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Intuitionistic Fuzzy Cone Metric Spaces and Fixed Point Theorems

Research Article

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Abstract: In this paper, the notion of intuitionistic fuzzy cone metric space is introduced. Some common fixed point theorems for occasionally weakly compatible mapping in intuitionistic fuzzy cone metric space are stated and proved.

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1. Introduction

Zadeh L.A [18] introduced the concept of fuzzy sets in 1965. Application of fuzzy set theory played an important role in all discipline of engineering like industrial, robotics, computer, nuclear, civil, electrical, mechanical etc. The notion of defining intuitionistic fuzzy sets (IFSs) for fuzzy set generalizations, introduced by Atanassov [1] , has proven interesting and useful in various application areas. Since this fuzzy set generalization can present the degrees of membership and non-membership with a degree of hesitancy, the knowledge and semantic representation becomes more meaningful and applicable. Huang and Zhang [7] introduced the notion of cone metric spaces by replacing real numbers with an ordering Banach space and proved some fixed point theorems for contractive mappings between these spaces. Kramosil and Michalek [9] introduced the fuzzy metric space by generalizing the concept of probabilistic metric spaces to fuzzy situation. George and Veeramani [6] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [9]. In 2015, Tarkan Oner et al.[11] introduced the notion of fuzzy cone metric space. N. Priyobarta1 et al.[12] discussed some fixed point results in fuzzy cone metric space. On the other hand, Park [13] has introduced and studied the notion of intuitionistic fuzzy metric spaces.

Further, Saadati et al.[14] proposed the idea of a continuous t-representable under the name modified intuitionistic fuzzy metric space which is a milestone in developing fixed point theory. In [3] Alaca, Turkoglu and Yildiz, proved the well known fixed point theorems of Banach and Edelstein in intuitionistic fuzzy metric spaces with the help of Grabiec [5]. The notion of compatible maps was introduced by G.Jungck [8], for a pair of self mapping. M.A. Al-Thagafi and N. Shahzad [4] introduced the concept of occasionally weakly compatible maps. In this paper, the concept of intuitionistic fuzzy cone metric space is introduced. Banach contraction theorem and some fixed point results on intuitionistic fuzzy cone metric space are stated and proved.

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2. Preliminaries

Definition 2.1. *A subset P of E is called a cone if*

1. *P is closed, non-empty and $P \neq \{0\}$;*
2. *If $a, b \in R$, $a, b \geq 0$ and $x, y \in P$ then $ax + by \in P$,*
3. *If both $x \in P$ and $-x \in P$ then $x = 0$.*

For a given cone, a partial ordering $\preccurlyeq$ on E via P is defined by $x \preccurlyeq y$ if only if $y - x \in P$. $x \prec y$ will stand for $x \preccurlyeq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$. Throughout this paper, we assume that all cones has non-empty interior.

Definition 2.2. *A cone metric space is an ordered (X, d), where X is any set and $d : X \times X \to E$ is a mapping satisfying:*

1. *$0 \preccurlyeq d(x, y)$for all $x, y \in X$,*
2. *$d(x, y) = 0$ if and only if $x = y$,*
3. *$d(x, y) = d(y, x)$ for all $x, y \in X$,*
4. *$d(x, z) \preccurlyeq d(x, y) + d(y, z)$ for all $x, y, z \in X$.*

Definition 2.3. *Let (X, d) be a cone metric space. Then, for each $c_1 \gg 0$ and $c_2 \gg 0$, $c_1, c_2 \in E$, there exists $c \gg 0$, $c \in E$ such that $c \ll c_1$ and $c \ll c_2$.*

Definition 2.4. *A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if $*$ satisfies the following conditions:*

1. *$*$ is associative and commutative,*
2. *$*$ is continuous,*
3. *$a * 1 = a$ for all $a \in [0, 1]$,*
4. *$a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.*

Definition 2.5. *A binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-conorm if $\diamond$ satisfies the following conditions:*

1. *$\diamond$ is associative and commutative,*
2. *$\diamond$ is continuous,*
3. *$a \diamond 0 = a$ for all $a \in [0, 1]$,*
4. *$a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0, 1]$.*

Definition 2.6. *Let X be a non-empty set. An element $x \in X$ is called a common fixed point of mappings $F : X \times X \to X$ and $T : X \to X$ if $x = T(x) = F(x, x)$.*

Definition 2.7. *Let X be a non-empty set. The mappings $F : X \times X \to X$ and $T : X \to X$ are called commutative if $T(F(x, y)) = F(T(x), T(y))$ for all $x, y \in X$.*

Definition 2.8. *Let X be a set, F, T self maps of X. A point x in X is called a coincidence point of F and T if and only if $F(x) = T(x)$. We shall call $w = F(x) = T(x)$ a point of coincidence of F and T.*

Definition 2.9. *A pair of maps U and V is called weakly compatible pair if they commute at coincidence points.*

Definition 2.10. *Two self maps F and T of a set X are occasionally weakly compatible if and only if there is a point x in X which is a coincidence point of F and T at which F and T commute.*

Lemma 2.1. *Let X be a set, F, T occasionally weakly compatible self maps of X. If F and T have a unique point of coincidence, $w = F(x) = T(x)$, then w is the unique common fixed point of F and T.*

3. Intuitionistic Fuzzy Cone Metric Space

Definition 3.1. *A 3-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy cone metric space if P is a cone of E, X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy set on $X^2 \times int(P)$ satisfying the following conditions:*
For all $x, y, z \in X$ and $t, s \in int(P)$ (that is $t \gg 0, s \gg 0$)

1. $M(x, y, t) + N(x, y, t) \leq 1$;
2. $M(x, y, t) > 0$;
3. $M(x, y, t) = 1$ *if and only if* $x = y$;
4. $M(x, y, t) = M(y, x, t)$;
5. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
6. $M(x, y, .) : int(P) \to (0, 1]$ *is continuous*;
7. $N(x, y, t) > 0$;
8. $N(x, y, t) = 0$ *if and only if* $x = y$;
9. $N(x, y, t) = N(y, x, t)$;
10. $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$;
11. $N(x, y, .) : int(P) \to (0, 1]$ *is continuous*;

Then (M, N) is called an intuitionistic fuzzy cone metric on X. The functions $M(x, y, t)$ and $N(x, y, t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Example 3.1. *If we take $E = \mathrm{R}, P = [0, \infty)$ and $a * b = \min\{a, b\}, a \diamond b = \max\{a, b\}$, then every intuitionistic fuzzy metric spaces became an intuitionistic fuzzy cone metric spaces.*

Example 3.2. *Let P be an any cone, $X = \mathrm{N}$, $a * b = \min\{a, b\}, a \diamond b = \max\{a, b\}$, $M, N : X^2 \times int(P) \to [0, 1]$ defined by*

$$M(x, y, t) = \begin{cases} \frac{x}{y} & if \quad x \leq y \\ \frac{y}{x} & if \quad y \leq x \end{cases} \qquad N(x, y, t) = \begin{cases} \frac{y-x}{y} & if \quad x \leq y \\ \frac{x-y}{y} & if \quad y \leq x \end{cases} \tag{1}$$

for all $x, y \in X$ and $t \gg 0$. Then $(X, M, N, *, \diamond)$ is an intuitionistic fuzzy cone metric spaces.

Example 3.3. *Let $E = \mathbb{R}^2$. Then $P = \{(k_1, k_2) : k_1, k_2 \geq 0\} \subset E$ is a normal cone with normal constant $K = 1$. Let $X = \mathbb{R}, a * b = \min\{a, b\}, a \diamond b = \max\{a, b\}$ and $M, N : X^2 \times int(P) \to (0, 1]$ defined by $M(x, y, t) = \frac{1}{e^{\frac{|x-y|}{\|t\|}}}$ and $N(x, y, t) = \frac{e^{\frac{|x-y|}{\|t\|}} - 1}{e^{\frac{|x-y|}{\|t\|}}}$ for all $x, y \in X$ and $t \gg 0$. Then $(X, M, N, *, \diamond)$ is an intuitionistic fuzzy cone metric spaces.*

Definition 3.2. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy cone metric space. For $t \gg 0$, the open ball $B(x, r, t)$ with center x and radius $r \in (0, 1)$ is defined by $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r, N(x, y, t) < r\}$.*

Definition 3.3. *Let $(X, M, N, *, \diamond)$ be an intuitionsitic fuzzy cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to converge to x if for any $t \gg 0$ and any $r \in (0, 1)$ there exists a natural number n_0 such that $M(x_n, x, t) > 1 - r, N(x_n, x, t) < r$ for all $n \geq n_0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.*

Definition 3.4. *Let $(X, M, N, *, \diamond)$ be an intuitionsitic fuzzy cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to Cauchy sequence if for any $0 < \epsilon < 1$ and any $t \gg 0$ there exists a natural number n_0 such that $M(x_n, x_m, t) > 1 - \epsilon, N(x_n, x_m, t) < \epsilon$ for all $n, m \geq n_0$.*

Definition 3.5. *$(X, M, N, *, \diamond)$ is called complete if every Cauchy sequence is convergent.*

Definition 3.6. *Let $(X, M, N, *, \diamond)$ be an intuitionsitic fuzzy cone metric space. A subset A of X is said to be FC-bounded if there exists $t \gg \theta$ and $r \in (0, 1)$ such that $M(x, y, t) > 1 - r, N(x, y, t) < r$ for all $x, y \in A$.*

Definition 3.7. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy cone metric space and $f : X \to X$ is a self mapping. Then f is said intuitionistic fuzzy cone contractive if there exists $k \in (0, 1)$ such that*

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k(\frac{1}{M(x, y, t)} - 1)$$

$$N(f(x), f(y), t) \leq kN(x, y, t)$$

for each $x, y \in X$ and $t \gg 0$. k is called the contractive constant of f.

Lemma 3.8. *If for two points $x, y \in X$ and $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x, y, t), N(x, y, kt) \leq N(x, y, t)$ then $x = y$.*

Theorem 3.9. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy cone metric space. Define $\tau = \{A \subseteq X : x \in A$ if and only if there exists $r \in (0, 1)$ and $t \gg 0$ such that $B(x, r, t) \subset A\}$, then τ is a topology on X.*

Proof. If $x \in \phi$, then $\phi = B(x, r, t) \subset \phi$ Hence $\phi \in \tau$. Since for any $x \in X$, any $r \in (0, 1)$ and any $t \gg 0, B(x, r, t) \subset X$, then $X \in \tau$.

Let $A, B \in \tau$ and $x \in A \cap B$ Then $x \in A$ and $x \in B$, so there exist $t_1 \gg 0, t_2 \gg 0$ and $r_1, r_2 \in (0, 1)$ such that $B(x, r_1, t_1) \subset A$ and $B(x, r_2, t_2) \subset B$.

From Proposition 2.3, For $t_1 \gg 0, t_2 \gg 0$, there exists $t \gg 0$ such that $t \gg t_1, t \gg t_2$ and take $r = \min\{r_1, r_2\}$. Then $B(x, r, t) \subset B(x, r_1, t_1) \cap B(x, r_2, t_2) \subset A \cap B$. Thus $A \cap B \in \tau$.

Let $A_i \in \tau$ for each $i \in I$ and $x \in \cup_{i\in I} A_i$. Then there exists $i_0 \in I$such that $x \in A_{i_0}$. So, there exist $t \gg 0$ and $r \in (0, 1)$ such that $B(x, t, r) \subset A_{i_0}$. Since $A_{i_0} \subset \cup_{i\in I} A_i, B(x, r, t) \subset \cup_{i\in I} A_i$. Thus $\cup_{i\in I} A_i \in \tau$. Hence, τ is a topology on X. □

Theorem 3.10. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy cone metric space. Then (X, τ) is Hausdorff.*

Proof. Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy cone metric space.

Let x, y be the two distinct points of X. Then $0 < M(x, y, t) < 1$ and $0 < N(x, y, t) < 1$.

Let $M(x, y, t) = r_1, N(x, y, t) = r_2$ and $r = \max\{r_1, r_2\}$.

Then for each $r_0 \in (r, 1)$, there exists r_3 and r_4 such that $r_3 * r_3 \geq r_0$ and $(1 - r_4) \diamond (1 - r_4) \leq (1 - r_0)$.

Put $r_5 = \max\{r_3, r_4\}$ and consider the open balls $B(x, 1 - r_5, t/2)$ and $B(y, 1 - r_5, t/2)$.

Then clearly $B(x, 1 - r_5, t/2) \cap B(y, 1 - r_5, t/2) = \phi$.

Suppose that $B(x, 1 - r_5, t/2) \cap B(y, 1 - r_5, t/2) \neq \phi$.

Then there exists $z \in B(x, 1 - r_5, t/2) \cap B(y, 1 - r_5, t/2)$.

$$\begin{aligned} r_1 &= M(x, y, t) \\ &\geq M(x, z, t/2) * M(z, y, t/2) \\ &\geq r_5 * r_5 \\ &\geq r_3 * r_3 \\ &\geq r_0 > r_1 \end{aligned} \tag{2}$$

and

$$\begin{aligned} r_2 &= N(x, y, t) \\ &\leq N(x, z, t/2) \diamond N(z, y, t/2) \\ &\leq (1 - r_5) \diamond (1 - r_5) \\ &\leq (1 - r_4) \diamond (1 - r_4) \\ &\leq 1 - r_0 < r_2 \end{aligned} \tag{3}$$

This is a contradiction. Hence $(X, M, N, *, \diamond)$ is hausdroff. □

Theorem 3.11. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy cone metric space, $x \in X$ and (x_n) be a sequence in X.Then (x_n) converges to x if and only if $M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$, for each $t \gg 0$.*

Proof. Suppose that, $x_n \to x$. Then, for each $t \gg 0$ and $r \in (0, 1)$, there exists a natural number n_0 such that $M(x_n, x, t) > 1 - r, N(x_n, x, t) < r$ for all $n \gg n_0$. We have $1 - M(x_n, x, t) < r$ and $N(x_n, x, t) < r$. Hence $M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$. Conversely, Suppose that $M(x_n, x, t) \to 1$ as $n \to \infty$. Then, for each $t \gg 0$ and $r \in (0, 1)$, there exists a natural number n_0 such that $1 - M(x_n, x, t) < r$ and $N(x_n, x, t) < r$ for all $n \geq n_0$. In that case, $M(x_n, x, t) > 1 - r$ and $N(x_n, x, t) < r$ Hence $x_n \to x$ as $n \to \infty$. □

4. Fixed Point Theorems

Theorem 4.1. *Let $(X, M, N, *, \diamond)$ be a complete fuzzy cone metric space in which fuzzy cone contractive sequences are Cauchy. Let $T : X \to X$ be a fuzzy cone contractive mapping being k the contractive constant. Then T has a unique fixed point.*

Proof. Fix $x \in X$ and let $x_n = T^n(x), n \in N$ For $t \gg 0$, we have

$$\frac{1}{M(T(x), T^2(x), t)} - 1 \leq k(\frac{1}{M(x, x_1, t)} - 1),$$

$$N(T(x), T^2(x), t) \leq kN(x, x_1, t).$$

and by induction

$$\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \leq k(\frac{1}{M(x_n, x_{n+1}, t)} 1),$$

$$N(x_{n+1}, x_{n+2}, t) \leq kN(x_n, x_{n+1}, t), \forall n \in N$$

Then (x_n) is a fuzzy contractive sequence, by assumptions it is a Cauchy sequence and (x_n) converges to y, for some $y \in X$. By Theorem 3.11, we have

$$\frac{1}{M(T(y), T(x_n), t)} - 1 \leq k(\frac{1}{M(y, x_n, t)} - 1) \to 0$$

,

$$N(T(y), T(x_n), t) \leq kN(y, x_n, t) \to 0$$

as $n \to \infty$. Then for each $t \gg 0$,

$$\lim_{n\to\infty} M(T(y), T(x_n), t) = 1, \lim_{n\to\infty} N((T(y), T(x_n), t)) = 0$$

and hence $\lim_{n\to\infty} T(x_n) = T(y)$,

i.e., $\lim_{n\to\infty} x_{n+1} = T(y)$ and $T(y) = y$.

Now we show uniqueness.

Assume $T(z) = z$ for some $z \in Z$. For $t \gg 0$, we have

$$\begin{aligned} \frac{1}{M(y, z, t)} - 1 &= \frac{1}{M(T(y), T(z), t)} - 1 \\ &\leq k(\frac{1}{M(y, z, t)} - 1) \\ &= k(\frac{1}{M(T(y), T(z), t)} - 1) \\ &\leq k^2(\frac{1}{(M(y, z, t))} - 1) \\ &\leq ... \leq k^n(\frac{1}{(M(y, z, t))} - 1) \to 0 \quad as \quad n \to \infty \end{aligned} \tag{4}$$

$$\begin{aligned} N(y, z, t) &= N(T(y), T(z), t) \\ &\leq kN(y, z, t) \\ &= kN(T(y), T(z), t) \\ &\leq k^2 N(y, z, t) \\ &\leq ... \leq k^n N(y, z, t) \to 0 \quad as \quad n \to \infty. \end{aligned} \tag{5}$$

Hence $M(y, z, t) = 1$ and $N(y, z, t) = 0$ and $y = z$. □

Theorem 4.2. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy cone metric space and let A, B, U and V be self-mappings of X. Let the pairs $\{A, U\}$ and $\{B, V\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that*

$$\begin{aligned} M(Ax, By, k(t)) \geq \min\{&M(U(x), V(y), t), M(U(x), A(x), t) \\ &M(B(y), V(y), t), M(A(x), V(y), t), M(B(y), U(x), t)\} \end{aligned} \tag{6}$$

$$N(Ax,By,k(t)) \leq \max\{N(U(x),V(y),t), N(U(x),A(x),t),$$
$$N(B(y),V(y),t), N(A(x),V(y),t), N(B(y),U(x),t)\}$$

for all $x, y \in X$ and for all $t \gg \theta$,then there exists a unique point $w \in X$ such that $A(w) = U(w) = w$ and a unique point $z \in X$ such that $B(z) = V(z) = z$. Moreover $z = w$, so that there is a unique common fixed point of A, B, U and V.

Proof. Let the pairs $\{A, U\}$and $\{B, V\}$ be occasionally weakly compatible,
so there are points $x, y \in X$ such that $A(x) = U(x)$and $B(y) = V(y)$.
We claim that $A(x) = B(y)$.
By inequality (6),

$$\begin{aligned}
M(A(x),B(y),k(t)) &\geq \min\{M(U(x),V(y),t), M(U(x),A(x),t), \\
&\quad M(B(y),V(y),t), M(A(x),V(y),t), M(B(y),U(x),t)\} \\
&= \min\{M(A(x),B(y),t), M(A(x),A(x),t), \\
&\quad M(B(y),B(y),t), M(A(x),B(y),t), M(B(y),A(x),t)\} \\
&= M(A(x),B(y),t).
\end{aligned}$$

$$\begin{aligned}
N(A(x),B(y),k(t)) &\leq \max\{N(U(x),V(y),t), N(U(x),A(x),t), \\
&\quad N(B(y),V(y),t), N(A(x),V(y),t), N(B(y),U(x),t)\} \\
&= \max\{N(A(x),B(y),t), N(A(x),A(x),t), \\
&\quad N(B(y),B(y),t), N(A(x),B(y),t), N(B(y),A(x),t)\} \\
&= N(A(x),B(y),t).
\end{aligned}$$

By lemma 3.8, $A(x) = B(y)$, i.e. $A(x) = U(x) = B(y) = V(y)$. Suppose that there is another point z such that $A(z) = U(z)$ then by (6), we have $A(z) = U(z) = B(y) = V(y)$, so $A(x) = A(z)$ and $w = A(x) = U(x)$ is the unique point of coincidence of A and U.
By Lemma 2.1, w is the only common fixed point of A and U.
Similarly there is a unique point $z \in X$ such that $z = B(z) = V(z)$.
Assume that $w \neq z$, we have

$$\begin{aligned}
M(w,z,k(t)) &= M(A(w),B(z),k(t)), \\
&\geq \min\{M(U(w),V(z),t), M(U(w),A(z),t), M(B(z),V(z),t), \\
&\quad M(A(w),V(z),t), M(B(z),U(w),t)\} \\
&= \min\{M(w,z,t), M(w,z,t), M(z,z,t), M(w,z,t), M(z,w,t)\} \\
&= M(w,z,t).
\end{aligned}$$

$$\begin{aligned}
N(w,z,k(t)) &= N(A(w),B(z),k(t)) \\
&\leq \max\{N(U(w),V(z),t), N(U(w),A(z),t), N(B(z),V(z),t), \\
&\quad N(A(w),V(z),t), N(B(z),U(w),t)\} \\
&= \max\{N(w,z,t), N(w,z,t), N(z,z,t), N(w,z,t), N(z,w,t)\} \\
&= N(w,z,t).
\end{aligned}$$

Therefore we have $z = w$ by Lemma 2.1 and z is a common fixed point of A, B, U and V. The uniqueness of the fixed point holds from (8) □

Theorem 4.3. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy cone metric space and let A, B, U and V be self-mappings of X. Let the pairs $\{A, U\}$ and $\{B, V\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that*

$$\begin{aligned} M(A(x), B(y), k(t)) \geq \phi[\min\{&M(U(x), V(y), t), M(U(x), A(x), t), \\ &M(B(y), V(y), t), M(A(x), V(y), t), M(B(y), U(x), t)\}] \end{aligned} \tag{7}$$

$$\begin{aligned} N(A(x), B(y), k(t)) \leq \zeta[\max\{&N(U(x), V(y), t), N(U(x), A(x), t), \\ &N(B(y), V(y), t), N(A(x), V(y), t), N(B(y), U(x), t)\}] \end{aligned}$$

for all $x, y \in X$ and $\phi, \zeta : [0, 1] \to [0, 1]$ such that $\phi(t) > t, \zeta(t) < t$ for all $0 \ll t < 1$, then there exists a unique common fixed point of A, B, U and V.

Proof. The proof follows from Theorem 4.2 □

Theorem 4.4. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy cone metric space and let A, B, U and V be self-mappings of X. Let the pairs $\{A, U\}$ and $\{B, V\}$ be occasionally weakly compatible. If there exists $k \in (0, 1)$ such that*

$$\begin{aligned} M(A(x), B(y), k(t)) \geq \phi(&M(U(x), V(y), t), M(U(x), A(x), t), M(B(y), V(y), t), \\ &M(A(x), V(y), t), M(B(y), U(x), t)) \end{aligned} \tag{8}$$

$$\begin{aligned} N(A(x), B(y), k(t)) \leq \zeta(&N(U(x), V(y), t), N(U(x), A(x), t), N(B(y), V(y), t), \\ &N(A(x), V(y), t), N(B(y), U(x), t)) \end{aligned}$$

for all $x, y \in X$ and $\phi, \zeta : [0, 1]^5 \to [0, 1]$ such that $\phi(t, 1, 1, t, t) > t, \zeta(t, 0, 0, t, t) < t$ for all $0 \ll t < 1$, then there exists a unique common fixed point of A, B, U and V.

Proof. Let the pairs $\{A, U\}$ and $\{B, V\}$ are occasionally weakly compatible, there are points $x, y \in X$ such that $A(x) = U(x)$ and $B(y) = V(y)$.
We claim that $A(x) = B(y)$. By inequality 8,we have

$$\begin{aligned} M(A(x), B(y), k(t)) &\geq \phi(M(U(x), V(y), t), M(U(x), A(x), t), M(B(y), V(y), t), \\ &\quad M(A(x), V(y), t), M(B(y), U(x), t)) \\ &= \phi(M(A(x), B(y), t), M(A(x), A(x), t), M(B(y), B(y), t), \\ &\quad M(A(x), B(y), t), M(B(y), A(x), t)) \\ &= \phi(M(A(x), B(y), t), 1, 1, M(A(x), B(y), t), M(B(y), A(x), t)) \\ &> M(A(x), B(y), t) \end{aligned}$$

$$\begin{aligned} N(A(x), B(y), k(t)) &\leq \zeta(N(U(x), V(y), t), N(U(x), A(x), t), N(B(y), V(y), t), \\ &\quad N(A(x), V(y), t), N(B(y), U(x), t)) \\ &= \zeta(N(A(x), B(y), t), N(A(x), A(x), t), N(B(y), B(y), t), \\ &\quad N(A(x), B(y), t), N(B(y), A(x), t)) \\ &= \zeta(N(A(x), B(y), t), 0, 0, N(A(x), B(y), t), N(B(y), A(x), t)) \\ &< N(A(x), B(y), t) \end{aligned}$$

a contradiction, therefore $A(x) = B(y)$, i.e. $A(x) = U(x) = B(y) = V(y)$. Suppose that there is a another point z such that $A(z) = U(z)$ then by (8) we have $A(z) = U(z) = B(y) = V(y)$, so $A(x) = A(z)$ and $w = A(x) = V(x)$ is the unique point of coincidence of A and U. By Lemma 2.1, w is a unique common fixed point of A and U. Similarly there is a unique point $z \in X$ such that $z = B(z) = V(z)$. Thus z is a common fixed point of A, B, U and V. The uniqueness of the fixed point holds from (8). □

Theorem 4.5. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy cone metric space and let A, B, U and V be self-mappings of X. Let the pairs $\{A, U\}$ and $\{B, V\}$ are occasionally weakly compatible. If there exists a point $k \in (0, 1)$ for all $x, y \in X$ and $t \gg 0$ satisfying*

$$M(A(x), B(y), k(t)) \geq M(U(x), V(y), t) * M(A(x), U(x), t) * M(B(y), V(y), t) * M(A(x), V(y), t) \tag{9}$$

$$N(A(x), B(y), k(t)) \leq N(U(x), V(y), t) \diamond N(A(x), U(x), t) \diamond N(B(y), V(y), t) \diamond N(A(x), V(y), t)$$

then there exists a unique common fixed point of A, B, U and V.

Proof. Let the pairs $\{A, U\}$ and $\{B, V\}$ are occasionally weakly compatible, there are points $x, y \in X$ such that $A(x) = U(x)$ and $B(y) = V(y)$.
We claim that $A(x) = B(y)$. By inequality (9), we have

$$\begin{aligned}
M(A(x), B(y), k(t)) &\geq M(U(x), B(y), t) * M(A(x), U(x), t) * M(By, B(y), t) * M(A(x), B(y), t) \\
&= M(A(x), B(y), t) * M(A(x), A(x), t) * M(B(y), B(y), t) * M(A(x), B(y), t) \\
&\geq M(A(x), B(y), t) * 1 * 1 * M(A(x), B(y), t) \\
&\geq M(A(x), B(y), t)
\end{aligned}$$

$$\begin{aligned}
N(A(x), B(y), k(t)) &\leq N(U(x), B(y), t) \diamond N(A(x), U(x), t) \diamond N(B(y), B(y), t) \diamond N(A(x), B(y), t) \\
&= N(A(x), B(y), t) \diamond N(A(x), A(x), t) \diamond N(B(y), B(y), t) \diamond N(A(x), B(y), t) \\
&\leq N(A(x), B(y), t) \diamond 0 \diamond 0 \diamond N(A(x), B(y), t) \\
&\leq N(A(x), B(y), t)
\end{aligned}$$

By lemma 3.8, we have $A(x) = B(y)$, i.e. $A(x) = U(x) = B(y) = V(y)$. Suppose that there is a another point z such that $A(z) = U(z)$ then by (9), we have $A(z) = U(z) = B(y) = V(y)$, so $A(x) = A(z)$ and $w = A(x) = U(x)$ is the unique point of coincidence of A and U. Similarly there is a unique point $z \in X$ such that $z = B(z) = V(z)$. Thus w is a common fixed point of A, B, U and V. □

Theorem 4.6. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy cone metric space and let A, B, U and V be self-mappings of X. Let the pairs $\{A, U\}$ and $\{B, V\}$ are occasionally weakly compatible. If there exists a point $k \in (0, 1)$ for all $x, y \in X$ and $t \gg 0$ satisfying*

$$\begin{aligned}
M(A(x), B(y), k(t)) \geq M(U(x), V(y), t) * M(A(x), U(x), t) * M(B(y), V(y), t) \\
* M(B(y), U(x), 2t) * M(A(x), V(y), t)
\end{aligned} \tag{10}$$

$$\begin{aligned}
N(A(x), B(y), k(t)) \leq N(U(x), V(y), t) \diamond N(A(x), U(x), t) \diamond N(B(y), V(y), t) \\
\diamond N(B(y), U(x), 2t) \diamond N(A(x), V(y), t)
\end{aligned}$$

then there exists a unique common fixed point of A, B, U and V.

Proof. We have,

$$\begin{aligned} M(A(x),B(y),k(t)) &\geq M(U(x),V(y),t) * M(A(x),U(x),t) * M(B(y),V(y),t) \\ &\quad * M(B(y),U(x),2t) * M(A(x),V(y),t) \\ &\geq M(U(x),V(y),t) * M(A(x),U(x),t) * M(B(y),V(y),t) \\ &\quad * M(U(x),V(y),t) * M(Ty,B(y),t) * M(A(x),V(y),t) \\ &\geq M(U(x),V(y),t) * M(A(x),U(x),t) * M(B(y),V(y),t) \\ &\quad * M(A(x),V(y),t) \end{aligned}$$

$$\begin{aligned} N(A(x),B(y),k(t)) &\leq N(U(x),V(y),t) \diamond N(A(x),U(x),t) \diamond N(B(y),V(y),t) \\ &\quad \diamond N(B(y),U(x),2t) \diamond N(A(x),V(y),t) \\ &\leq N(U(x),V(y),t) \diamond N(A(x),U(x),t) \diamond N(B(y),V(y),t) \\ &\quad \diamond N(U(x),V(y),t) \diamond N(Ty,B(y),t) \diamond N(A(x),V(y),t) \\ &\leq N(U(x),V(y),t) \diamond N(A(x),U(x),t) \\ &\quad \diamond N(B(y),V(y),t) \diamond N(A(x),V(y),t) \end{aligned}$$

and therefore from Theorem 4.6, A, B, U and V have a common fixed point. □

Theorem 4.7. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy cone metric space. Then continuous self mappings U and V of X have a common fixed point in X if and only if there exists a self mapping A of X such that the following conditions are satisfied*

1. $AX \subset VX \cap UX$,
2. *the pairs $\{A, U\}$ and $\{B, V\}$ are weakly compatible,*
3. *there exists a point $k \in (0, 1)$ such that for every $x, y \in X$ and $t \gg 0$*

$$\begin{aligned} M(A(x),A(y),k(t)) &\geq M(U(x),V(y),t) * M(A(x),U(x),t) \\ &\quad * M(A(y),V(y),t) * M(A(x),V(y),t) \end{aligned} \tag{11}$$

$$\begin{aligned} N(A(x),A(y),k(t)) &\leq N(U(x),V(y),t) \diamond N(A(x),U(x),t) \\ &\quad \diamond N(A(y),V(y),t) \diamond N(A(x),V(y),t) \end{aligned}$$

Then A, U and V have a unique common fixed point.

Proof. Since compatible implies occasionally weakly compatible, the result follows from Theorem 4.6 □

Theorem 4.8. *Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy cone metric space and let A and B be self-mappings of X. Let the A and B are occasionally weakly compatible. If there exists a point $k \in (0, 1)$ for all $x, y \in X$ and $t \gg 0$*

$$\begin{aligned} M(U(x),U(y),k(t)) &\geq aM(A(x),A(y),t) + b\min\{M(A(x),A(y),t), \\ &\quad M(U(x),A(x),t), M(U(y),A(y),t)\} \end{aligned} \tag{12}$$

$$\begin{aligned} N(U(x),U(y),k(t)) &\leq aN(A(x),A(y),t) + b\max\{N(A(x),A(y),t), \\ &\quad N(U(x),A(x),t), N(U(y),A(y),t)\} \end{aligned}$$

for all $x, y \in X$, where $a, b > 0, a + b > 1$. Then A and U have a unique common fixed point.

Proof. Let the pairs $\{A, U\}$ be occasionally weakly compatible, so there is a point $x \in X$ such that $A(x) = U(x)$. Suppose that there exist another point $y \in X$ for which $A(y) = U(y)$. We claim that $U(x) = U(y)$. By inequality (12), we have

$$\begin{aligned} M(U(x), U(y), k(t)) &= aM(A(x), A(y), t) + b\min\{M(A(x), A(y), t), \\ &\quad M(U(x), A(x), t), M(U(y), A(y), t)\} \\ &= aM(U(x), U(y), t) + b\min\{M(U(x), U(y), t), \\ &\quad M(U(x), U(x), t), M(U(y), U(y), t)\} \\ &= (a+b)M(U(x), U(y), t) \end{aligned}$$

$$\begin{aligned} N(U(x), U(y), k(t)) &= aN(A(x), A(y), t) + b\min\{N(A(x), A(y), t), \\ &\quad N(U(x), A(x), t), N(U(y), A(y), t)\} \\ &= aN(U(x), U(y), t) + b\max\{N(U(x), U(y), t), \\ &\quad N(U(x), U(x), t), N(U(y), U(y), t)\} \\ &= (a+b)N(U(x), U(y), t) \end{aligned}$$

a contradiction, since $(a + b) > 1$. Therefore $U(x) = U(y)$. Therefore $A(x) = A(y)$ and $A(x)$ is unique. From Lemma 2.1, A and U have a unique fixed point. □

5. Conclusion

In this paper, the concept of intuitionistic fuzzy cone metric space is introduced. Some fixed point theorems on intuitionistic fuzzy cone metric space are stated and proved.

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