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$m(\alpha_i)-$Series to circulate Functions Arising From Generalized Alpha Difference Equation

Research Article

M.Maria Susai Manuel[1], G.Dominic Babu[2], G.Britto Antony Xavier[3]* and M.Meganathan[3]

1 Department of Mathematics, R.M.D. Engineering College, Kavaraipettai, Tamil Nadu, India.
2 Department of Mathematics, Annai Velankanni College, Tholaiyavattam, Kanyakumari, Tamil Nadu, India.
3 Department of Mathematics, Sacred Heart College, Tirupattur, Vellore, Tamil Nadu, India.

Abstract: In this paper, authors investigate the summation and closed form relations to certain types of generalized higher order alpha difference equations for finding the values of alpha multi series to circular functions in the field of finite difference methods. We provide examples verified by MATLAB to illustrate the alpha multi series to circular functions.

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Keywords: Alpha-difference equation, Alpha multi-series, Closed form solution, Circular functions.

1. Introduction

The operator Δ_α with its relevant theorems found in [1] was introduced by Jerzy Ponenda in 1984 [3] and was extended to $\Delta_{\alpha(\ell)}$ by defining $\Delta_{\alpha(\ell)}u(k) = u(k+\ell) - \alpha u(k)$ for $\ell \in (0,\infty)$ [7]. This operator $\Delta_{\alpha(\ell)}$ becomes difference operator Δ when $\ell = 1$ and $\alpha = 1$. Higher order difference equations are obtained by varying m in Δ^m. Correspondingly, for $\alpha_i \neq 0$ and $\ell_i > 0$, the operator $\Delta_{\alpha_1(\ell_1)}\Delta_{\alpha_2(\ell_2)}\cdots\Delta_{\alpha_m(\ell_m)}$ can yield higher order alpha difference equations. By taking $\alpha_i = \ell_i = 1$ for $i = 1, 2, \cdots, m$ we get Δ^m. Having incorporated additional dimensions in $\Delta_{\alpha_i(\ell_i)}$ we get more applications than the theory involving Δ. Several relations and formulae on multi-series involving circular functions have been derived in [2]. The linear expression of power of sine and cosine functions given as $\sin^{2m}(\tau/2) = 2^{1-2m}\sum\limits_{\nu=1}^{m}(-1)^\nu\binom{2m}{m-\nu}\cos\nu\tau + 2^{-2m}\binom{2m}{m}$ is used in [4]. Here, we introduce alpha multi series as below. Let $u(k) = 0$ for $k < 0$. Then,

$$1(\alpha_1)-series : u_{1\alpha_1(\ell_1)}(k) = u(k-\ell_1) + \alpha_1 u(k-2\ell_1) + \cdots + \alpha_1^{[\frac{k}{\ell_1}]-1} u\left(k - \left[\frac{k}{\ell_1}\right]\ell_1\right),$$
$$2(\alpha_2)-series : u_{2\alpha_2(\ell_2)}(k) = u_{1\alpha_2(\ell_2)}(k-\ell_2) + \alpha_2 u_{1\alpha_2(\ell_2)}(k-2\ell_2) + \cdots + \alpha_2^{[\frac{k}{\ell_2}]-1} u_{1\alpha_2(\ell_2)}\left(k - \left[\frac{k}{\ell_2}\right]\ell_2\right)$$

and in general $m(\alpha_i)-$series:

$$u_{(m-1)\alpha_i(\ell_i)}(k-\ell_i) + \alpha_i u_{(m-1)\alpha_i(\ell_i)}(k-2\ell_i) + \cdots + \alpha_i^{[u_{m\alpha_i(\ell_i)}(k)=\frac{k}{\ell_i}]-1} u_{(m-1)\alpha_i(\ell_i)}\left(k - \left[\frac{k}{\ell_i}\right]\ell_i\right). \tag{1}$$

* E-mail: brittoshc@gmail.com

By denoting $\underset{(\alpha,\ell)_{1\to n}}{\Delta} = \Delta_{\alpha_1(\ell_1)}\Delta_{\alpha_2(\ell_2)}\cdots\Delta_{\alpha_m(\ell_m)}$, we find that the $m(\alpha_i)-$series mentioned above is a summation solution of generalized alpha difference equation

$$\underset{(\alpha,\ell)_{1\to n}}{\Delta} v(k) = u(k),\ \ k \in [0,\infty),\ \ \ell_i > 0, \tag{2}$$

In this paper, our main aim is to find formula for getting the value of $m(\alpha_i)$- series given in (1) for circular function $u(k)$, using $\underset{(\alpha,\ell)_{1\to n}}{\Delta^{-1}} u(k) = \underset{\alpha_1(\ell_1)}{\Delta^{-1}}\underset{\alpha_2(\ell_2)}{\Delta^{-1}} \ldots \underset{\alpha_n(\ell_n)}{\Delta^{-1}} u(k)$.

2. Basic Definition

Definition 2.1 ([5]). *Let $u(k)$, $k \in [0.\infty)$ be a real valued function and $\ell \in (0,\infty)$ be fixed. Then the generalized α-difference operator on $u(k)$ is defined as:*

$$\Delta_{\alpha(\ell)}u(k) = u(k+\ell) - \alpha u(k). \tag{3}$$

$$\text{and if}\quad \Delta_{\alpha(\ell)}v(k) = u(k),\ \text{then}\ \Delta^{-1}_{\alpha(\ell)}u(k) = v(k) - \alpha^{[\frac{k}{\ell}]}v(j). \tag{4}$$

Lemma 2.2 ([2]). *Let $n \in N(1)$, $k \in [0,\infty)$ and p,q are constants. Then*

$$\sin^n pk = \frac{1}{2^{n-1}}\begin{cases}\sum\limits_{r=0}^{\frac{n-1}{2}} (-1)^{\frac{n-1}{2}+r}\frac{n^{(r)}}{r!} sinp(n-2r)k, & \text{if } n \text{ is odd.}\\ \sum\limits_{r=0}^{\frac{n-2}{2}} (-1)^{\frac{n-1}{2}+r}\frac{n^{(r)}}{r!} cosp(n-2r)k + \frac{n^{(\frac{n}{2})}}{2(\frac{n}{2})!}, & \text{if } n \text{ is even.}\end{cases} \tag{5}$$

$$\text{and}\ \ \cos^n pk = \frac{1}{2^{n-1}}\begin{cases}\sum\limits_{r=0}^{\frac{n-1}{2}} \frac{n^{(r)}}{r!} cosp(n-2r)k, & \text{if } n \text{ is odd.}\\ \sum\limits_{r=0}^{\frac{n-2}{2}} \frac{n^{(r)}}{r!} cosp(n-2r)k + \frac{n^{(\frac{n}{2})}}{2(\frac{n}{2})!}, & \text{if } n \text{ is even.}\end{cases} \tag{6}$$

Lemma 2.3 ([2]). *Let $P = p(n_1 - 2r_1) + q(n_2 - 2r_2)$, $\overline{P} = p(n_1 - 2r_1) - q(n_2 - 2r_2)$ and P, $\overline{P}$ are varying with respect to n_1,n_2,r_1,r_2,p and q.*

1. *If n_1 and n_2 are odd positive integers, then*

$$sin^{n_1}pkcos^{n_2}qk = \frac{(-1)^{\frac{n_1-1}{2}}}{2^{n_1+n_2-1}} \sum_{r_1=0}^{\frac{n_1-1}{2}}\sum_{r_2=0}^{\frac{n_2-1}{2}} (-1)^{r_1}\frac{n_1^{(r_1)}}{r_1!}\frac{n_2^{(r_2)}}{r_2!}\left\{sinPk + sin\overline{P}k\right\} \tag{7}$$

2. *If n_1 is an odd positive integer and n_2 is an even positive integer, then*

$$sin^{n_1}pkcos^{n_2}qk = \frac{1}{2^{n_1+n_2-1}}\sum_{r_1=0}^{\frac{n_1-1}{2}}(-1)^{\frac{n_1-1}{2}+r_1}\frac{n_1^{(r_1)}}{r_1!}\left\{\sum_{r_2=0}^{\frac{n_2-1}{2}}\frac{n_2^{(r_2)}}{r_2!}(sinPk + sin\overline{P}k) + \frac{n_2^{(\frac{n_2}{2})}}{(\frac{n_2}{2})!}sin\left(\frac{P+\overline{P}}{2}\right)k\right\} \tag{8}$$

3. *If n_1 is an even positive integer and n_2 is an odd positive integer, then*

$$sin^{n_1}pkcos^{n_2}qk = \frac{1}{2^{n_1+n_2-1}}\sum_{r_2=0}^{\frac{n_2-1}{2}}\frac{n_2^{(r_2)}}{r_2!}\left\{\sum_{r_1=0}^{\frac{n_2-1}{2}}(-1)^{\frac{n_1}{2}+r_1}\frac{n_1^{(r_1)}}{r_1!}(cosPk + cos\overline{P}k) + \frac{n_1^{(\frac{n_1}{2})}}{(\frac{n_1}{2})!}cos\left(\frac{P-\overline{P}}{2}\right)k\right\} \tag{9}$$

4. If n_1 and n_2 are even positive integers, then

$$sin^{n_1}pkcos^{n_2}qk = \frac{1}{2^{n_1+n_2-1}}\left\{\left(\sum_{r_1=0}^{\frac{n_2-1}{2}}(-1)^{\frac{n_1}{2}+r_1}\frac{n_1^{(r_1)}}{r_1!}\left(\sum_{r_2=0}^{\frac{n_2-1}{2}}\frac{n_2^{(r_2)}}{r_2!}(cosPk+cos\overline{P}k)\right.\right.\right.$$

$$\left.\left.\left.+\frac{n_1^{(\frac{n_1}{2})}}{(\frac{n_1}{2})!}cos\left(\frac{P-\overline{P}}{2}\right)k\right)+\frac{n_2^{(\frac{n_2}{2})}}{(\frac{n_2}{2})!}cos\left(\frac{P+\overline{P}}{2}\right)k\right)+\frac{1}{2}\frac{n_1^{(\frac{n_1}{2})}}{(\frac{n_1}{2})!}\frac{n_2^{(\frac{n_2}{2})}}{(\frac{n_2}{2})!}\right\}. \tag{10}$$

Proof. The proof of (7),(8),(9) and (10) are obtained by combining (5) and (6) and using properties of trigonometric funtions. □

Theorem 2.4 ([6] Multi-infinite alpha series formula)**.** *If for $t = 1,2,...,n$, $\lim_{r_t\to\infty}\left\{\frac{1}{\alpha_t^{r_t}}(\prod_{i=1}^{t}\Delta^{-1}_{\alpha_i(l_i)}u(k+r_tl_t))\right\}=0$, then we have*

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{u(k+\sum_{i=1}^{n}r_i\ell_i)}{\alpha_1^{r_1}\alpha_2^{r_2}...\alpha_n^{r_n}}=\left(\prod_{i=1}^{n}(-\alpha_i)\Delta^{-1}_{\alpha_i(\ell_i)}\right)u(k). \tag{11}$$

3. Main Result

In this section, we use power set notations: For $A = \{m_1, m_2, m_3, ..., m_n\} \subset N(1)$, we denote, $\alpha_A = \alpha_{m_1}\alpha_{m_2}\alpha_{m_3}...\alpha_{m_n}$, $\ell_A = \ell_{m_1} + \ell_{m_2} + ... + \ell_{m_n}$,$L = \{1, 2, 3, ...n\}$, $A^c = L - A$ and $p(L) = \{\{\}, \{1\}, \{2\}, ..., \{1, 2\}, \{2, 3\}, ...\{L\}\}$.

Theorem 3.1. *Consider the notations mentioned in preliminaries. Let p be any real number such that $1+\alpha_i^2 \neq 2\alpha_i cosp\ell_i$ for all $i = 1, 2, ..., n$. Then we have*

$$\Delta^{-1}_{(\alpha,\ell)_{1\to n}} sinpk = \frac{\sum_{A\in p(L)}(-1)^{o(A)}\alpha_A sinp(k-\ell_{A^c})}{\prod_{i=1}^{n}(1+\alpha_i^2-2\alpha_i cosp\ell_i)} \tag{12}$$

and

$$\Delta^{-1}_{(\alpha,\ell)_{1\to n}} cospk = \frac{\sum_{A\in p(L)}(-1)^{o(A)}\alpha_A cosp(k-\ell_{A^c})}{\prod_{i=1}^{n}(1+\alpha_i^2-2\alpha_i cosp\ell_i)}. \tag{13}$$

Proof. Replacing u(k) by $\sin pk$ and ℓ by ℓ_1 in (3), we find

$$\Delta_{\alpha_1(\ell_1)} sinpk = sinp(k+\ell_1) - \alpha_1 sinpk, \tag{14}$$

which yields $\Delta_{\alpha_1(\ell_1)} sinp(k-\ell_1) = sinpk - \alpha_1 sinp(k-\ell_1)$. Since $\Delta_{\alpha_1(\ell_1)}[sinp(k-\ell_1)-\alpha_1 sinpk] = sinpk[1+\alpha_1^2-2\alpha_1 cosp\ell_1]$, and α_1,ℓ_1,p are constants, we have

$$\Delta^{-1}_{\alpha_1(\ell_1)} sinpk = \frac{sinp(k-\ell_1)-\alpha_1 sinpk}{[1+\alpha_1^2-2\alpha_1 cosp\ell_1]}, \tag{15}$$

Taking $\Delta^{-1}_{\alpha_2(\ell_2)}$ on both sides, we get

$$\Delta^{-1}_{\alpha_2(\ell_2)}(\Delta^{-1}_{\alpha_1(\ell_1)} sinpk) = \frac{\Delta^{-1}_{\alpha_2(\ell_2)} sinp(k-\ell_1) - \alpha_1 \Delta^{-1}_{\alpha_2(\ell_2)} sinpk}{1+\alpha_1^2-2\alpha_1 cosp\ell_1}. \tag{16}$$

Substituting (15) in (16), we derive

$$\Delta^{-1}_{\alpha_2(\ell_2)}(\Delta^{-1}_{\alpha_1(\ell_1)} sinpk) = \frac{sinp(k-\ell_2-\ell_1)-\alpha_2 sinp(k-\ell_1)-\alpha_1 sinp(k-\ell_2)+\alpha_1\alpha_2 sinpk}{(1+\alpha_1^2-2\alpha_1 cosp\ell_1)(1+\alpha_2^2-2\alpha_2 cosp\ell_2)}$$

In general, we write

$$\Delta^{-1}_{(\alpha,\ell)_{1\to n}} sinpk = \frac{\sum\limits_{A\in p(L)} (-1)^{o(A)}\alpha_A sinp(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i cosp\ell_i)}.$$

Similarly, we obtain (13) for $cospk$. □

Theorem 3.2. *Let $m\in N(1)$, $k\in[0,\infty)$ and p is constant. Then*

$$\Delta^{-1}_{(\alpha,\ell)_{1\to n}} \sin^m pk = \frac{1}{2^{m-1}}\begin{cases} \sum\limits_{r=1}^{[\frac{m}{2}]+1} \frac{(-1)^{r+1}\binom{m}{[\frac{m}{2}]+r} \sum\limits_{A\in p(L)}(-1)^{o(A)}\alpha_A \sin(2r-1)p(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i\cos(2r-1)p\ell_i)}, \\ \qquad \text{if } m \text{ is odd.} \\ \frac{\binom{m}{m/2}}{2\prod\limits_{i=2}^{n}(1-\alpha_i)} + \sum\limits_{r=1}^{m/2} \frac{(-1)^r\binom{m}{\frac{m}{2}-r}\sum\limits_{A\in p(L)}(-1)^{o(A)}\alpha_A\cos 2rp(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i\cos 2rp\ell_i)}, \\ \qquad \text{if } m \text{ is even.}\end{cases} \tag{17}$$

and

$$\Delta^{-1}_{(\alpha,\ell)_{1\to n}} \cos^m pk = \frac{1}{2^{m-1}}\begin{cases} \sum\limits_{r=1}^{[\frac{m}{2}]+1} \frac{\binom{m}{[\frac{m}{2}]+r} \sum\limits_{A\in p(L)}(-1)^{o(A)}\alpha_A \sin(2r-1)p(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i\cos(2r-1)p\ell_i)} \\ \qquad \text{if } m \text{ is odd.} \\ \frac{\binom{m}{m/2}}{2\prod\limits_{i=2}^{n}(1-\alpha_i)} + \sum\limits_{r=1}^{m/2} \frac{\binom{m}{\frac{m}{2}-r}\sum\limits_{A\in p(L)}(-1)^{o(A)}\alpha_A\cos 2rp(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i\cos 2rp\ell_i)} \\ \qquad \text{if } m \text{ is even.}\end{cases} \tag{18}$$

are closed form solutions of (2) when $u(k)=\sin^m pk$ and $\cos^m pk$ respectively.

Proof. Since $\sin^2 pk = \frac{1}{2}(1-cos2pk)$, taking $\Delta^{-1}_{(\alpha,\ell)_{1\to n}}$ on $\sin^2 pk = \frac{1}{2}(1-cos2pk)$ and applying (13), we obtain

$$\Delta^{-1}_{(\alpha,\ell)_{1\to n}} sin^2 pk = \frac{1}{2}\left\{\frac{1}{\prod\limits_{i=1}^{n}(1-\alpha_i)} - \frac{\sum\limits_{A\in p(L)}(-1)^{o(A)}\alpha_A cos2p(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i cos2p\ell_i)}\right\}$$

Since $\sin^3 pk = \frac{1}{2^2}\{3\sin pk - \sin 3pk\}$, from (12),we get

$$\Delta^{-1}_{(\alpha,\ell)_{1\to n}} sin^3 pk = \frac{1}{2^2}\left\{3\frac{\sum\limits_{A\in p(L)}(-1)^{o(A)}\alpha_A sinp(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i cosp\ell_i)} - \frac{\sum\limits_{A\in p(L)}(-1)^{o(A)}\alpha_A sin3p(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i cos3p\ell_i)}\right\}.$$

Continuing this process we get (17) and similarly (18) follows by applying (12) on RHS of (5). □

Lemma 3.3. *Let $m\in N(1)$, $k\in[0,\infty)$ and p is constant. Then*

$$\Delta^{-1}_{(\alpha,\ell)_{1\to n}} \sin^m pk\cos^m pk = \frac{1}{2^{2m-1}}\begin{cases} \sum\limits_{r=1}^{[\frac{m}{2}]+1} \frac{(-1)^{r+1}\binom{m}{[\frac{m}{2}]+r} \sum\limits_{A\in p(L)}(-1)^{o(A)}\alpha_A \sin 2(2r-1)p(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i\cos 2(2r-1)p\ell_i)}, \\ \qquad \text{if } m \text{ is odd.} \\ \frac{\binom{m}{m/2}}{2\prod\limits_{i=1}^{n}(1-\alpha_i)} + \sum\limits_{r=1}^{[\frac{m}{2}]} \frac{(-1)^r\binom{m}{\frac{m}{2}-r}\sum\limits_{A\in p(L)}(-1)^{o(A)}\alpha_A\cos 4rp(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i\cos 4rp\ell_i}, \\ \qquad \text{if } m \text{ is even.}\end{cases} \tag{19}$$

is a closed form solution of equation (2) when $u(k)=\sin^m pk\cos^m pk$ respectively.

Proof. The proof follows by taking $q=p$ and operating $\Delta^{-1}_{(\alpha,\ell)_{1\to n}}$ on Lemma 2. □

Lemma 3.4. *Let $m \in N(1)$, $k \in [0,\infty)$ and p is constant. Then*

$$\underset{(\alpha,\ell)_{1\to n}}{\Delta^{-1}} \sin pk \cos^m pk = \frac{1}{2^m}\begin{cases} \sum\limits_{r=1}^{\left[\frac{m}{2}\right]} \left\{ \binom{m}{\left[\frac{m}{2}\right]+1-r} - \binom{m}{\left[\frac{m}{2}\right]-r} \right\} \dfrac{\sum\limits_{A\in p(\ell)} (-1)^{o(A)} \sin 2rp(k-\ell_{A^c})}{\prod\limits_{i=1}^{n} I(1+\alpha_i^2-2\alpha_i \cos 2rp\ell_i)} \\ \quad + (-1)^{\left[\frac{m}{2}\right]} \dfrac{\sum\limits_{A\in p(\ell)} (-1)^{o(A)} \sin(m+1)p(k-\ell_{A^c})}{\prod\limits_{i=1}^{n} I(1+\alpha_i^2-2\alpha_i cos(m+1)p\ell_i)}, \text{if } m \text{ is odd}, m>1. \\ \sum\limits_{r=1}^{\left[\frac{m}{2}\right]} \left\{ \binom{m}{\left[\frac{m}{2}\right]+1-r} - \binom{m}{\left[\frac{m}{2}\right]-r} \right\} \dfrac{\sum\limits_{A\in p(\ell)} (-1)^{o(A)} \sin(2r-1)p(k-\ell_{A^c})}{\prod\limits_{i=1}^{n} I(1+\alpha_i^2-2\alpha_i \cos(2r-1)p\ell_i)} \\ \quad + (-1)^{\left[\frac{m}{2}\right]} \dfrac{\sum\limits_{A\in p(\ell)} (-1)^{o(A)} \sin(m+1)p(k-\ell_{A^c})}{\prod\limits_{i=1}^{n} I(1+\alpha_i^2-2\alpha_i cos(m+1)p\ell_i)}, \text{if } m \text{ is even.} \end{cases}$$

is a closed form solution of equation (2) when $u(k) = \sin pk \cos^m pk$.

Proof. By taking $p = q$ and $\underset{(\alpha,\ell)_{1\to n}}{\Delta^{-1}}$ on Lemma 2, we get the result. □

Lemma 3.5. *Let $m \in N(1)$, $k \in [0,\infty)$ and p is constant. Then*

$$\underset{(\alpha,\ell)_{1\to n}}{\Delta^{-1}} \cos pk \sin^m pk = \frac{1}{2^m}\begin{cases} \sum\limits_{r=1}^{\left[\frac{m}{2}\right]} (-1)^{r-1} \left\{ \binom{m}{\left[\frac{m}{2}\right]+1-r} - \binom{m}{\left[\frac{m}{2}\right]-r} \right\} \dfrac{\sum\limits_{A\in p(\ell)} (-1)^{o(A)} \sin 2rp(k-\ell_{A^c})}{\prod\limits_{i=1}^{n} I(1+\alpha_i^2-2\alpha_i \cos 2rp\ell_i)} \\ \quad + (-1)^{\left[\frac{m}{2}\right]} \dfrac{\sum\limits_{A\in p(\ell)} (-1)^{o(A)} \sin(m+1)p(k-\ell_{A^c})}{\prod\limits_{i=1}^{n} I(1+\alpha_i^2-2\alpha_i cos(m+1)p\ell_i)}, \text{if } m \text{ is odd}, m>1. \\ \sum\limits_{r=1}^{\left[\frac{m}{2}\right]} (-1)^{r-1} \left\{ \binom{m}{\left[\frac{m}{2}\right]+1-r} - \binom{m}{\left[\frac{m}{2}\right]-r} \right\} \dfrac{\sum\limits_{A\in p(\ell)} (-1)^{o(A)} \cos(2r-1)p(k-\ell_{A^c})}{\prod\limits_{i=1}^{n} I(1+\alpha_i^2-2\alpha_i \cos(2r-1)p\ell_i)} \\ \quad + (-1)^{\left[\frac{m}{2}\right]} \dfrac{\sum\limits_{A\in p(\ell)} (-1)^{o(A)} \cos(m+1)p(k-\ell_{A^c})}{\prod\limits_{i=1}^{n} I(1+\alpha_i^2-2\alpha_i cos(m+1)p\ell_i)}, \text{if } m \text{ is even.} \end{cases}$$

is a closed form solution of equation (2) when $u(k) = \cos pk \sin^m pk$.

Proof. The proof follows by operating $\underset{(\alpha,\ell)_{1\to n}}{\Delta^{-1}}$ and $p = q$ on Lemma 2. □

Lemma 3.6. *Let $m \in N(1)$, $k \in [0,\infty)$ and p, q are constants. Then*

$$\underset{(\alpha,\ell)_{1\to n}}{\Delta^{-1}} \sin pk \cos^m qk = \frac{1}{2^m}\begin{cases} \sum\limits_{r=1}^{\left[\frac{m}{2}\right]+1} \dfrac{\binom{m}{r+\left[\frac{m}{2}\right]} \sum\limits_{A\in p(\ell)} (-1)^{o(A)} \alpha_A \sin(p+(2r-1)q)(k-\ell_{A^c})}{\prod\limits_{i=1}^{n} (1+\alpha_i^2-2\alpha_i \cos(p+(2r-1)q)\ell_i)} \\ + \sum\limits_{r=1}^{\left[\frac{m}{2}\right]+1} \dfrac{\binom{m}{r+\left[\frac{m}{2}\right]} \sum\limits_{A\in p(\ell)} (-1)^{o(A)} \alpha_A \sin(p+(1-2r)q)(k-\ell_{A^c})}{\prod\limits_{i=1}^{n} (1+\alpha_i^2-2\alpha_i \cos(p+(1-2r)q)\ell_i)}, \\ \qquad \text{if } m \text{ is odd.} \\ \sum\limits_{r=-\left[\frac{m}{2}\right]}^{\left[\frac{m}{2}\right]} \dfrac{\binom{m}{r+m/2} \sum\limits_{A\in p(\ell)} (-1)^{o(A)} \alpha_A \sin(p+2rq)(k-\ell_{A^c})}{\prod\limits_{i=1}^{n} (1+\alpha_i^2-2\alpha_i \cos(p+2rq)\ell_i)}, \\ \qquad \text{if } m \text{ is even.} \end{cases}$$

is a closed form solution of equation (2) when $u(k) = \sin pk \cos^m qk$.

Proof. By operating $\underset{(\alpha,\ell)_{1\to n}}{\Delta^{-1}}$ on Lemma 2, we get the result. □

Lemma 3.7. *Let $m \in N(1)$, $k \in [0,\infty)$ and p, q are constants. Then*

$$\underset{(\alpha,\ell)_{1\to n}}{\Delta^{-1}} \cos pk \sin^m qk = \frac{1}{2^m}\begin{cases} \sum\limits_{r=1}^{\left[\frac{m}{2}\right]+1} \binom{m}{\left[\frac{m}{2}\right]+r} (-1)^r \left\{ \dfrac{\sum\limits_{A\in p(\ell)} (-1)^{o(A)} \alpha_A \sin(p+(1-2r)q)(k-\ell_{A^c})}{\prod\limits_{i=1}^{n} (1+\alpha_i^2-2\alpha_i \cos(p+(1-2r)q)\ell_i)} \right. \\ \left. - \dfrac{\sum\limits_{A\in p(\ell)} (-1)^{o(A)} \alpha_A \sin(p+(2r-1)q)(k-\ell_{A^c})}{\prod\limits_{i=1}^{n} (1+\alpha_i^2-2\alpha_i \cos(p+(2r-1)q)\ell_i)} \right\}, \text{if } m \text{ is odd.} \\ \sum\limits_{r=-\left[\frac{m}{2}\right]}^{\left[\frac{m}{2}\right]} \dfrac{(-1)^r \binom{m}{\left[\frac{m}{2}\right]+r} \sum\limits_{A\in p(\ell)} (-1)^{o(A)} \alpha_A \cos(p+2rq)(k-\ell_{A^c})}{\prod\limits_{i=1}^{n} (1+\alpha_i^2-2\alpha_i \cos(p+2rq)\ell_i)}, \\ \qquad \text{if } m \text{ is even.} \end{cases}$$

is a closed form solution of equation (2) when $u(k) = \cos pk \sin^m qk$.

Proof. The proof directly follows by taking $\underset{(\alpha,\ell)_{1\to n}}{\Delta^{-1}}$ on Lemma 2. □

Lemma 3.8. *Let $k \in [0, \infty)$ and p is constant. Then*

$$\underset{(\alpha,\ell)_{1\to n}}{\Delta^{-1}} ksinpk = \frac{\sum\limits_{A\in p(\ell)} (-1)^{o(A)}\alpha_A(k-\ell_{A^c})sinp(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i cosp\ell_i)}$$

$$+2\sum_{i=1}^{n}\frac{\alpha_i\ell_i sinp\ell_i}{(1+\alpha_i^2-2\alpha_i cosp\ell_i)}\left(\frac{\sum\limits_{A\in p(\ell)}(-1)^{o(A)}\alpha_A cosp(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i cosp\ell_i)}\right) \tag{20}$$

and

$$\underset{(\alpha,\ell)_{1\to n}}{\Delta^{-1}} kcospk = \frac{\sum\limits_{A\in p(\ell)} (-1)^{o(A)}\alpha_A(k-\ell_{A^c})cosp(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i cosp\ell_i)}$$

$$+2\sum_{i=1}^{n}\frac{\alpha_i\ell_i sinp\ell_i}{(1+\alpha_i^2-2\alpha_i cosp\ell_i)}\left(\frac{\sum\limits_{A\in p(\ell)}(-1)^{o(A)}\alpha_A sinp(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i cosp\ell_i)}\right). \tag{21}$$

Proof. Replacing u(k) by $ksinpk$ and $kcospk$ in (3), we find

$$\underset{\alpha_1(\ell_1)}{\Delta} ksinpk = (k+\ell_1)sinp(k+\ell_1) - \alpha_1 ksinpk, \tag{22}$$

which yields

$$\underset{\alpha_1(\ell_1)}{\Delta}(k-\ell_1)sinp(k-\ell_1) = ksinpk - \alpha_1(k-\ell_1)sinp(k-\ell_1). \tag{23}$$

Since

$$\underset{\alpha_1(\ell_1)}{\Delta}[(k-\ell_1)sinp(k-\ell_1)-\alpha_1 ksinpk] = ksinpk[1+\alpha_1^2-2\alpha_1 cosp\ell_1] - 2\alpha_1\ell_1 cospksinp\ell_1,$$

and α_1,ℓ_1,p are constants, we have

$$\underset{\alpha_1(\ell_1)}{\Delta^{-1}} ksinpk = \frac{(k-\ell_1)sinp(k-\ell_1)-\alpha_1 ksinpk}{[1+\alpha_1^2-2\alpha_1 cosp\ell_1]} + \frac{2\alpha1\ell_1[cosp(k-\ell_1)-\alpha_1 cospk]sinp\ell_1}{1+\alpha_1^2-2\alpha_1 cosp\ell_1]^2} \tag{24}$$

Taking $\underset{\alpha_2(\ell_2)}{\Delta^{-1}} \cdots \underset{\alpha_n(\ell_n)}{\Delta^{-1}}$ on (24),we get (20) and similarly we get (21). □

Lemma 3.9. *Let $m \in N(1)$, $k \in [0, \infty)$ and p is constant. Then*

$$\underset{(\alpha,\ell)_{1\to n}}{\Delta^{-1}} k\sin^m pk = \frac{1}{2^{n-1}}\begin{cases} \sum\limits_{r=1}^{[\frac{m}{2}]+1}(-1)^{1+r}\binom{m}{[\frac{m}{2}]+r}\left\{\frac{\sum\limits_{A\in p(\ell)}(-1)^{o(A)}\alpha_A(k-\ell_{A^c}\sin(2r-1)p(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i\cos(2r-1)p\ell_i)}\right. \\ +2\sum\limits_{i=1}^{n}\frac{\alpha_i\ell_i sin(2r-1)p\ell_i}{(1+\alpha_i^2-2\alpha_i cos(2r-1)\ell_i)} \\ \left.\left(\frac{\sum\limits_{A\in p(\ell)}(-1)^{o(A)}\alpha_A(k-\ell_{A^c}\cos(2r-1)p(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i\cos(2r-1)p\ell_i)}\right)\right\}, \text{if } n \text{ is odd.} \\ \binom{m}{\frac{m}{2}}\left\{\frac{k}{\prod\limits_{i=1}^{n}(1-\alpha_i)}-\frac{\ell_i}{(1-\alpha_i)\prod\limits_{i=1}^{n}(1-\alpha_i)}\right\}+\sum\limits_{r=1}^{\frac{m}{2}}(-1)^r\binom{m}{\frac{m}{2}-r} \\ \left\{\frac{\sum\limits_{A\in p(\ell)}(-1)^{o(A)}\alpha_A(k-\ell_{A^c})\cos 2rp(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i\cos 2rp\ell_i)}+2\sum\limits_{i=1}^{n}\frac{\alpha_i\ell_i sin2rp\ell_i}{(1+\alpha_i^2-2\alpha_i cos(2r-1)\ell_i)}\right. \\ \left.\left\{\frac{\sum\limits_{A\in p(\ell)}(-1)^{o(A)}\alpha_A(k-\ell_{A^c}\sin 2rp(k-\ell_{A^c})}{\prod\limits_{i=1}^{n}(1+\alpha_i^2-2\alpha_i\cos(2r-1)p\ell_i)}\right\}\right\}, \text{if } n \text{ is even.} \end{cases}$$

is a closed form solution of equation (2) when $u(k) = k\sin^m pk$.

Proof. The proof follows by using (5), (6), (20) and (21). □

Example 3.10. *If $u(k) = sin^2 pk$ in (11), then we have*

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\cdots\sum_{r_n=0}^{\infty}\frac{sin^2(k+\sum_{i=1}^{n} r_i l_i)}{\alpha_1^{r_1}\alpha_2^{r_2}...\alpha_n^{r_n}} = \left(\prod_{i=1}^{n}(-\alpha_i)\,\Delta^{-1}_{\alpha_i(l_i)}\right) sin^2 pk. \tag{25}$$

Taking $n = 1, k = 0, \alpha_1 = 2, \ell_1 = 3, p = 1$ in (25),we get

$$(-\alpha_1)\,\Delta^{-1}_{\alpha_1\ell_1} sin^2 pk = (-\alpha_1)\frac{1}{2}\left\{\frac{1}{(1-\alpha_1)} - \frac{cos2p(k-\ell_1) - \alpha_1 cos2pk}{1+\alpha_1^2 - 2\alpha_1 cos2p\ell_1}\right\}$$
$$= (-2)\frac{1}{2}\left\{\frac{1}{(1-2)} - \frac{cos(-6) - 2cos(0)}{1+4-4cos6}\right\}$$

$$Now,\ \sum_{r_1=0}^{\infty}\frac{sin^2(k+r_1\ell_1)}{\alpha_1^{r_1}} = \sum_{r_1=0}^{\infty}\frac{sin^2(r_1\ell_1)}{\alpha_1^{r_1}} = \sum_{r_1=0}^{\infty}\frac{sin^2(3r_1)}{2^{r_1}}$$
$$= \frac{sin^2(0)}{2^0} + \frac{sin^2 3}{2} + \frac{sin^2 6}{2^2} + \frac{sin^2 9}{2^3} + \frac{sin^2 12}{2^4} + ...$$

4. Conclusion

In this paper, applying the solutions of Generalized α_i-difference equations, we have derived several formulae for alpha multi series involving circular functions. Particularly, taking $\ell_i = 1$ and $\alpha_i = 1$ we obtain the solutions of higher order difference equations.

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