



The Laplace-Adomian Decomposition Method Applied to the Kundu-Eckhaus Equation

Research Article

O. González-Gaxiola^{1*}¹ Departamento de Matemáticas Aplicadas y Sistemas, Universidad Autónoma Metropolitana-Cuajimalpa, Cuajimalpa, México D.F.

Abstract: The Kundu-Eckhaus equation is a nonlinear partial differential equation which seems in the quantum field theory, weakly nonlinear dispersive water waves and nonlinear optics. In spite of its importance, exact solution to this nonlinear equation are rarely found in literature. In this work, we solve this equation and present a new approach to obtain the solution by means of the combined use of the Adomian Decomposition Method and the Laplace Transform (LADM). Besides, we compare the behaviour of the solutions obtained with the only exact solutions given in the literature through fractional calculus. Moreover, it is shown that the proposed method is direct, effective and can be used for many other nonlinear evolution equations in mathematical physics.

MSC: 35Q40, 35A25, 37L65.

Keywords: Kundu-Eckhaus equation, Nonlinear Schrödinger equation, Adomian decomposition method.

© JS Publication.

1. Introduction

Most of the phenomena that arise in the real world can be described by means of nonlinear partial and ordinary differential equations and, in some cases, by integral or integro-differential equations. However, most of the mathematical methods developed so far, are only capable to solve linear differential equations. In the 1980's, George Adomian (1923-1996) introduced a powerful method to solve nonlinear differential equations. Since then, this method is known as the Adomian decomposition method (ADM) [3, 4]. The technique is based on a decomposition of a solution of a nonlinear differential equation in a series of functions. Each term of the series is obtained from a polynomial generated by a power series expansion of an analytic function. The Adomian method is very simple in an abstract formulation but the difficulty arises in calculating the polynomials that becomes a non-trivial task. This method has widely been used to solve equations that come from nonlinear models as well as to solve fractional differential equations [13, 14, 25].

The Kundu-Eckhaus equation has been studied by many researchers and those studies done through varied and different methods have yielded much information related to the behavior of their solutions. The mathematical structure of Kundu-Eckhaus equation was studied for the first time in [18] and [22]. For example, in [27] the authors applied Bäcklund transformation for obtaining bright and dark soliton solutions to the Eckhaus-Kundu equation with the cubic-quintic nonlinearity. After much research have been related to the equation by various methods, many of them can be found in [9] and [26] and some applications of the equation nonlinear optics can be found in [21] and [12]. Recently, in [19] the authors obtain

* E-mail: ogonzalez@correo.cua.uam.mx

obtain some new complex analytical solutions to the Kundu-Eckhaus equation which seems in the quantum field theory, weakly nonlinear dispersive water waves and nonlinear optics using improved Bernoulli sub-equation function method.

In the present work we will utilize the Adomian decomposition method in combination with the Laplace transform (LADM) [30] to solve the Kundu-Eckhaus equation. This equation is a nonlinear partial differential equation that, in nonlinear optics, is used to model some dispersion phenomena. We will decompose the nonlinear terms of this equation using the Adomian polynomials and then, in combination with the use of the Laplace transform, we will obtain an algorithm to solve the problem subject to initial conditions. Finally, we will illustrate our procedure and the quality of the obtained algorithm by means of the solution of an example in which the Kundu-Eckhaus equation is solved for some initial condition and we will compare the results with previous results reported in the literature.

Our work is divided in several sections. In “The Adomian Decomposition Method Combined With Laplace Transform” section, we present, in a brief and self-contained manner, the LADM. Several references are given to delve deeper into the subject and to study its mathematical foundation that is beyond the scope of the present work. In “The nonlinear Kundu-Eckhaus Equation” section, we give a brief introduction to the model described by the Kundu-Eckhaus equation and we will establish that LADM can be used to solve this equation in its nonlinear version. In “The General Solution of the Nonlinear Kundu-Eckhaus Equation Through LADM” and the “Numerical Example”, we will show by means of numerical examples, the quality and precision of our method, comparing the obtained results with the only exact solutions available in the literature [34]. Finally, in the “Conclusion” section, we summarise our findings and present our final conclusions.

2. The Adomian Decomposition Method Combined With Laplace Transform

The ADM is a method to solve ordinary and partial nonlinear differential equations. Using this method is possible to express analytic solutions in terms of a series [4, 31]. In a nutshell, the method identifies and separates the linear and nonlinear parts of a differential equation. Inverting and applying the highest order differential operator that is contained in the linear part of the equation, it is possible to express the solution in terms of the rest of the equation affected by the inverse operator. At this point, the solution is proposed by means of a series with terms that will be determined and that give rise to the Adomian Polynomials [29]. The nonlinear part can also be expressed in terms of these polynomials. The initial (or the border conditions) and the terms that contain the independent variables will be considered as the initial approximation. In this way and by means of a recurrence relations, it is possible to find the terms of the series that give the approximate solution of the differential equation. Given a partial (or ordinary) differential equation

$$Fu(x, t) = g(x, t) \quad (1)$$

with initial condition

$$u(x, 0) = f(x) \quad (2)$$

where F is a differential operator that could, in general, be nonlinear and therefore includes some linear and nonlinear terms. In general, equation (1) could be written as

$$L_t u(x, t) + Ru(x, t) + Nu(x, t) = g(x, t) \quad (3)$$

where $L_t = \frac{\partial}{\partial t}$, R is a linear operator that includes partial derivatives with respect to x , N is a nonlinear operator and g is a non-homogeneous term that is u -independent. Solving for $L_t u(x, t)$, we have

$$L_t u(x, t) = g(x, t) - Ru(x, t) - Nu(x, t). \quad (4)$$

The Laplace transform \mathcal{L} is an integral transform discovered by Pierre-Simon Laplace and is a powerful and very useful technique for solving ordinary and partial differential Equations, which transforms the original differential equation into an elementary algebraic equation [17]. Before using the Laplace transform combined with Adomian decomposition method we review some basic definitions and results on it.

Definition 2.1. Given a function $f(t)$ defined for all $t \geq 0$, the Laplace transform of f is the function F defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st} dt \tag{5}$$

for all values of s for which the improper integral converges. In particular $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$.

It is well known that there exists a bijection between the set of functions satisfying some hypotheses and the set of their Laplace transforms. Therefore, it is quite natural to define the inverse Laplace transform of $F(s)$.

Definition 2.2. Given a continuous function $f(t)$, if $F(s) = \mathcal{L}\{f(t)\}$, then $f(t)$ is called the inverse Laplace transform of $F(s)$ and denoted $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

The Laplace transform has the derivative properties:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0), \tag{6}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s), \tag{7}$$

where the superscript (n) denotes the n -th derivative with respect to t for $f^{(n)}(t)$, and with respect to s for $F^{(n)}(s)$. The LADM consists of applying Laplace transform [30] first on both sides of Eq. (4), obtaining

$$\mathcal{L}\{L_t u(x, t)\} = \mathcal{L}\{g(x, t) - Ru(x, t) - Nu(x, t)\} \tag{8}$$

An equivalent expression to (8) is

$$su(x, s) - u(x, 0) = \mathcal{L}\{g(x, t) - Ru(x, t) - Nu(x, t)\} \tag{9}$$

In the homogeneous case, $g(x, t) = 0$, we have

$$u(x, s) = \frac{f(x)}{s} - \frac{1}{s} \mathcal{L}\{Ru(x, t) + Nu(x, t)\} \tag{10}$$

now, applying the inverse Laplace transform to equation (10)

$$u(x, t) = f(x) - \mathcal{L}^{-1}\left[\frac{1}{s} \mathcal{L}\{Ru(x, t) + Nu(x, t)\}\right]. \tag{11}$$

The ADM method proposes a series solution $u(x, t)$ given by,

$$u(x, t) = \sum_{n=0}^\infty u_n(x, t) \tag{12}$$

The nonlinear term $Nu(x, t)$ is given by

$$Nu(x, t) = \sum_{n=0}^\infty A_n(u_0, u_1, \dots, u_n) \tag{13}$$

where $\{A_n\}_{n=0}^{\infty}$ is the so-called Adomian polynomials sequence established in [29] and [7] and, in general, give us term to term:

$$\begin{aligned} A_0 &= N(u_0) \\ A_1 &= u_1 N'(u_0) \\ A_2 &= u_2 N'(u_0) + \frac{1}{2} u_1^2 N''(u_0) \\ A_3 &= u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!} u_1^3 N^{(3)}(u_0) \\ A_4 &= u_4 N'(u_0) + \left(\frac{1}{2} u_2^2 + u_1 u_3\right) N''(u_0) + \frac{1}{2!} u_1^2 u_2 N^{(3)}(u_0) + \frac{1}{4!} u_1^4 N^{(4)}(u_0) \\ &\vdots \end{aligned}$$

Other polynomials can be generated in a similar way. For more details, see [29] and [7] and references therein. Some other approaches to obtain Adomian's polynomials can be found in [15, 16]. Using (12) and (13) into equation (11), we obtain,

$$\sum_{n=0}^{\infty} u_n(x, t) = f(x) - \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \left\{ R \sum_{n=0}^{\infty} u_n(x, t) + \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n) \right\} \right], \tag{14}$$

From the equation (14) we deduce the following recurrence formulas

$$\begin{cases} u_0(x, t) = f(x), \\ u_{n+1}(x, t) = -\mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \{ R u_n(x, t) + A_n(u_0, u_1, \dots, u_n) \} \right], \quad n = 0, 1, 2, \dots \end{cases} \tag{15}$$

Using (15) we can obtain an approximate solution of (1), (2) using

$$u(x, t) \approx \sum_{n=0}^k u_n(x, t), \quad \text{where} \quad \lim_{k \rightarrow \infty} \sum_{n=0}^k u_n(x, t) = u(x, t). \tag{16}$$

It becomes clear that, the Adomian decomposition method, combined with the Laplace transform needs less work in comparison with the traditional Adomian decomposition method. This method decreases considerably the volume of calculations. The decomposition procedure of Adomian will be easily set, without linearising the problem. In this approach, the solution is found in the form of a convergent series with easily computed components; in many cases, the convergence of this series is very fast and only a few terms are needed in order to have an idea of how the solutions behave. Convergence conditions of this series are examined by several authors, mainly in [1, 2, 10, 11]. Additional references related to the use of the Adomian Decomposition Method, combined with the Laplace transform, can be found in [5, 20, 30].

3. The Nonlinear Kundu-Eckhaus Equation

In mathematical physics, the Kundu-Eckhaus equation is a nonlinear partial differential equations within the nonlinear Schrödinger class [18, 22]:

$$i u_t + u_{xx} + 2(|u|^2)_x u + |u|^4 u = 0. \tag{17}$$

In the equation (17) the dependent variable $u(x, t)$ is a complex-valued function of two real variables x and t . The equation (17) is a basic model that describes optical soliton propagation in Kerr media [24]. The complete integrability and multi-soliton solutions, breather solutions, and various types of rogue wave solutions associated with the Kundu-Eckhaus equation have been widely reported by many authors [6, 8, 23, 24]. Nevertheless, in optic fiber communications systems, one always

has to increase the intensity of the incident light field to produce ultrashort (femtosecond) optical pulses [33]. In this case, the simple NLS equation is inadequate to accurately describe the phenomena, and higher-order nonlinear terms, such as third-order dispersion, self-steepening, and self-frequency shift, must be taken into account [27, 28, 32]. Explicitly calculating the derivatives that appear in equation (17), we obtain

$$u_t = iu_{xx} + 2i(|u|^2)_x u + i|u|^4 u. \quad (18)$$

To make the description of the the problem complete, we will consider some initial condition

$$u(x, 0) = f(x)$$

In the following section we will develop an algorithm using the method described in section 2 in order to solve the nonlinear Kundu-Eckhaus equation (18) without resort to any truncation or linearization.

4. The General Solution of the Nonlinear Kundu-Eckhaus Equation Through LADM

Comparing (18) with equation (4) we have that $g(x, t) = 0$, L_t and R becomes:

$$L_t u = \frac{\partial}{\partial t} u, \quad R u = i \frac{\partial^2 u}{\partial x^2}, \quad (19)$$

while the nonlinear term is given by

$$N u = i[2(|u|^2)_x + |u|^4]u. \quad (20)$$

By using now equation (15) through the LADM method we obtain recursively

$$\begin{cases} u_0(x, t) = f(x), \\ u_{n+1}(x, t) = \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \{ R u_n(x, t) + A_n(u_0, u_1, \dots, u_n) \} \right], \quad n = 0, 1, 2, \dots \end{cases} \quad (21)$$

Note that, the nonlinear term $N u = i[2(|u|^2)_x + |u|^4]u$ can be split into three terms to facilitate calculations

$$N_1 u = iu^3 \bar{u}^2, \quad N_2 u = 2iu^2 \bar{u}_x, \quad N_3 u = 2iu_x u \bar{u}$$

from this, we will consider the decomposition of the nonlinear terms into Adomian polynomials as

$$N_1 u = iu^3 \bar{u}^2 = \sum_{n=0}^{\infty} P_n(u_0, u_1, \dots, u_n) \quad (22)$$

$$N_2 u = 2iu^2 \bar{u}_x = \sum_{n=0}^{\infty} Q_n(u_0, u_1, \dots, u_n). \quad (23)$$

$$N_3 u = 2iu_x u \bar{u} = \sum_{n=0}^{\infty} R_n(u_0, u_1, \dots, u_n). \quad (24)$$

Calculating, we obtain

$$\begin{aligned}
 P_0 &= i\bar{u}_0^2 u_0^3, \\
 P_1 &= 3i\bar{u}_0^2 u_0^2 u_1 + 2i\bar{u}_0 u_1 u_0^3, \\
 P_2 &= 3i\bar{u}_0^2 u_0^2 u_2 + 3i\bar{u}_0^2 u_0 u_1^2 + 6i\bar{u}_0 \bar{u}_1 u_0^2 u_1 + 2i\bar{u}_0 \bar{u}_2 u_0^3 + i\bar{u}_1^2 u_0^3, \\
 P_3 &= 3i\bar{u}_0^2 u_0^2 u_3 + 6i\bar{u}_0^2 u_0 u_1 u_2 + i\bar{u}_0^2 u_1^3 + 6i\bar{u}_0 u_1 u_0^2 u_2 + 6i\bar{u}_0 \bar{u}_1 u_0 u_1^2 + 2i\bar{u}_0 \bar{u}_3 u_0^3 + 3i\bar{u}_1^2 u_0^2 u_1 + 2i\bar{u}_1 \bar{u}_2 u_0^3 + 6i\bar{u}_0 \bar{u}_2 u_0^2 u_1, \\
 P_4 &= 3i\bar{u}_0^2 u_0^2 u_4 + 3i\bar{u}_0^2 u_1^2 u_2 + 6i\bar{u}_0^2 u_0 u_1 u_3 + 3i\bar{u}_0^2 u_0 u_2^2 + 6i\bar{u}_0 \bar{u}_1 u_0^2 u_3 + 2i\bar{u}_0 \bar{u}_1 u_1^3 + 12i\bar{u}_0 \bar{u}_1 u_0 u_1 u_2 + 6i\bar{u}_0 \bar{u}_2 u_0 u_1^2 \\
 &\quad + 6i\bar{u}_0 \bar{u}_2 u_0^2 u_2 + 6i\bar{u}_0 \bar{u}_3 u_0^2 u_1 + 2i\bar{u}_0 \bar{u}_4 u_0^3 + 3i\bar{u}_1^2 u_0^2 u_2 + 3i\bar{u}_1^2 u_0 u_1^2 + 6i\bar{u}_1 u_2 u_0^2 u_1 + 2i\bar{u}_1 \bar{u}_3 u_0^3 + i\bar{u}_2^2 u_0^3, \\
 &\quad \vdots \\
 Q_0 &= 2iu_0^2 \bar{u}_{0x}, \\
 Q_1 &= 2iu_0^2 \bar{u}_{1x} + 4iu_0 u_1 \bar{u}_{0x}, \\
 Q_2 &= 2iu_1^2 \bar{u}_{0x} + 2iu_0^2 \bar{u}_{2x} + 4iu_0 u_1 \bar{u}_{1x} + 4iu_0 u_2 \bar{u}_{0x}, \\
 Q_3 &= 2iu_1^2 \bar{u}_{1x} + 2iu_0^2 \bar{u}_{3x} + 4iu_0 u_1 \bar{u}_{2x} + 4iu_0 u_2 \bar{u}_{1x} + 4iu_0 u_3 \bar{u}_{0x} + 4iu_1 u_2 \bar{u}_{0x}, \\
 Q_4 &= 2iu_2^2 \bar{u}_{0x} + 2iu_1^2 \bar{u}_{2x} + 4iu_0 u_1 \bar{u}_{3x} + 4iu_0 u_2 \bar{u}_{2x} + 4iu_0 u_3 \bar{u}_{1x} + 4iu_0 u_4 \bar{u}_{0x} + 4iu_1 u_2 \bar{u}_{1x} + 4iu_1 u_3 \bar{u}_{0x}, \\
 &\quad \vdots \\
 R_0 &= 2iu_0 \bar{u}_0 u_{0x}, \\
 R_1 &= 2iu_0 \bar{u}_0 u_{1x} + 2iu_0 \bar{u}_1 u_{0x} + 2iu_1 \bar{u}_0 u_{0x}, \\
 R_2 &= 2iu_0 \bar{u}_0 u_{2x} + 2iu_0 \bar{u}_1 u_{1x} + 2iu_0 \bar{u}_2 u_{0x} + 2iu_1 \bar{u}_0 u_{1x} + 2iu_1 \bar{u}_1 u_{0x} + 2iu_2 \bar{u}_0 u_{0x}, \\
 R_3 &= 2iu_0 \bar{u}_0 u_{3x} + 2iu_0 \bar{u}_1 u_{2x} + 2iu_0 \bar{u}_2 u_{1x} + 2iu_0 \bar{u}_3 u_{0x} + 2iu_1 \bar{u}_0 u_{2x} + 2iu_1 \bar{u}_1 u_{1x} + 2iu_1 \bar{u}_2 u_{0x} + 2iu_2 \bar{u}_0 u_{1x} \\
 &\quad + 2iu_2 \bar{u}_1 u_{0x} + 2iu_3 \bar{u}_0 u_{0x}, \\
 R_4 &= 2iu_0 \bar{u}_0 u_{4x} + 2iu_0 \bar{u}_1 u_{3x} + 2iu_0 \bar{u}_2 u_{2x} + 2iu_0 \bar{u}_3 u_{1x} + 2iu_0 \bar{u}_4 u_{0x} + 2iu_1 \bar{u}_0 u_{3x} + 2iu_1 \bar{u}_1 u_{2x} + 2iu_1 \bar{u}_2 u_{1x} \\
 &\quad + 2iu_1 \bar{u}_3 u_{0x} + 2iu_2 \bar{u}_0 u_{2x} + 2iu_2 \bar{u}_1 u_{1x} + 2iu_2 \bar{u}_2 u_{0x} + 2iu_3 \bar{u}_0 u_{1x} + 2iu_3 \bar{u}_1 u_{0x} + 2iu_4 \bar{u}_0 u_{0x}, \\
 &\quad \vdots
 \end{aligned}$$

Now, considering (22), (23) and (24), we have

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \dots, u_n) = \sum_{n=0}^{\infty} ((P_n + Q_n + R_n)(u_0, u_1, \dots, u_n)), \tag{25}$$

then, the Adomian polynomials corresponding to the nonlinear part $Nu = i[2(|u|^2)_x + |u|^4]u$ are

$$\begin{aligned}
 A_0 &= i\bar{u}_0^2 u_0^3 + 2iu_0^2 \bar{u}_{0x} + 2iu_0 \bar{u}_0 u_{0x}, \\
 A_1 &= 3i\bar{u}_0^2 u_0^2 u_1 + 2i\bar{u}_0 u_1 u_0^3 + 2iu_0^2 \bar{u}_{1x} + 4iu_0 u_1 \bar{u}_{0x} + 2iu_0 \bar{u}_0 u_{1x} + 2iu_0 \bar{u}_1 u_{0x} + 2iu_1 \bar{u}_0 u_{0x}, \\
 A_2 &= 3i\bar{u}_0^2 u_0^2 u_2 + 3i\bar{u}_0^2 u_0 u_1^2 + 6i\bar{u}_0 \bar{u}_1 u_0^2 u_1 + 2i\bar{u}_0 \bar{u}_2 u_0^3 + i\bar{u}_1^2 u_0^3 + 2iu_1^2 \bar{u}_{0x} + 2iu_0^2 \bar{u}_{2x} + 4iu_0 u_1 \bar{u}_{1x} \\
 &\quad + 4iu_0 u_2 \bar{u}_{0x} + 2iu_0 \bar{u}_0 u_{2x} + 2iu_0 \bar{u}_1 u_{1x} + 2iu_0 \bar{u}_2 u_{0x} + 2iu_1 \bar{u}_0 u_{1x} + 2iu_1 \bar{u}_1 u_{0x} + 2iu_2 \bar{u}_0 u_{0x}, \\
 A_3 &= 3i\bar{u}_0^2 u_0^2 u_3 + 6i\bar{u}_0^2 u_0 u_1 u_2 + i\bar{u}_0^2 u_1^3 + 6i\bar{u}_0 u_1 u_0^2 u_2 + 6i\bar{u}_0 \bar{u}_1 u_0 u_1^2 + 2i\bar{u}_0 \bar{u}_3 u_0^3 + 3i\bar{u}_1^2 u_0^2 u_1 + 2i\bar{u}_1 \bar{u}_2 u_0^3 + 6i\bar{u}_0 \bar{u}_2 u_0^2 u_1 \\
 &\quad + 2iu_1^2 \bar{u}_{1x} + 2iu_0^2 \bar{u}_{3x} + 4iu_0 u_1 \bar{u}_{2x} + 4iu_0 u_2 \bar{u}_{1x} + 4iu_0 u_3 \bar{u}_{0x} + 4iu_1 u_2 \bar{u}_{0x} + 2iu_0 \bar{u}_0 u_{3x} + 2iu_0 \bar{u}_1 u_{2x} + 2iu_0 \bar{u}_2 u_{1x} \\
 &\quad + 2iu_0 \bar{u}_3 u_{0x} + 2iu_1 \bar{u}_0 u_{2x} + 2iu_1 \bar{u}_1 u_{1x} + 2iu_1 \bar{u}_2 u_{0x} + 2iu_2 \bar{u}_0 u_{1x} + 2iu_2 \bar{u}_1 u_{0x} + 2iu_3 \bar{u}_0 u_{0x},
 \end{aligned}$$

$$\begin{aligned}
A_4 = & 3i\bar{u}_0^2 u_0^2 u_4 + 3i\bar{u}_0^2 u_1^2 u_2 + 6i\bar{u}_0^2 u_0 u_1 u_3 + 3i\bar{u}_0^2 u_0 u_2^2 + 6i\bar{u}_0 \bar{u}_1 u_0^2 u_3 + 2i\bar{u}_0 \bar{u}_1 u_1^3 + 12i\bar{u}_0 \bar{u}_1 u_0 u_1 u_2 + 6i\bar{u}_0 \bar{u}_2 u_0 u_1^2 \\
& + 6i\bar{u}_0 \bar{u}_2 u_0^2 u_2 + 6i\bar{u}_0 \bar{u}_3 u_0^2 u_1 + 2i\bar{u}_0 \bar{u}_4 u_0^3 + 3i\bar{u}_1^2 u_0^2 u_2 + 3i\bar{u}_1^2 u_0 u_1^2 + 6i\bar{u}_1 u_2 u_0^2 u_1 + 2i\bar{u}_1 \bar{u}_3 u_0^3 + i\bar{u}_2^2 u_0^3 + 2iu_2^2 \bar{u}_{0x} + 2iu_1^2 \bar{u}_{2x} \\
& + 4iu_0 u_1 \bar{u}_{3x} + 4iu_0 u_2 \bar{u}_{2x} + 4iu_0 u_3 \bar{u}_{1x} + 4iu_0 u_4 \bar{u}_{0x} + 4iu_1 u_2 \bar{u}_{1x} + 4iu_1 u_3 \bar{u}_{0x} + 2iu_0 \bar{u}_0 u_{4x} + 2iu_0 \bar{u}_1 u_{3x} + 2iu_0 \bar{u}_2 u_{2x} \\
& + 2iu_0 \bar{u}_3 u_{1x} + 2iu_0 \bar{u}_4 u_{0x} + 2iu_1 \bar{u}_0 u_{3x} + 2iu_1 \bar{u}_1 u_{2x} + 2iu_1 \bar{u}_2 u_{1x} + 2iu_1 \bar{u}_3 u_{0x} + 2iu_2 \bar{u}_0 u_{2x} + 2iu_2 \bar{u}_1 u_{1x} \\
& + 2iu_2 \bar{u}_2 u_{0x} + 2iu_3 \bar{u}_0 u_{1x} + 2iu_3 \bar{u}_1 u_{0x} + 2iu_4 \bar{u}_0 u_{0x}, \\
& \vdots
\end{aligned}$$

Using the expressions obtained above for equation (18), we will illustrate, with two examples, the effectiveness of LADM to solve the nonlinear Kundu-Eckhaus equation.

5. Numerical Example

Using Laplace Adomian decomposition method (LADM), we solve this Kundu-Eckhaus equation subject to the initial condition $u(0, x) = f(x) = \beta e^{ix}$. Here i is the imaginary unit and $\beta \in \mathbb{R}$ with $\beta \neq 0$. To use ADM, the equation (18) is decomposed in the operators (19) and (20). Through the LADM (21), we obtain recursively

$$\begin{aligned}
u_0(x, t) &= f(x), \\
u_1(x, t) &= \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \{ Ru_0 + A_0 \} \right], \\
u_2(x, t) &= \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \{ Ru_1 + A_1 \} \right], \\
&\vdots \\
u_{n+1}(x, t) &= \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \{ Ru_n + A_n \} \right].
\end{aligned}$$

Besides

$$\begin{aligned}
A_0 &= i\beta^5 e^{ix}, \\
A_1 &= -\beta^5 (1 - \beta^4) (3 + 2e^{2ix}) t e^{ix}, \\
A_2 &= \beta^3 (1 - \beta^4) \begin{pmatrix} 4i\beta^6 e^{-ix} + 11i\beta^6 e^{ix} + 6i\beta^6 e^{3ix} + 14\beta^4 e^{-ix} \\ -\beta^4 e^{ix} - 14\beta^4 e^{3ix} - i\beta^2 e^{ix} - 7e^{ix} \end{pmatrix} \frac{t^2}{2}, \\
A_3 &= \beta^3 (1 - \beta^4) \begin{pmatrix} 86i\beta^8 e^{-ix} - 143\beta^{10} e^{-ix} - 82\beta^{10} e^{3ix} \\ -36\beta^{10} e^{5ix} + 9i\beta^8 e^{ix} - 6i\beta^8 e^{3ix} \\ -96\beta^6 e^{-ix} + 110\beta^6 e^{ix} + 134\beta^6 e^{3ix} \\ +36\beta^6 e^{5ix} - 99i\beta^4 e^{ix} - 43\beta^2 e^{ix} \\ -18\beta^2 e^{3ix} + 24ie^{ix} + 8i\beta^4 e^{-ix} \end{pmatrix} \frac{t^3}{3!}.
\end{aligned}$$

With the above, we have

$$\begin{aligned}
u_0(x, t) &= \beta e^{ix}, \\
u_1(x, t) &= \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \{ iu_{0,xx} + i\beta^5 e^{ix} \} \right] = \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \{ -i\beta e^{ix} + i\beta^5 e^{ix} \} \right] \\
&= \mathcal{L}^{-1} \left[\frac{1}{s^2} \mathcal{L} \{ i\beta e^{ix} (\beta^4 - 1) \} \right] = -i\beta (1 - \beta^4) t e^{ix},
\end{aligned}$$

$$\begin{aligned}
 u_2(x, t) &= \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \{ iu_{1,xx} - \beta^5(1 - \beta^4)(3 + 2e^{2ix})e^{ix} \} \right] \\
 &= \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \{ \beta(1 - \beta^4)te^{ix} - \beta^5(1 - \beta^4)(3 + 2e^{2ix})e^{ix} \} \right] \\
 &= \mathcal{L}^{-1} \left[\frac{1}{s^3} \left((1 - \beta^4)(\beta e^{ix} - 3\beta^5 e^{ix} - 2\beta^5 e^{3ix}) \right) \right] \\
 &= [\beta(1 - \beta^4)(e^{ix} - 3\beta^4 e^{ix} - 2\beta^4 e^{3ix})] \frac{t^2}{2},
 \end{aligned}$$

$$\begin{aligned}
 u_3(x, t) &= \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \{ iu_{2,xx} + A_2 \} \right] \\
 &= -\beta(1 - \beta^4) \left(\begin{array}{c} 4i\beta^4 e^{ix} + 18i\beta^4 e^{3ix} - ie^{ix} + 7\beta^2 e^{ix} + 14\beta^6 e^{3ix} + \beta^6 e^{ix} \\ -14\beta^6 e^{-ix} - 6i\beta^8 e^{3ix} - 11i\beta^8 e^{ix} - 4i\beta^8 e^{-ix} \end{array} \right) \frac{t^3}{3!},
 \end{aligned}$$

$$\begin{aligned}
 u_4(x, t) &= \mathcal{L}^{-1} \left[\frac{1}{s} \mathcal{L} \{ iu_{3,xx} + A_3 \} \right] \\
 &= -\beta(1 - \beta^4) \left(\begin{array}{c} 143\beta^{12} e^{-ix} + 82\beta^{12} e^{3ix} + 36\beta^{12} e^{5ix} - 86i\beta^{10} e^{-ix} \\ -9i\beta^{10} e^{ix} + 6i\beta^{10} e^{3ix} + 100\beta^8 e^{-ix} - 99\beta^8 e^{ix} \\ -80\beta^8 e^{3ix} - 36\beta^8 e^{5ix} + 100i\beta^6 e^{ix} - 22i\beta^6 e^{-ix} \\ +126i\beta^6 e^{3ix} + 39\beta^4 e^{ix} - 144\beta^4 e^{3ix} - 17i\beta^2 e^{ix} \end{array} \right) \frac{t^4}{4!}.
 \end{aligned}$$

Thus, the solution approximate of the nonlinear Kundu-Eckhaus equation (18) is given by:

$$u_{LADM} = u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + u_4(x, t). \tag{26}$$

In the following examples, we will compare (26) with the exact solution of (18) found in [34], which is given for $\alpha = 1$ as

$$u(x, t) = \pm e^{ix} \cdot \frac{1}{\left(1 + \left(\frac{1}{u_0^4} - 1\right)e^{4it}\right)^{\frac{1}{4}}} \tag{27}$$

with the initial condition $u(x, 0) = u_0 e^{ix}$.

Example 5.1. In this numerical example, we will consider $\beta = \sqrt[16]{2}$. With this value for β we obtain

$$\begin{aligned}
 u_{LADM}(x, t) &= 1.04427e^{ix} + 0.19758ite^{ix} - 0.19758(-2.3784e^{3ix} - 2.5676e^{ix}) \frac{t^2}{2!} \\
 &\quad + 0.19758(-5.6569ie^{-ix} + 12.920ie^{3ix} - 18.156e^{-ix} + 18.156e^{3ix} - 10.802ie^{ix} \\
 &\quad + 8.9304e^{ix} - ie^{ix}) \frac{t^3}{3!} + 0.19758(-161.1606ie^{-ix} + 172.6550ie^{3ix} + 381.918e^{-ix} \\
 &\quad - 146.476e^{3ix} + 9.6329e^{5ix} + 97.2654ie^{ix} - 93.6281e^{ix}) \frac{t^4}{4!}.
 \end{aligned}$$

In Tables 1 and 2 we show, for different times, the values of the exact solution of (18) given in [34] and the values given by the approximation previous u_{LADM} . The expression for the exact solution of (18) is:

$$u_{exact}(x, t) = \frac{e^{ix}}{\left(1 + (2^{-\frac{1}{4}} - 1)e^{4it}\right)^{\frac{1}{4}}}. \tag{28}$$

Comparing the values of the exact solution with the values given by LADM, we conclude that the approximate solution is very close to the exact solution when time values are small. As time becomes greater.

The quality of the approximation is also shown in figures 1 and 2 where the real part of the exact u_{exact} and the approximate solution u_{LADM} (imaginary part of u_{exact} and the approximate solution u_{LADM} respectively) are depicted in the same figure.

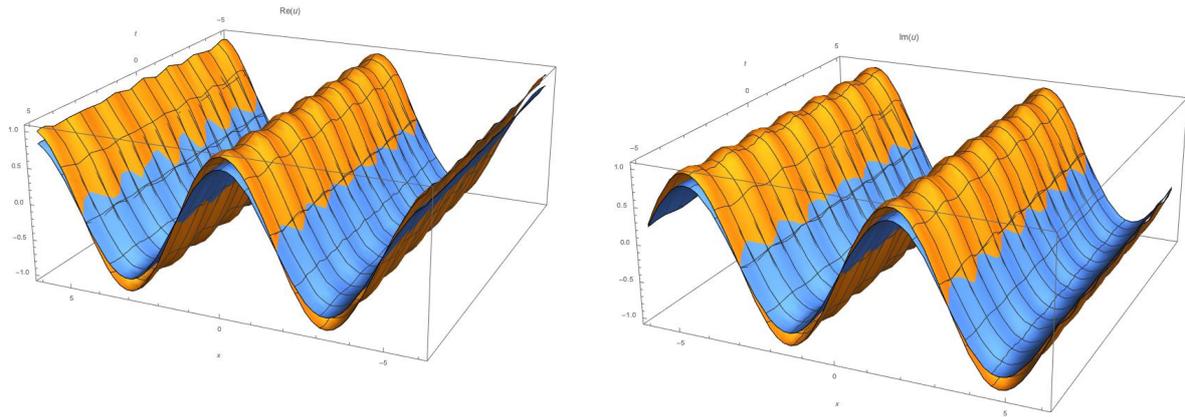


Figure 1. Plot of the real part (left) and imaginary part (right) of the approximate solution u_{LADM} versus the real part of the u_{exact} .

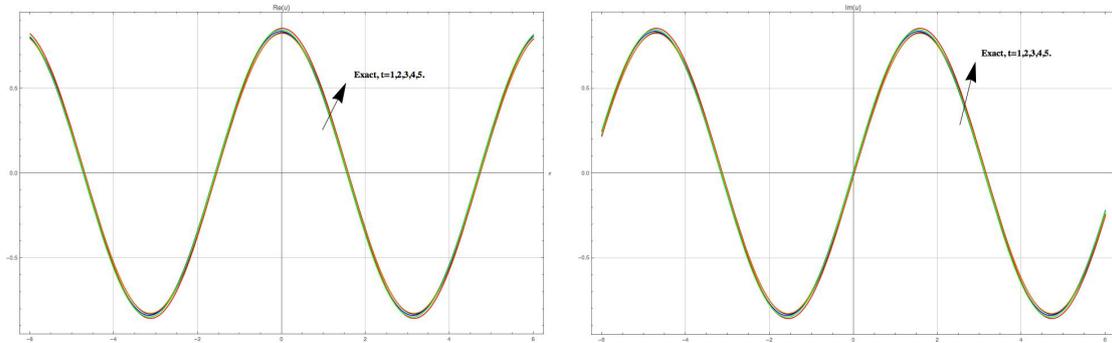


Figure 2. Graph of real part (left) and imaginary part (right) of u_{LADM} versus real part and imaginary part of u_{ex} for $t = 1, 2, 3, 4,$ and 5 .

From Tables 1 , 2, 3 and 4, we can conclude that the difference between the exact and the obtained LADM approximate solution is very small. This fact tells us about the effectiveness and accuracy of the LADM method.

x	$Re(u_{ex})$ [34]	$Re(u_{LADM})$	Error
0.5	0.867245034766	0.867450061289	2.05×10^{-4}
1.0	0.548389430252	0.548531893954	1.42×10^{-4}
1.5	0.095268967462	0.0953139882595	4.50×10^{-5}
2.0	-0.381176661184	-0.381240105952	6.34×10^{-5}
2.5	-0.764296949171	-0.764453326013	1.56×10^{-4}
3.0	-0.960290688213	-0.960501710624	2.11×10^{-4}
3.5	-0.921171775472	-0.921385777806	2.14×10^{-4}
4.0	-0.656517885107	-0.656682472129	1.64×10^{-4}
4.5	-0.231125519605	-0.231200394673	4.78×10^{-5}
5.0	0.250854433879	0.250887602795	3.31×10^{-5}

Table 1. Table for the real parts with $t = 1.0$.

x	$Re(u_{ex})$ [34]	$Re(u_{LADM})$	Error
0.5	0.851260107762	0.851444733853	1.84×10^{-4}
1.0	0.503432273480	0.503521108880	8.88×10^{-5}
1.5	0.0323466608364	0.0323179555396	2.87×10^{-5}
2.0	-0.446658542510	-0.446797760445	1.39×10^{-4}
2.5	-0.816306156888	-0.816521802055	2.15×10^{-4}
3.0	-0.986093554388	-0.986332829329	2.39×10^{-4}
3.5	-0.914450858558	-0.914655180424	2.04×10^{-4}
4.0	-0.618918699965	-0.619038043636	1.19×10^{-4}
4.5	-0.171853658076	-0.171858804059	5.14×10^{-6}
5.0	0.317287152916	0.317397464537	1.10×10^{-4}

Table 2. Table for the real parts with $t = 2.0$.

x	$Im(u_{ex})$ [34]	$Im(u_{LADM})$	Error
0.5	0.443634458364	0.443712601886	7.81×10^{-5}
1.0	0.805105282409	0.805272154752	1.66×10^{-4}
1.5	0.969458254291	0.969672999286	2.14×10^{-4}
2.0	0.896454034484	0.896664075067	2.10×10^{-4}
2.5	0.603966602108	0.604120513019	1.53×10^{-4}
3.0	0.163607081464	0.163667179944	6.00×10^{-5}
3.5	-0.316809158718	-0.316857586873	4.84×10^{-5}
4.0	-0.719659467741	-0.719804565630	1.45×10^{-4}
4.5	-0.946312040059	-0.946518282658	2.06×10^{-4}
5.0	-0.941274421185	-0.941491313113	2.16×10^{-4}

Table 3. Table for the imaginary parts with $t = 1.0$.

x	$Im(u_{ex})$ [34]	$Im(u_{LADM})$	Error
0.5	0.508147216007	0.508299876291	1.52×10^{-4}
1.0	0.854056971297	0.854279457764	2.22×10^{-4}
1.5	0.990863793735	0.991101633938	2.37×10^{-4}
2.0	0.885072601884	0.885267564246	1.94×10^{-4}
2.5	0.562584769106	0.562689120041	1.04×10^{-4}
3.0	0.102356564020	0.102344754781	1.18×10^{-5}
3.5	-0.382932097747	-0.383057175847	1.25×10^{-4}
4.0	-0.774465626762	-0.774673350242	2.07×10^{-4}
4.5	-0.976382959913	-0.976622470821	2.39×10^{-4}
5.0	-0.939247691931	-0.939460349642	2.12×10^{-4}

Table 4. Table for the imaginary parts with $t = 2.0$.

6. Conclusions

In order to show the accuracy and efficiency of our method, we have solved two examples, comparing our results with the exact solution of the equation that was obtained in [34]. Our results show that LADM produces highly accurate solutions in complicated nonlinear problems. We therefore, conclude that the Laplace-Adomian decomposition method is a notable non-sophisticated powerful tool that produces high quality approximate solutions for nonlinear partial differential equations using simple calculations and that attains converge with only few terms.

References

- [1] K.Abbasoui and Y.Cherruault, *Convergence of Adomian's method applied to differential equations*, Comput. Math. Appl., 28(5)(1994), 103-109.
- [2] K.Abbasoui and Y.Cherruault, *New ideas for proving convergence of decomposition methods*, Comput. Math. Appl., 29(7)(1995), 103-108.
- [3] G.Adomian, *Nonlinear Stochastic Operator Equations*, Academic Press, (1986).

-
- [4] G.Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, (1994).
- [5] G.Adomian, *A review of decomposition method and some recent results for nonlinear equation*, Math. Comput. Model., 13(7)(1992), 17-43.
- [6] A.Ankiewicz, J.K.Soto-Crespo and N.Akhmediev, *Rogue waves and rational solutions of the Hirota equation*, Phys. Rev. E., 81(2010).
- [7] E.Babolian and Sh.Javadi, *New method for calculating Adomian polynomials*, Applied Math. and Computation, 153(2004), 253-259.
- [8] U.Bandelow and N.Akhmediev, *Persistence of rogue waves in extended nonlinear Schrödinger equations: integrable Sasa-Satsuma case*, Phys. Lett. A., 376(2012), 1558-1561.
- [9] A.Bekir and A.Boz, *Exact solutions for nonlinear evolution equations using Exp-function method*, Phys. Lett. A., 372(2008), 1619-1625.
- [10] Y.Cherruault, *Convergence of Adomian's method*, Kybernetes, 18(2)(1989), 31-38.
- [11] Y.Cherruault and G.Adomian, *Decomposition methods: a new proof of convergence*, Math. Comput. Modelling, 18(12)(1993), 103-106.
- [12] P.A.Clarkson and J.A.Tuszynski, *Exact solutions of the multidimensional derivative nonlinear Schrödinger equation for many-body systems of criticality*, J. Phys. A., 23(1990), 4269-4288.
- [13] S.Das, *Generalized dynamic systems solution by decomposed physical reactions*, Int. J. Appl. Math. Stat., 17(2010), 44-75.
- [14] S.Das, *Functional Fractional Calculus, 2nd edition*, Springer-Verlag, (2011).
- [15] J.S.Duan, *Convenient analytic recurrence algorithms for the Adomian polynomials*, Appl. Math. Comput., 217(2011), 6337-6348.
- [16] J.S.Duan, *New recurrence algorithms for the nonclassic Adomian polynomials*, Appl. Math. Comput., 62(2011), 2961-2977.
- [17] J.L.Schiff, *The Laplace Transform: Theory and Applications*, Springer, (1999).
- [18] W.Eckhaus, *The long-time behaviour for perturbed wave-equations and related problems, in Trends in applications of pure mathematics to mechanics*, Lecture Notes in Physics, Springer, 249(1986), 168-194.
- [19] H.M.Baskonus and H.Bulut, *On the complex structures of Kundu-Eckhaus equation via improved Bernoulli sub-equation function method*, Waves in Random and Complex Media, 25(4)(2015), 720-728.
- [20] S.A.Khuri, *A Laplace Decomposition Algorithm Applied to a Class of Nonlinear Differential Equations*, J. of Applied Mathematics, 1(4)(2001), 141-155.
- [21] Y.Kodama, *Optical solitons in a monomode fiber*, J. Stat. Phys., 39(1985), 597-614.
- [22] A.Kundu, *Landau-Lifshitz and higher-order nonlinear systems gauge generated from nonlinear Schrödinger-type equations*, Journal of Math. Phys., 25(1984), 3433-3438.
- [23] D.H.Peregrine and J.Austral, *Water waves, nonlinear Schrödinger equations and their solutions*, Math. Soc. B: Appl. Math., 25(1983), 16-43.
- [24] K.Porsezian and K.Nakkeeran, *Optical solitons in presence of Kerr dispersion and self-frequency shift*, Phys. Rev. Lett., 76(2001), 39-55
- [25] S.Saha Ray and R.K.Bera, *An approximate solution of nonlinear fractional differential equation by Adomians decomposition method*, Appl. Math. Comput., 167(2005), 561-571.
- [26] N.Taghizadeh, M.Mirzazadeh and F.Tascan, *The first-integral method applied to the Eckhaus equation*, Appl. Math. Lett., 25(2012), 798802.

- [27] P.Wang et al, *Bright and dark soliton solutions and Bäcklund transformation for the Eckhaus-Kundu equation with the cubicquintic nonlinearity*, Appl. Math. Comput., 251(2015), 233-242.
- [28] L.H.Wang, K.Porsezian and J.S.He, *Breather and rogue wave solutions of a generalized nonlinear Schrödinger equation*, Phys. Rev. E., 87(2013), 053202.
- [29] A.M.Wazwaz, *A new algorithm for calculating Adomian polynomials for nonlinear operators*, Appl. Math. and Computation, 111(1)(2000), 33-51.
- [30] A.M.Wazwaz, *The combined Laplace transform-Adomian decomposition method for handling nonlinear Volterra integro-differential equations*, Appl. Math. Comput., 216(4)(2010), 1304-1309.
- [31] A.M.Wazwaz and S.M.El-Sayed, *A New Modification of the Adomian Decomposition Method for Linear and Nonlinear Operators*, Appl. Math. Comput., 122(3)(2001), 393-405.
- [32] A.M.Wazwaz, *A study on linear and nonlinear Schrödinger equations by the variational iteration method*, Chaos Soliton Fract., 37(2008), 1136-1142
- [33] Zhaqilao, *On Nth-order rogue wave solution to the generalized nonlinear Schrödinger equation*, Phys. Lett. A., 377(12)(2013), 855-859.
- [34] A.O.Unal, *Exact Solutions of Some Complex Partial Differential Equations of Fractional Order*, J. of Fractional Calculus and Applications, 5(2014), 209-217.