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# Similarity Reductions and Integrable Properties of Lax Pair for Boiti–Leon–Manna–Pempinelli Equation

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**Abstract:** Based on symbolic computational system Maple; classical Lie symmetry reductions of Lax pair for Boiti–Leon–Manna–Pempinelli(BLMP) equation are presented. We have obtained three interesting reductions for Lax pair along with one exact solution. Analysis of reductions have shown that, under similar symmetry group compatibility condition of reduced Lax pair is same as of reduced equation. Moreover, one of the reduced equation is analyzed with Bell polynomials for Bäcklund transformations and Lax pairs; several new traveling wave solutions are also constructed.

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## 1. Introduction

Most celebrated technique developed so far for solving nonlinear partial differential equations is Lie group analysis, where identification of Lie symmetries for given equation is a primary task to solve such equation. Although the involved calculations demands hard labour but with the availability of symbolic computational packages such as Maple, the calculations becomes quite manageable. Once the symmetries for given equation are obtained, one may proceed in two different ways to find exact solution: first; using finite transformation from known symmetry, general solution may be constructed starting with some seed solution, second; one may go for reduction in number of independent variables leading to group invariant solutions of equation. In the second case, the reductions are ordinary differential equations which may be integrated using Painlevé singularity analysis [1, 2] and if reductions able to pass Painlevé property then exact solutions to given equation may be obtained in term of Painlevé transcedents. In this paper, we would like to reduce for following (2+1)–dimensional Boiti–Leon–Manna–Pempinelli equation via its Lax pair.

$$u_{yt} - 3u_x u_{xy} - 3u_y u_{xx} + u_{xxxy} = 0. \tag{1}$$

The equation (1) which is also known as Asymmetric–Nizhnik–Novikov–Veselov equation and is (2+1)–dimensional extension of KdV equation [3] which can be seen by putting $y = x$. Generalized symmetries for equation (1) are obtained by constructing formal series symmetries [4], Painlevé analysis is performed on equation (1) and singular manifold method is used to construct Lax pair for this equation [5], interaction of dromions and invisible excitation called ghoston is also discussed for this equation

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[6], quasi-periodic wave solutions to this equation are also obtained by using Hirota bilinear method [7]. Now the equation (1) can also be re-written in the following system of equations:

$$u_x + v_y = 0, \tag{2a}$$

$$u_t - u_{xxx} + 3(uv)_x = 0. \tag{2b}$$

The system (2) represents model for incompressible fluid having $u$ and $v$ as velocity components. Employing singular manifold method, Estévez and Leble [5] able to construct following Lax pair for equation (1),

$$\psi_{xy} - u_y\psi - \psi_x = 0, \qquad \psi_t + \psi_{xxx} - 3u_x\psi_x = 0. \tag{3}$$

In literature, one may find numerous examples of reductions of partial differential equations using symmetry transformations, but reduction of associated linear problem or Lax pairs is not that frequent. For example, Legaré [8] has obtained symmetry reduction of Lax pair for self-dual Yang-Mills(SDYM for abbreviation) equations in four–dimension and the author has proved that compatibility of the reduced Lax pair is same as reduction of SDYM equation under same symmetry group, Estévez [9] has obtained five interesting reduction for Lax pair of generalized Hirota-Satsuma system along with some exact solutions, Mabrouk [10] have also obtained two parameter symmetry reduction of Lax pair of this generalized Hirota-Satsuma system, Hong-Yan [11] compared the reduction of Swada–Kotera equation with compatibility condition of its reduced Lax pair under same symmetry group and similar comparison is also carried out by the same author for Konopelchenko–Dubrovsky equation [12]. Hong–Yan has found that the symmetry reduction of an equation is not same as that of compatibility condition of reduced Lax pair under same symmetry group, Mabrouk and Kassem [13] has obtained two parameter symmetry reduction for (3) along with some exact solutions.

This paper is planned as follows. The Section 2 deal with symmetry determination of Lax pair (3) using Lie classical approach and in Section 3 three interesting reductions for Lax pair are obtained along with exact solution for (1), in Section 4 several traveling wave solutions are obtained for Section 1, one of the reduced equation is analyzed using Bell polynomials in Section 5. Finally, conclusion is drawn in Section 6 .

## 2. Lie Symmetry Analysis of Lax Pair (3)

In order to obtain reductions for Lax pair we use Lie group method of infinitesimal transformations [14–16]. We consider one parameter Lie group of infinitesimal transformations for $(x, y, t, u, \psi)$, defined by:

$$\begin{aligned}
&x^* = x + \epsilon\xi(x, y, t, u, \psi) + O(\epsilon^2), \quad y^* = y + \epsilon\tau(x, y, t, u, \psi) + O(\epsilon^2),\\
&t^* = t + \epsilon\eta(x, y, t, u, \psi) + O(\epsilon^2), \quad u^* = u + \epsilon\phi_1(x, y, t, u, \psi) + O(\epsilon^2),\\
&\psi^* = \psi + \epsilon\phi_2(x, y, t, u, \psi) + O(\epsilon^2),
\end{aligned}$$

where $\epsilon$ being group parameter and $\xi, \tau, \eta, \phi_1, \phi_2$ are infinitesimals of transformation having explicit dependence on $x, y, t, u$ and $\psi$. These infinitesimals can be determined from overdetermined system of partial differential equations which one obtain when solution of (3) is invariant under infinitesimal transformation. The vector field associated with group of infinitesimal transformation may be written as follows:

$$V = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial t} + \phi_1\frac{\partial}{\partial u} + \phi_2\frac{\partial}{\partial \psi}, \tag{4}$$

To apply Lie classical method [15, 17] we rewrite the system (3) as follow:

$$\Delta_1 = \psi_{xy} - u_y\psi - \psi_x = 0,$$
$$\Delta_2 = \psi_t + \psi_{xxx} - 3u_x\psi_x = 0,$$

the admitted Lie symmetries for (3) can be obtained, if and only if, following condition is satisfied:

$$V^{(3)}(\Delta_1, \Delta_2)\big|_{\Delta_1,\Delta_2=0} = 0. \tag{5}$$

The condition (5) is nothing but invariance criteria for symmetry determination, and $V^{(3)}$ being third order prolongation of vector field $V$. The invariance condition (5) simplifies to following overdetermined system of partial differential equations for $\xi, \tau, \eta, \phi_1$ and $\phi_2$:

$$\begin{aligned}
&\xi_\psi = 0, \quad \xi_u = 0, \quad 3\,\xi_x - \eta_t = 0, \quad \xi_y = 0,\\
&\tau_t = 0, \quad \tau_\psi = 0, \quad \tau_u = 0, \quad \tau_x = 0, \quad \eta_\psi = 0, \quad \eta_u = 0,\\
&\eta_x = 0, \quad \eta_y = 0, \quad \phi_{2,t} = 0, \quad \psi\phi_{2,\psi} - \phi_2 = 0,\\
&\phi_{2,u} = 0, \quad \phi_{2,x} = 0, \quad \phi_{2,y} - \tau_y\,\psi = 0, \quad \phi_{1,\psi} = 0,\\
&3\,\phi_{1,u} + \eta_t = 0, \quad 3\,\phi_{1,x} + \xi_t = 0, \quad \phi_{1,y} = 0.
\end{aligned} \tag{6}$$

The determining equations (6) are determined in algorithmic manner as described by various authors [14, 15, 18, 19] and underlying procedure is efficiently implemented in symbolic language [20]. The solution of determining system (6) can be obtained as follow:

$$\begin{aligned}
&\xi = -2\frac{dF_1(t)}{dt}\cdot x + F_2(t), \quad \tau = \alpha(y) + \lambda, \quad \eta = -6F_1(t),\\
&\phi_1 = \alpha(y)\psi, \quad \phi_2 = \frac{1}{3}\frac{d^2F_1(t)}{dt^2}\cdot x^2 - \frac{1}{3}\frac{dF_2(t)}{dt}\cdot x + F_3(t) + 2\frac{dF_1(t)}{dt}\cdot u,
\end{aligned} \tag{7}$$

where $F_1(t), F_2(t)$ and $F_3(t)$ are arbitrary functions of $t$ and $\alpha(y)$ is arbitrary function of $y$ such that the corresponding Lie algebra is infinite dimensional. We shall see in the next section, that the constant $\lambda$ that appears in (7) plays role of spectral parameter for reduced Lax pair (3).

Once the infinitesimals are determined, one can go for reduction of Lax pair (3) by solving characteristics equations

$$\frac{dx}{-2\frac{dF_1(t)}{dt}\cdot x + F_2(t)} = \frac{dy}{\alpha(y)+\lambda} = \frac{dt}{-6F_1(t)} = \frac{du}{\frac{1}{3}\frac{d^2F_1(t)}{dt^2}\cdot x^2 - \frac{1}{3}\frac{dF_2(t)}{dt}\cdot x + F_3(t) + 2\frac{dF_1(t)}{dt}\cdot u} = \frac{d\psi}{\alpha(y)\psi}, \tag{8}$$

solution of these characteristics equation is nothing but solution of linear equations in Lagranges form:

$$\xi\frac{\partial u}{\partial x} + \tau\frac{\partial u}{\partial y} + \eta\frac{\partial u}{\partial t} - \phi_1 = \xi\frac{\partial \psi}{\partial x} + \tau\frac{\partial \psi}{\partial y} + \eta\frac{\partial \psi}{\partial t} - \phi_2 = 0. \tag{9}$$

The equations (9) are also known as invariant surface conditions and from characteristics equations (8) we can obtain three different types similarity variables hence three reductions of (3) by imposing conditions on arbitrary functions, that we shall see in next section.

## 3. Similarity Reduction of Lax Pair

In order to obtain reductions for Lax pair (3) we impose restrictions on arbitrary functions which we discuss in following cases:

- For $F_1(t) \neq 0$, $\alpha(y) \neq 0$.

  Solving characteristics equations (8) we find similarity variables as follow:

$$z_1 = \frac{x}{F_1^{\frac{1}{3}}(t)} + \frac{1}{6}\int \frac{F_2(t)}{F_1^{\frac{4}{3}}(t)}dt, \quad z_2 = \int \frac{1}{\alpha(y)+\lambda}dy + \frac{1}{6}\int \frac{1}{F_1(t)}dt, \tag{10}$$

  and the reduction fields

$$\psi = \Psi(z_1, z_2)\cdot \exp\left(\int \frac{\alpha(y)}{\alpha(y)+\lambda}dy\right), \quad u = \frac{U(z_1, z_2)}{F_1(t)^{\frac{1}{3}}} - \frac{1}{6F_1(t)}\int F_1^{\frac{1}{3}}(t)\cdot Q(z_1, t)dt, \tag{11}$$

  where

$$\begin{aligned} Q(z_1, t) = {} & \frac{1}{3F_1^{\frac{1}{3}}(t)}\frac{d^2F_1(t)}{dt^2}\cdot z_1^2 - \left(\frac{1}{9}\frac{d^2F_1(t)}{dt^2}\int \frac{F_2(t)}{F_1^{\frac{4}{3}}(t)}dt + \frac{1}{3F_1^{\frac{2}{3}}(t)}\frac{dF_2(t)}{dt}\right)\cdot z_1 + \frac{F_1^{\frac{1}{3}}(t)}{108}\frac{d^2F_1(t)}{dt^2}\left(\int \frac{F_2(t)}{F_1^{\frac{4}{3}}(t)}dt\right)^2 \\ & + \frac{1}{18F_1^{\frac{1}{3}}(t)}\frac{dF_2(t)}{dt}\int \frac{F_2(t)}{F_1^{\frac{4}{3}}(t)}dt + \frac{F_3(t)}{F_1(t)}, \end{aligned} \tag{12}$$

  by using ansatz (10) and (11) in (3) reduction of Lax pair is obtained follow

$$\Psi_{z_1z_2} - U_{z_2}\Psi - \lambda\Psi_{z_1} = 0, \tag{13a}$$

$$\Psi_{z_1} + \frac{1}{6}\Psi_{z_2} + \Psi_{z_1z_1z_1} - 3U_{z_1}\Psi_{z_1} = 0, \tag{13b}$$

  further simplification of (13) gives

$$6\Psi_{z_1z_1z_1z_1} + 6\Psi_{z_1z_1} - 18U_{z_1}\Psi_{z_1z_1} - 18U_{z_1z_1}\Psi_{z_1} + U_{z_2}\Psi + \lambda\Psi_{z_1} = 0, \tag{14a}$$

$$\Psi_{z_1} + \frac{1}{6}\Psi_{z_2} + \Psi_{z_1z_1z_1} - 3U_{z_1}\Psi_{z_1} = 0, \tag{14b}$$

  which is fourth order spectral problem. In order to obtain compatibility condition for Lax pair (14), we use compatibility condition $\Psi_{z_1z_1z_1z_1z_2} = \Psi_{z_2z_1z_1z_1z_1}$ and after one by one elimination of derivatives of $\Psi$ we arrive at compatibility condition

$$6U_{z_1z_2} + U_{z_2z_2} + 6U_{z_1z_1z_1z_2} - 18U_{z_1}U_{z_1z_2} - 18U_{z_1z_1}U_{z_2} = 0. \tag{15}$$

- $F_1(t) = 0$, $F_2(t) \neq 0$, $\alpha(y) \neq 0$.

  Solving characteristics equations (8) we find similarity variables as follow:

$$z_1 = \frac{x}{F_2(t)} - \int \frac{1}{\alpha(y)+\lambda}dy, \quad z_2 = t, \tag{16}$$

  and the reduction fields

$$\psi = \Psi(z_1, z_2)\cdot \exp\left(\int \frac{\alpha(y)}{\alpha(y)+\lambda}dy\right), \quad u = U(z_1, z_2) - \frac{1}{6F_2(t)}\frac{dF_2(t)}{dt}\cdot x^2 + \frac{F_3(t)}{F_2(t)}\cdot x, \tag{17}$$

substituting ansatz (16) and (17) into (3), the reduction of Lax pair followed as

$$\Psi_{z_1 z_1} - F_2(z_2)U_{z_1}\Psi + \lambda\Psi_{z_1} = 0, \tag{18a}$$

$$\Psi_{z_1 z_1 z_1} + F_2^3(z_2)\Psi_{z_2} - 3F_2(z_2)U_{z_1}\Psi_{z_1} - 3F_2(z_2)F_3(z_2)\Psi_{z_1} = 0, \tag{18b}$$

using compatibility condition $\Psi_{z_2 z_1 z_1} = \Psi_{z_1 z_1 z_2}$ for Lax pair (18) we arrive at

$$\left(\frac{3F_3(z_2)}{F_2(z_2)}U_{z_1} - U\frac{dF_2(z_2)}{dz_2} - F_2(z_2)U_{z_2} + \frac{3}{F_2(z_2)}U_{z_1}^2 - \frac{1}{F_2^2(z_2)}U_{z_1 z_1 z_1}\right)_{z_1} = 0. \tag{19}$$

- $F_1(t) = 0, F_2(t) = 0,\ \alpha(y) \neq 0.$

  Solving characteristics equations (8) we find similarity variables as follow:

$$z_1 = x, \quad z_2 = t, \tag{20}$$

and reduction fields are found as

$$\psi = \Psi(z_1, z_2)\cdot \exp\left(\int \frac{\alpha(y)}{\alpha(y)+\lambda}dy\right), \quad u = U(z_1, z_2) + F_3(z_2)\cdot\int \frac{1}{\alpha(y)+\lambda}dy, \tag{21}$$

substituting ansatz (20) and (21) into (3), the reduction of Lax pair is found as

$$F_3(z_2)\Psi + \lambda\Psi_{z_1} = 0, \tag{22a}$$

$$\Psi_{z_2} + \Psi_{z_1 z_1 z_1} - 3U_{z_1}\Psi_{z_1} = 0, \tag{22b}$$

compatibility condition for reduced Lax pair (22) is found as under:

$$3F_3(z_2)U_{z_1 z_1} - \frac{dF_1(z_2)}{dz_2} = 0, \tag{23}$$

equation (23) easily integrated to

$$U(z_1, z_2) = \frac{1}{6F_3(z_2)}\frac{dF_3(z_2)}{dz_2}\cdot z_1^2 + \frac{1}{3F_3(z_2)}f_1(z_2)\cdot z_1 + \frac{1}{3F_3(z_2)}f_2(z_2), \tag{24}$$

where $f_1(z_2)$ and $f_2(z_2)$ are arbitrary function, using (24) equations (22) and (23) can be easily integrated to

$$\Psi(z_1, z_2) = c\cdot\exp\left(-\frac{F_3(z_2)}{\lambda}z_1 - \frac{1}{\lambda}\int f_1(z_2)dz_2 + \frac{1}{\lambda^3}\int F_3^3(z_2)dz_2\right), \tag{25}$$

from (24) and (25) exact solution to Lax pair (3) may be written as

$$\psi(x, y, t) = c\cdot\exp\left(-\frac{F_3(t)}{\lambda}x - \frac{1}{\lambda}\int f_1(t)dt + \frac{1}{\lambda^3}\int F_3^3(t)dt\right)\cdot\exp\left(\frac{\alpha(y)}{\alpha(y)+\lambda}dy\right), \tag{26}$$

$$u(x, y, t) = \frac{1}{6F_3(t)}\frac{dF_3(t)}{dt}\cdot x^2 + \frac{1}{3F_3(t)}f_1(t)\cdot x + \frac{1}{3F_3(t)}f_2(t) + F_3(t)\int\frac{1}{\alpha(y)+\lambda}dy, \tag{27}$$

where (27) serve as solution to equation (1) and $f_1(t), f_2(t), F_3(t)$ and $\alpha(y)$ are all arbitrary functions.

**Remark 3.1.** *Here we want to add remark related to comments in paper [8] that symmetry reduction of SDYM equation is same as that of compatibility condition of its reduced Lax pair under similar symmetry group. On substitution ansätz (10) along with reduction field u from (11) in equation (1), the reduction is found as under:*

$$6U_{z_1 z_2} + U_{z_2 z_2} + 6U_{z_1 z_1 z_1 z_2} - 18U_{z_1}U_{z_1 z_2} - 18U_{z_1 z_1}U_{z_2} = 0, \tag{28}$$

*which is same as compatibility condition (15) of reduced Lax pair (14) and this agrees with the comments of Legaré [**?** ] related to reduction of Lax pairs for SDYM equations. It is straight forward procedure to verify that similar equivalence also holds for other reductions as well. The equivalence that we have observed is perhaps due to isomorphism between Lie algebra of BLMP equation and its zero-curvature(Lax pair) representation.*

## 4. Traveling Wave Solutions for (28)

Based on tanh-function method as described in [21–24] and symbolic manipulation program Maple to carry out tedious calculations, we have constructed following traveling wave solutions for reduced equation (28); hence for main equation (1) using invariant transformations (10) and (11)

$$
\begin{aligned}
U_1(z_1,z_2) &= \frac{a_{-1}}{\csc(\lambda_1 z_1+\delta)} + a_0 + a_1\csc(\lambda_1 z_1+\delta),\\
u_1(x,y,t) &= \frac{1}{F_1(t)^{\frac{1}{3}}}\left(\frac{a_{-1}}{\csc(\lambda_1 z_1+\delta)} + a_0 + a_1\csc(\lambda_1 z_1+\delta)\right) - \Gamma(x,t),\\
U_2(z_1,z_2) &= \frac{a_{-1}}{\operatorname{csch}(\lambda_1 z_1+\delta)} + a_0 + a_1\operatorname{csch}(\lambda_1 z_1+\delta),\\
u_2(x,y,t) &= \frac{1}{F_1(t)^{\frac{1}{3}}}\left(\frac{a_{-1}}{\operatorname{csch}(\lambda_1 z_1+\delta)} + a_0 + a_1\operatorname{csch}(\lambda_1 z_1+\delta)\right) - \Gamma(x,t),\\
U_3(z_1,z_2) &= \frac{a_{-1}}{\sec(\lambda_1 z_1+\delta)} + a_0 + a_1\sec(\lambda_1 z_1+\delta),\\
u_3(x,y,t) &= \frac{1}{F_1(t)^{\frac{1}{3}}}\left(\frac{a_{-1}}{\sec(\lambda_1 z_1+\delta)} + a_0 + a_1\sec(\lambda_1 z_1+\delta)\right) - \Gamma(x,t),\\
U_4(z_1,z_2) &= \frac{a_{-1}}{\operatorname{sech}(\lambda_1 z_1+\delta)} + a_0 + a_1\operatorname{sech}(\lambda_1 z_1+\delta),\\
u_4(x,y,t) &= \frac{1}{F_1(t)^{\frac{1}{3}}}\left(\frac{a_{-1}}{\operatorname{sech}(\lambda_1 z_1+\delta)} + a_0 + a_1\operatorname{sech}(\lambda_1 z_1+\delta)\right) - \Gamma(x,t),\\
U_5(z_1,z_2) &= a_0 + 2\lambda_1\tan\left[\left(24\lambda_1{}^3 - 6\lambda_1\right)z_2 + \lambda_1 z_1 + \delta\right],\\
u_5(x,y,t) &= \frac{1}{F_1(t)^{\frac{1}{3}}}\left(a_0 + 2\lambda_1\tan\left[\left(24\lambda_1{}^3 - 6\lambda_1\right)z_2 + \lambda_1 z_1 + \delta\right]\right) - \Gamma(x,t),\\
U_6(z_1,z_2) &= -\frac{2\lambda_1}{\tan\left[\left(24\lambda_1{}^3 - 6\lambda_1\right)z_2 + \lambda_1 z_1 + \delta\right]} + a_0,\\
u_6(x,y,t) &= \frac{1}{F_1(t)^{\frac{1}{3}}}\left(-\frac{2\lambda_1}{\tan\left[\left(24\lambda_1{}^3 - 6\lambda_1\right)z_2 + \lambda_1 z_1 + \delta\right]} + a_0\right) - \Gamma(x,t),\\
U_7(z_1,z_2) &= a_0 - 2\lambda_1\tanh\left(\left(-24\lambda_1{}^3 - 6\lambda_1\right)z_2 + \lambda_1 z_1 + \delta\right),\\
u_7(x,y,t) &= \frac{1}{F_1(t)^{\frac{1}{3}}}\left(a_0 - 2\lambda_1\tanh\left(\left(-24\lambda_1{}^3 - 6\lambda_1\right)z_2 + \lambda_1 z_1 + \delta\right)\right) - \Gamma(x,t),\\
U_8(z_1,z_2) &= -\frac{2\lambda_1}{\tanh\left[\left(-24\lambda_1{}^3 - 6\lambda_1\right)z_2 + \lambda_1 z_1 + \delta\right]} + a_0,\\
u_8(x,y,t) &= \frac{1}{F_1(t)^{\frac{1}{3}}}\left(-\frac{2\lambda_1}{\tanh\left[\left(-24\lambda_1{}^3 - 6\lambda_1\right)z_2 + \lambda_1 z_1 + \delta\right]} + a_0\right) - \Gamma(x,t),\\
U_9(z_1,z_2) &= \frac{a_{-1}}{\operatorname{cn}(\lambda_1 z_1+\delta,\omega)} + a_0 + a_1\operatorname{cn}(\lambda_1 z_1+\delta,\omega),\\
u_9(x,y,t) &= \frac{1}{F_1(t)^{\frac{1}{3}}}\left(\frac{a_{-1}}{\operatorname{cn}(\lambda_1 z_1+\delta,\omega)} + a_0 + a_1\operatorname{cn}(\lambda_1 z_1+\delta,\omega)\right) - \Gamma(x,t),\\
U_{10}(z_1,z_2) &= \frac{a_{-1}}{\operatorname{sn}(\lambda_1 z_1+\delta,\omega)} + a_0 + a_1\operatorname{sn}(\lambda_1 z_1+\delta,\omega)\\
u_{10}(x,y,t) &= \frac{1}{F_1(t)^{\frac{1}{3}}}\left(\frac{a_{-1}}{\operatorname{sn}(\lambda_1 z_1+\delta,\omega)} + a_0 + a_1\operatorname{sn}(\lambda_1 z_1+\delta,\omega)\right) - \Gamma(x,t),
\end{aligned}
$$

where $F_1(t), F_2(t), F_3(t)$ are arbitrary functions; $\Gamma(x,t) = \frac{1}{6F_1(t)}\int F_1^{\frac{1}{3}}(t)\cdot Q(z_1,t)dt$ and $Q(z_1,t)$ is given by (12). The traveling solutions presented above for equation (1) are new and to best of our knowledge, these solutions have never been reported before.

## 5. Further Analysis of Reduced Equation(28)

In order to explore the integrable aspect of reduced equation (28) we shall analyze equation (28) from slightly different perspective. Based on Bell polynomial approach(for details see [25, 26] and "Appendix" as well), it possible to bilinearize equation (28) by introducing transformation

$$u = c\,(q)_{z_1}, \text{ for } q = q(z_1, z_2), \tag{29}$$

where $c$ being arbitrary constant to be determined later. Substituting (29) into (28) and integrating once wrt to $z_1$, we get

$$E(q) = -18\,c\,q_{z_1,z_2}q_{z_1 z_1} + q_{z_2,z_2} + 6\,q_{z_1,z_1,z_1,z_2} + 6\,q_{z_1,z_2} = 0, \tag{30}$$

where constant of integration is taken as zero. In order to connect (30) with $\mathscr{P}$-polynomials(also known as even order Bell polynomials), we must choose $c = -1$, so that using formulae for even order Bell polynomials as given in [25] (see also (48) in "Appendix"), we have

$$E(q) = 6\,\mathscr{P}_{z_2,z_1}(q) + 6\,\mathscr{P}_{z_2,3z_1}(q) + \mathscr{P}_{2z_2}(q) = 0, \tag{31}$$

through transformation $q = 2\log f$ and on account of relation between Hirota's D-operator and even order Bell polynomial (see formula (47) in "Appendix"). The equation (31) reduce to bilinear equation as follow:

$$\left(6\,D_{z_2}D_{z_1}^3 + D_{z_2}^2 + 6\,D_{z_2}D_{z_1}\right) f\cdot f = 0. \tag{32}$$

The bilinear equation (32) will play vital role in construction of Bäcklund transformations and Lax pairs for reduced Equation(28).

### 5.1. Bilinear Bäcklund transformations

In order to find bilinear BT for equation (28), suppose $\tilde{q} = 2\log\tilde{f}$ be another solution of (31), then corresponding two field condition can be written as follows:

$$E(\tilde{q}) - E(q) = [6\,\mathscr{P}_{z_2,z_1}(q) + 6\,\mathscr{P}_{z_2,3z_1}(q) + \mathscr{P}_{2z_2}(q)] - [6\,\mathscr{P}_{z_2,z_1}(\tilde{q}) + 6\,\mathscr{P}_{z_2,3\,z_1}(\tilde{q}) + \mathscr{P}_{2z_2}(\tilde{q})] \tag{33}$$

To proceed further, we introduce new variables

$$W = \log f\tilde{f}, \quad V = \log\frac{\tilde{f}}{f}, \quad \tilde{q} = 2\log\tilde{f} = W + V, \quad q = \log f = W - V, \tag{34}$$

so that equation (33) is written as

$$\begin{aligned} E(\tilde{q}) - E(q) &= [6\,\mathscr{P}_{z_2,z_1}(W-V) - 6\,\mathscr{P}_{z_2,z_1}(W+V)] + [6\,\mathscr{P}_{z_2,3z_1}(W-V) - 6\,\mathscr{P}_{z_2,3z_1}(W+V)] + [\mathscr{P}_{2z_2}(W-V) - \mathscr{P}_{2z_2}(W+V)] = 0 \\ &= 36\,V_{z_1,z_2}W_{2\,z_1} + 36\,W_{z_1,z_2}V_{2\,z_1} + 2\,V_{2\,z_2} + 12\,V_{z_1,z_2} + 12\,V_{3\,z_1,z_2} = 0. \end{aligned} \tag{35}$$

Next thing that we have to do is to express (35) in linear combination of binary Bell polynomials $\mathscr{Y}$-polynomials(see [25] for detailed discussion), for this purpose we choose additional constraint of lowest possible weight

$$C[V_{z_1}^2 + W_{z_1,z_2}] + V_{z_2} = 0. \tag{36}$$

On account of constraint (36), the two field equation (35) decouples into

$$C\,\mathscr{Y}_{2\,z_1}(V,W) + \mathscr{Y}_{z_2}(V,W) = 0, \tag{37a}$$

$$6\,\mathscr{Y}_{z_2}(V,W) - C\,\mathscr{Y}_{z_1,z_2}(V,W) + 6\,\mathscr{Y}_{2\,z_1,z_2}(V,W) = 0. \tag{37b}$$

It is straightforward to write (37) in bilinear Bäcklund transformations

$$(C\,D_{z_1}^2 + D_{z_2})f\cdot\tilde{f} = 0, \tag{38a}$$

$$(-C\,D_{z_1}D_{z_2} + 6\,D_{z_1}^2 D_{z_2} + 6\,D_{z_2})f\cdot\tilde{f} = 0. \tag{38b}$$

It is quite interesting to note that the Bell system (37) can be linearize using Cole-Hopf transformation $V = \log\psi$ which result in Lax pairs for (28) as follow

$$C\,\psi_{2\,z_1} - C\,\psi\,U_{z_1} + \psi_{z_2} - \lambda\psi = 0, \tag{39a}$$

$$C\psi\,U_{z_2} - \psi_{z_1 z_2} - 12\,U_{z_2}\,\psi_{z_1} - 6\,U_{z_1}\,\psi_{z_2} + 6\,\psi_{2\,z_1,z_2} + 6\,\psi_{z_2} = 0, \tag{39b}$$

here $\lambda$ plays role of spectral parameter.

# 6. Conclusion

To conclude, starting with symmetry reduction of Lax pair of equation (1) we have obtained three reductions along with one exact solution which contains several arbitrary functions. It is quite remarkable to note that the process of reducing Lax pair from (2+1)–dimension to (1+1)–dimensional spectral problem is useful since there are some methods which works better in lower dimension and sometime as in third case it is possible to directly integrate the reduced spectral problem for exact solution. During investigation of reductions, we came to realise that, under similar symmetry group the reduced equation (1) is same as compatibility condition of its reduced Lax pair which is ofcourse a trivial conclusion. It would be a matter of great importance if there exist symmetry for Lax pair (3) such that symmetry reduction of equation (1) and compatibility of reduced Lax pair under the symmetry are inequivalent. Moreover, by symmetry reduction of Lax for equation (1) we have obtained its exact solution (27) which may be useful to analyze dynamics of incompressible fluids by appropriately choosing arbitrary functions. Nevertheless, the reduced equation (28) is thoroughly investigated for traveling wave solutions and integrable properties such as Bäcklund transformations and Lax pairs.

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# Appendix

For multi-variable $C^\infty$ function $f = f(x_1, x_2, ..., x_n)$, the multi-dimensional Bell polynomial is defined as follow:

$$Y_{n_1 x_1,...,n_l x_l}(f) = Y_{n_1,...,n_l}(f_{r_1 x_1,...,r_l x_l}) = e^{-f}\partial_{x_1}^{n_1}...\partial_{x_l}^{n_l} e^f \tag{40}$$

where we have considered $f_{r_1x_1,...,r_lx_l} = \partial_{x_1}^{r_1}...\partial_{x_l}^{r_l} f$, $r_i = 0, 1, ..., n_i$ and $i = 1, 2, ..., l$. These polynomials admits generalised Faà di Bruno formula

$$Y_{n_1x_1,...,n_lx_l}(f) = \sum \frac{n_1!n_2!\cdots n_l!}{b_1!b_2!\cdots b_k!} \prod_{j=1}^{k} \left( \frac{f_{r_{1j},...,r_{lj}}}{r_{1j}!,...,r_{lj}!} \right)^{b_j} \tag{41}$$

Using partitional formula (41), first few Bell polynomials can be written as follows:

$$\begin{aligned} &Y_{x_1}(f) = f_{x_1}, \quad Y_{2x_1}(f) = f_{2x_1} + f_{x_1}^2, \quad Y_{x_1,x_2}(f) = f_{x_1,x_2} + f_{x_1}f_{x_2} \\ &Y_{3x_1}(f) = f_{3x_1} + 3f_{x_1}f_{2x_1} + f_{x_1}^3, \quad Y_{2x_1,x_2} = f_{2x_1,x_2} + f_{2x_1}f_{x_2} + 2f_{x_1,x_2}f_{x_1} + f_{x_1}^2 f_{x_2}, ..... \end{aligned} \tag{42}$$

The multi-dimensional binary Bell polynomials can be defined as follows:

$$\mathscr{Y}_{n_1x_1,...,n_lx_l}(V, W) = Y_{n_1x_1,...,n_lx_l}(f)\big|_{f_{r_1x_1,...,r_lx_l}} = \begin{cases} V_{r_1x_1,...,r_lx_l}, & \sum_{i=1}^{l} r_i \quad \text{is odd} \\ W_{r_1x_1,...,r_lx_l}, & \sum_{i=1}^{l} r_i \quad \text{is even} \end{cases} \tag{43}$$

Based on the above definition, the first few lowest order binary Bell polynomials are

$$\begin{aligned} &\mathscr{Y}_x(V) = V_x, \quad \mathscr{Y}_{2x}(V, W) = W_{2x} + V_x^2, \quad \mathscr{Y}_{x,t}(V, W) = W_{x,t} + V_xV_t \\ &\mathscr{Y}_{3x}(V, W) = V_{3x} + 3V_xW_{2x} + V_x^3, \quad \mathscr{Y}_{2x,t}(V, W) = V_{2x,t} + V_tW_{2x} + 2V_xW_{x,t} + V_x^2V_t \\ &\mathscr{Y}_{4x}(V, W) = W_{4x} + 4V_xV_{3x} + 3W_{2x}^2 + 6V_x^2W_{2x} + V_x^4, ..... \end{aligned} \tag{44}$$

The direct link between binary Bell polynomials and standard Hirota D-operator is given by

$$\mathscr{Y}_{n_1x_1,...,n_lx_l}(V, W)\Big|_{V=\log\frac{f}{g}, W=\log fg} = (fg)^{-1} D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} f \cdot g \tag{45}$$

where the D-operator is defined by

$$D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} f \cdot g = (\partial_{x_1} - \partial_{x_1'})^{n_1} \cdots (\partial_{x_l} - \partial_{x_l'})^{n_l} f(x_1, ..., x_l) \cdot g(x_1', ..., x_l')\big|_{x_1'=x_1,...,x_l'=x_l} \tag{46}$$

For $f = g$, the identity (45) reduces to particular case

$$(f)^{-2} D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} f \cdot f = \mathscr{Y}_{n_1x_1,...,n_lx_l}(0, q = 2\log f) = \begin{cases} 0, & \sum_{i=1}^{l} n_i \quad \text{is odd} \\ \mathscr{P}_{n_1x_1,...,n_lx_l}(q), & \sum_{i=1}^{l} n_i \quad \text{is even} \end{cases} \tag{47}$$

where these even ordered $\mathscr{Y}$-polynomials are called $\mathscr{P}$-polynomials and first few of them are given by following formulas

$$\mathscr{P}_{2x}(q) = q_{2x}, \quad \mathscr{P}_{x,t}(q) = q_{x,t}, \quad \mathscr{P}_{3x,t}(q) = q_{3x,t} + 3q_{x,t}q_{2x}, ...... \tag{48}$$

The binary Bell polynomials $\mathscr{Y}_{n_1x_1,...,n_lx_l}(V, W)$ can be separated into generalised Bell polynomials $Y_{n_1x_1,...,n_lx_l}(V)$ and $\mathscr{P}$-polynomials

$$(fg)^{-1} D_{x_1}^{n_1} \cdots D_{x_l}^{n_l} f \cdot g = \mathscr{Y}_{n_1x_1,...,n_lx_l}(V, W)\Big|_{V=\log\frac{f}{g}, W=\log fg}$$

$$
\begin{aligned}
&= \mathscr{Y}_{n_1x_1,\ldots,n_lx_l}(V, V+q)\Big|_{V=\log\frac{f}{g},\, q=2\log g} \\
&= \sum_{r_1=0}^{n_1}\cdots\sum_{r_l=0}^{n_l}\prod_{i=1}^{l}\begin{pmatrix} n_i \\ r_i \end{pmatrix}\mathscr{P}_{r_1x_1,\ldots,r_lx_l}(q)Y_{(n_1-r_1)x_1,\ldots,(n_l-r_l)x_l}(V) \qquad (49)
\end{aligned}
$$

The key observation here is that, the generalised Bell polynomial can be linearised by Cole-Hopf transformation $V = \log\psi$ i.e.

$$
Y_{n_1x_1,\ldots,n_lx_l}(V = \log\psi) = \frac{\psi_{n_1x_1,\ldots,n_lx_l}}{\psi} \qquad (50)
$$

such that (49)

$$
(fg)^{-1}D_{x_1}^{n_1}\cdots D_{x_l}^{n_l} f\cdot g\big|_{g=\exp(\frac{q}{2}),\,\frac{f}{g}=\psi} = \psi^{-1}\sum_{r_1=0}^{n_1}\cdots\sum_{r_l=0}^{n_l}\prod_{i=1}^{l}\begin{pmatrix} n_i \\ r_i \end{pmatrix}\mathscr{P}_{r_1x_1,\ldots,r_lx_l}(q)\psi_{(n_1-r_1)x_1,\ldots,(n_l-r_l)x_l}.
$$

The formula (51) helps to construct associated Lax system for non-linear equations in shortest possible way.