sd} \cup \{v\}$ is not dominated by any vertex in $D_{sd} \cup \{v\}$. If $u \in V - D_{sd}$ and u is not dominated by $D_{sd} - \{v\}$, but is dominated by D_{sd}, then u is adjacent only to vertex v in D_{sd}, i.e $N(v) \cap D_{sd} = \{v\}$. This implies that the set D_{sd} described above is of minimum cardinality and since $|D_{sd}| = 2(\lfloor \frac{n}{2} \rfloor + 1)$ it follows that $\gamma(G^{''}) = 2(\lfloor \frac{n}{2} \rfloor + 1)$. Hence the proof. $\square$

We now provide results related to the shadow distance graphs.

Theorem 2.6. *For $n \geq 4$,* $\gamma(D_{sd}\{I_{mg}(C_n), \{2\}\}) = \begin{cases} n & \text{where } n \equiv 0 \ (mod\ 4) \\ 2\lceil \frac{n}{2} \rceil & \text{where } n \equiv 1,3 \ (mod\ 4) \\ n+2 & \text{where } n \equiv 2 \ (mod\ 4) \end{cases}$

Proof. Consider two copies of $G = I_{mg}(C_n)$, one G itself and the other denoted by $G^{'}$. Let $V(G) = \{v_i, v_j^{'}\}$, where $i, j = 1, 2, 3, \ldots, n$ and $E(G) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{e_1, e_2, e_3, , \ldots, e_n\}$ such that $e_i = (v_i, v_{i+1})$ and $e_n = (v_n, v_1)$, where $1 \leq i \leq n-1$, $E_2 = \{e_1^{'}, e_2^{'}, e_3^{'}, \ldots, e_n^{'}\}$ such that $e_i^{'} = (v_i^{'}, v_{i+1}^{'})$ and $e_n^{'} = (v_n, v_1)$ for $i = 1, 2, \ldots, n-1$, $E_3 = (v_i, v_j^{'})$, where $i, j = 1, 2, 3, \ldots, n$. Let $V(G^{'}) = v_i^{''}, v_j^{'''}$, where $i, j = 1, 2, 3, \ldots, n$ and $E(G) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{e_1^{''}, e_2^{''}, e_3^{''}, , \ldots, e_n^{''}\}$ such that $e_i = (v_i^{''}, v_{i+1}^{''})$ and $e^{''}(v_n^{''}, V_1^{''})$, where $i = 1, 2, \ldots, n-1$, $E_2 = \{e_1^{'''}, e_2^{'''}, e_3^{'''}, \ldots, e_n^{'''}\}$ such that $e_i^{'} = (v_i^{'''}, v_{i+1}^{'''})$ and $e^{'''}(v_n^{'''}, V_1^{'''})$, where $i = 1, 2, \ldots, n-1$, $E_3 = (v_i^{''}, v_i^{'''})$, where $i = 1, 2, 3, \ldots, n$.

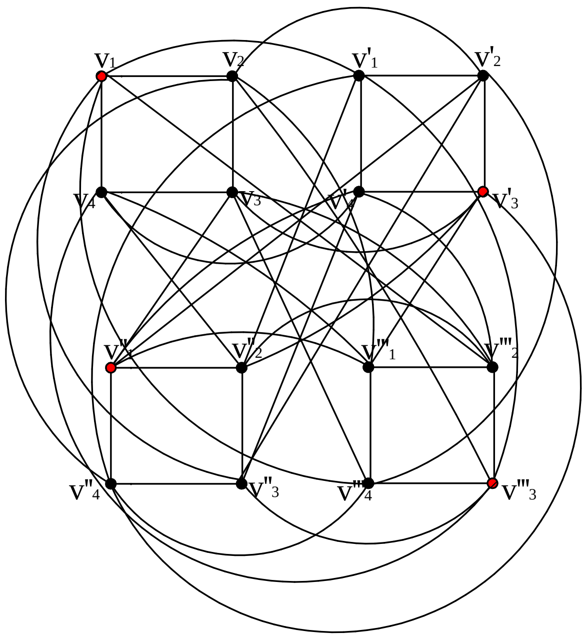

Figure 5. $\gamma(D_{sd}\{I_{mg}(C_n), \{2\}\}) = 4$ **where** $D = \{v_1, v_3^{'}, v_1^{''}, v_3^{'''}\}$

Let $G^{''} = D_{sd}\{I_{mg}(C_n), \{2\}\}$. For $n = 4$, the set $D_{sd} = \{v_1, v_3^{'}, v_1^{''}, v_3^{'''}\}$ is a minimal edge dominating set with minimum cardinality and hence $\gamma(G^{''}) = 4$. Let $n \geq 5$. Consider the set $D_{sd} = V_1 \cup V_2$, where $V_1 = \{v_{4j-3}\} \cup \{v_{4j-3}^{''}\}$ and

$$V_2 = \begin{cases} \{v_{4k-1}^{'}\} \cup \{v_{4k-1}^{'''}\}, & \text{where } n \equiv 0, 1, 3 \ (mod\ 4) \\ \{v_{4p-1}^{'}\} \cup \{v_{4p-1}^{'''}\} \cup \{v_n^{'}\} \cup \{v_n^{'''}\}, & \text{where } n \equiv 2 \ (mod\ 4) \end{cases}$$

where $1 \leq j \leq \lceil \frac{n}{4} \rceil$, $1 \leq k \leq \lceil \frac{n}{4} \rceil - 1$, $1 \leq p \leq \lfloor \frac{n}{4} \rfloor$. This set D_{sd} is a minimal dominating set with minimum cardinality since for any vertex $v \in D_{sd}$, $D_{sd} - \{v\}$ is not a dominating set. Thus, some vertex u in $V - D_{sd} \cup \{v\}$ is not dominated by any vertex in $D_{sd} \cup \{v\}$. If $u \in V - D_{sd}$ and u is not dominated by $D_{sd} - \{v\}$, but is dominated by D_{sd}, then u is adjacent only to vertex v in D_{sd}, i.e $N(v) \cap D_{sd} = \{v\}$. This implies that the set D_{sd} described above is of minimum cardinality and since

$$|D_{sd}| = \begin{cases} n, & \text{where } n \equiv 0 \ (mod\ 4) \\ 2\lceil \frac{n}{2} \rceil, & \text{where } n \equiv 1, 3 \ (mod\ 4) \\ n+2, & \text{where } n \equiv 2 \ (mod\ 4) \end{cases},$$

the result follows. Hence the proof. □

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