International Journal of Mathematics And its Applications
Volume 5, Issue 1–A (2017), 37–43.
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

α, k-Laplace Transform with Poly Binomial Numbers

Research Article

G.Britto Antony Xavier[1]*, B.Govindan[1], S.John Borg[1] and M.Meganathan[1]

1 Department of Mathematics, Sacred Heart College, Tirupattur, Vellore, Taminadu, India.

Abstract: In this paper, we introduce α, k-Laplace Transform which is an extension of Generalized Laplace Transform (GLT) obtained by an inverse difference operator in the field of Digital Signal Processing (DSP). It provides a better platform to study the nature and functioning of relevant signal. Here we take k as variable and α as parameter. Suitable examples with relevant diagrams which are generated and verified using MATLAB are inserted to validate our main results.

MSC: 49A10, 26D05, 39A70.

Keywords: Generalized Laplace Transform, Generalized difference operator, factorials and Polynomial-binomial.

1. Introduction

In 2015, Aleksandar Ivic discussed the Laplace Transform of $P^2(x)$ and obtained expression $\int\limits_0^\infty P(x)e^{-sx}dx = \pi s^{-2}\sum\limits_{n=1}^{\infty} r(n)e^{-\pi^2/n}$ [2]. In practice, many applications of Laplace Transform $L[f(x)] = \int\limits_0^\infty f(x)e^{-sx}dx$, and the forward Discrete Laplace Transform $L[f(n)] = \sum\limits_{n=0}^{\infty} f(n)e^{-sn}$ are discussed and mentioned by several authors and in the citations [2, 6–8]. A new type generalized Laplace Transform defined as

$$L_\ell u(k) = \bar{u}_\ell(s) = \ell\Delta_\ell^{-1}u(k)e^{sk}\Big|_0^\infty = \ell\sum_{r=0}^{\infty} u(r\ell)e^{-sr\ell}. \tag{1}$$

This transform is introduced in [4]. Outcomes of the Generalized Laplace Transform lies in between the outcomes of the Discrete Laplace Transform and Laplace Transform. The Generalized Laplace Transform becomes the Discrete Laplace Transform and the Laplace Transform when $\ell = 1$ and $\ell \to 0$ respectively [1, 2]. The general theory on Δ_ℓ, Δ_α, $\Delta_{\alpha(\ell)}$ and $\Delta_{k(\ell)}$ one can refer [3]. From α difference operator [9], if $\underset{\alpha(\ell)}{\Delta} v(k) = u(k)$ then we have

$$v(k) = \Delta_{\alpha(\ell)}^{-1}u(k) - \alpha^{\left[\frac{k}{\ell}\right]}\Delta_{\alpha(\ell)}^{-1}u(\hat{\ell}(k)) = \sum_{r=1}^{\left[\frac{k}{\ell}\right]} \alpha^{r-1}u(k-r\ell),\ \hat{\ell}(k) = k - [k/\ell]\ell \tag{2}$$

By replacing the parameter α by variable k, we define k-Difference operator with variable coefficient as

$$\underset{k(\ell)}{\Delta} v(k) = v(k+\ell) - kv(k) \tag{3}$$

* *E-mail: brittoshc@gmail.com*

The α-Laplace Transform of $u(k)$ is defined as

$$\underset{\alpha(\ell)}{L} u(k) = \underset{\alpha(\ell)}{\bar{u}}(s) = \ell \overset{-1}{\underset{\alpha(\ell)}{\Delta}} u(k)e^{sk}\Big|_0^\infty. \tag{4}$$

and the k-Laplace Transform defined as

$$\underset{k(\ell)}{L} u(k) = \underset{k(\ell)}{\bar{u}}(s) = \ell \overset{-1}{\underset{k(\ell)}{\Delta}} u(k)e^{-sk}\Big|_0^\infty. \tag{5}$$

In this paper, we establish α, k-Discrete Laplace Transform for certain functions using the above said operators.

2. Preliminaries

In this section, we present basic theory of the Generalized difference operator $\Delta_\ell, \Delta_\alpha(\ell), \Delta_k(\ell)$ for getting results on k-Discrete Laplace Transform. Let s_r^m and S_r^m are Stirling numbers of first and second kinds respectively, $\ell > 0$, m is non-negative integer and $k_\ell^{(m)} = k(k-\ell)(k-2\ell)\cdots(k-(m-1)\ell)$. From [5] we use the following identities:

$$(i)\ k_\ell^{(m)} = \sum_{r=1}^{m} s_r^m \ell^{m-r} k^r, (ii)\ k^m = \sum_{r=1}^{m} S_r^m \ell^{m-r} k_\ell^{(r)}, (iii)\ \Delta_\ell k_\ell^{(m)} = (m\ell)k_\ell^{(m-1)}, \tag{6}$$

$$(iv)\ \Delta_\ell^{-1} k_\ell^{(m)} = \frac{k_\ell^{(m+1)}}{\ell(m+1)}\ (v)\ \Delta_\ell^{-1} k^m = \sum_{r=1}^{m} \frac{S_r^m \ell^{m-r} k_\ell^{(r)}}{(r+1)\ell}\ (vi)\ \Delta_\ell^{-1} e^{isk} = \frac{e^{isk}}{(e^{is\ell}-1)}, \tag{7}$$

$$(vii)\ \Delta_\ell^{-1} u(k)\Big|_a^b = \sum_{r=0}^{M-1} u(a+r\ell),\ M = \frac{b-a}{\ell}\ \text{and}\ (viii)\ \Delta_\ell^{-1} u(k)\Big|_0^\infty = \sum_{r=0}^{\infty} u(r\ell). \tag{8}$$

Lemma 2.1 ([5]). *Let $\ell > 0$ and $u(k)$, $w(k)$ are real valued bounded functions. Then*

$$\Delta_\ell^{-1}(u(k)w(k)) = u(k)\Delta_\ell^{-1} w(k) - \Delta_\ell^{-1}(\Delta_\ell^{-1} w(k+\ell)\Delta_\ell u(k)). \tag{9}$$

Lemma 2.2 ([3]). *(1(k)-series of $u(k)$): The first order generalized k-difference equation $v(k+\ell) - kv(k) = u(k)$, $k \in [\ell, \infty)$, $\ell > 0$, has a summation solution of the form*

$$\sum_{r=0}^{\left[\frac{k}{\ell}\right]} k_\ell^{(r)} u(k-r\ell) = \overset{-1}{\underset{(k+\ell)(\ell)}{\Delta}} u(k+\ell) - k_\ell^{([\frac{k}{\ell}]+1)} \overset{-1}{\underset{\hat{\ell}(k)(\ell)}{\Delta}} u(\hat{\ell}(k)). \tag{10}$$

Lemma 2.3. *Let $\ell > 0$ and $u(k)$, $v(k)$ are real valued bounded functions. Then*

$$\overset{-1}{\underset{k(\ell)}{\Delta}}(u(k)v(k)) = u(k) \overset{-1}{\underset{k(\ell)}{\Delta}} v(k) - \overset{-1}{\underset{k(\ell)}{\Delta}}(\overset{-1}{\underset{k(\ell)}{\Delta}} v(k+\ell)\Delta_\ell u(k)). \tag{11}$$

Proof. From (3), we get

$$\underset{k(\ell)}{\Delta}(u(k)w(k)) = u(k) \underset{k(\ell)}{\Delta} w(k) + w(k+\ell)\Delta_\ell u(k). \tag{12}$$

By taking $\underset{k(\ell)}{\Delta} w(k) = v(k)$ and $w(k) = \overset{-1}{\underset{k(\ell)}{\Delta}} v(k)$ in equation (12), we obtain (11) □

Theorem 2.4. *Let $k \in (-\infty, \infty)$ and $\ell > 0$. Then we have*

$$\overset{-1}{\underset{\alpha(\ell)}{\Delta}}(e^{-sk}\cos ak) = \frac{e^{-sk}(e^{-s\ell}\cos a(k-\ell)) - \alpha\cos ak}{e^{-2s\ell} - 2\alpha e^{-s\ell}\cos a\ell + \alpha^2}, \tag{13}$$

$$\overset{-1}{\underset{\alpha(\ell)}{\Delta}}(e^{-sk}\sin ak) = \frac{e^{-sk}(e^{-s\ell}\sin a(k-\ell) - \alpha\sin ak)}{e^{-2s\ell} - 2\alpha e^{-s\ell}\cos a\ell + \alpha^2}. \tag{14}$$

Proof. The proof follows by the definition of $\overset{-1}{\underset{\alpha(\ell)}{\Delta}}$ and solving the following relations:

$$\underset{\alpha(\ell)}{\Delta}(e^{-sk}\cos ak) = e^{-sk}\cos ak(e^{-s\ell}\cos a\ell - \alpha) - e^{-sk}e^{-s\ell}\sin ak\sin a\ell,$$
$$\underset{\alpha(\ell)}{\Delta}(e^{-sk}\sin ak) = e^{-sk}\sin ak(e^{-s\ell}\cos a\ell - \alpha) + e^{-sk}e^{-s\ell}\cos ak\sin a\ell.$$

□

3. Generalized Alpha Laplace Transform and its Properties

In this section we derive α-Laplace and k-Laplace Transforms for the the parameter α and variable k. When $\alpha = 1$ and $\ell \to 0$, the α-Laplace Transform becomes Laplace Transform.

Theorem 3.1. *For $n \in N(0)$, the α-ifference equation $\underset{\alpha(\ell)}{\Delta} v(k) = k^n$ has a closed form solution of the form*

$$\overset{-1}{\underset{\alpha(\ell)}{\Delta}} k^n = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m \ell^m k^{n-m}}{(1-\alpha)^{m+1}} (\ell_m^{(\alpha)}). \tag{15}$$

Where

$$\ell_m^{(\alpha)} = \sum_{r=1}^{m} (-1)^{r+1} \binom{m}{r} (\ell_{m-r}^{(\alpha)})(1-\alpha)^{r-1} \text{ and } \ell_0^{(\alpha)} = \ell_1^{(\alpha)} = 1, \ \alpha \neq 1, \ m \in N(0). \tag{16}$$

Proof. Taking $v(k) = k^0$ in (2), we get $\underset{\alpha(\ell)}{\Delta} k^0 = (k+\ell)^0 - \alpha k^0$. Taking ℓ as constant and using $\overset{-1}{\underset{\alpha(\ell)}{\Delta}}$, we arrive

$$\overset{-1}{\underset{\alpha(\ell)}{\Delta}} k^0 = \frac{1}{1-\alpha} = \frac{1}{1-\alpha}(\ell_0^{(\alpha)}), \text{ where } \ell_0^{(\alpha)} = 1. \tag{17}$$

Taking $v(k) = k$ in (2), we get $\underset{\alpha(\ell)}{\Delta} k = (k+\ell) - \alpha k = (1-\alpha)k + \ell$, which gives

$$\overset{-1}{\underset{\alpha(\ell)}{\Delta}} k = \frac{k}{1-\alpha} - \frac{\ell}{(1-\alpha)} \overset{-1}{\underset{\alpha(\ell)}{\Delta}} 1. \tag{18}$$

By substituting (17) in (18), we get

$$\overset{-1}{\underset{\alpha(\ell)}{\Delta}} k = \frac{k}{1-\alpha} - \frac{\ell}{(1-\alpha)^2} = \frac{k}{1-\alpha}(\ell_0^{(\alpha)}) - \frac{\ell}{(1-\alpha)^2}(\ell_1^{(\alpha)}), \text{ where } \ell_0^{(\alpha)} = \ell_1^{(\alpha)} = 1. \tag{19}$$

Now taking $v(k) = k^2$ in (2), we get $\underset{\alpha(\ell)}{\Delta} k^2 = (k+\ell)^2 - \alpha k^2$, which gives

$$\overset{-1}{\underset{\alpha(\ell)}{\Delta}} k^2 = \frac{k^2}{1-\alpha} - \frac{2\ell}{(1-\alpha)} \overset{-1}{\underset{\alpha(\ell)}{\Delta}} k - \frac{\ell^2}{(1-\alpha)} \overset{-1}{\underset{\alpha(\ell)}{\Delta}} 1, \tag{20}$$

By substituting (17) and (19)in (20), we get

$$\overset{-1}{\underset{\alpha(\ell)}{\Delta}} k^2 = \frac{k^2}{1-\alpha} - \frac{2\ell k}{(1-\alpha)^2} + \frac{\ell^2}{(1-\alpha)^3}(1+\alpha) = \binom{2}{0}\frac{k^2}{1-\alpha}(\ell_0^{(\alpha)}) - \binom{2}{1}\frac{\ell k}{(1-\alpha)^2}(\ell_1^{(\alpha)}) + \binom{2}{2}\frac{\ell^2}{(1-\alpha)^3}(\ell_2^{(\alpha)}), \tag{21}$$

where $\ell_2^{(\alpha)} = 1 + \alpha = \binom{2}{1}(\ell_1^{(\alpha)}) - \binom{2}{2}(\ell_0^{(\alpha)})(1-\alpha)$. Taking $v(k) = k^3$ in (2), we get $\underset{\alpha(\ell)}{\Delta} k^3 = (k+\ell)^3 - \alpha k^3$, which gives

$$\overset{-1}{\underset{\alpha(\ell)}{\Delta}} k^3 = \frac{k^3}{1-\alpha} - \frac{3\ell}{(1-\alpha)} \overset{-1}{\underset{\alpha(\ell)}{\Delta}} k^2 + \frac{3\ell^2}{(1-\alpha)} \overset{-1}{\underset{\alpha(\ell)}{\Delta}} k - \frac{\ell^3}{(1-\alpha)} \overset{-1}{\underset{\alpha(\ell)}{\Delta}} 1 \tag{22}$$

From (17), (19)and (21), we arrive

$$\overset{-1}{\underset{\alpha(\ell)}{\Delta}} k^3 = \frac{k^3}{1-\alpha} - \frac{3\ell k^2}{(1-\alpha)^2} + \frac{3\ell^2 k(1+\alpha)}{(1-\alpha)^3} - \frac{\ell^3(\alpha^2+4\alpha+1)}{(1-\alpha)^4} \tag{23}$$

$$\overset{-1}{\underset{\alpha(\ell)}{\Delta}} k^3 = \binom{3}{0}\frac{k^3}{1-\alpha}(\ell_0^{(\alpha)}) - \binom{3}{1}\frac{\ell k^2}{(1-\alpha)^2}(\ell_1^{(\alpha)}) + \binom{3}{2}\frac{\ell^2 k(\ell_2^{(\alpha)})}{(1-\alpha)^3} - \frac{\ell^3(\ell_3^{(\alpha)})}{(1-\alpha)^4} \tag{24}$$

where $\ell_3^{(\alpha)} = 1 + 4\alpha + \alpha^2 = \binom{3}{1}(\ell_2^{(\alpha)}) - \binom{3}{2}(\ell_1^{(\alpha)})(1-\alpha) + \binom{3}{3}(\ell_0^{(\alpha)})(1-\alpha)^2$. Continuing this process, we get the proof of the theorem. The expression given in (16) is called poly-binomial numbers. Which generates the following table of poly-binomial numbers for ℓ_m for $m = 0, 1, 2, ..., n$.

Table of poly-binomial numbers

$\ell_0^{(\alpha)}$:1

$\ell_1^{(\alpha)}$:1

$\ell_2^{(\alpha)} : 1+\alpha$

$\ell_3^{(\alpha)} : 1+4\alpha+\alpha^2$

$\ell_4^{(\alpha)} : 1+11\alpha+11\alpha^2+\alpha^3$

$\ell_5^{(\alpha)} : 1+26\alpha+66\alpha^2+26\alpha^3+\alpha^4$

....

....

□

Corollary 3.2. *The α-difference equation $\underset{\alpha(\ell)}{\Delta} v(k) = k_\ell^{(n)}$ has a closed form solution of the form*

$$\underset{\alpha(\ell)}{\overset{-1}{\Delta}} k_\ell^{(p)} = \sum_{n=1}^{p} s_n^p \ell^{p-n} \left\{ \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m \ell^m k^{n-m}}{(1-\alpha)^{m+1}} (\ell_m^\alpha) \right\}. \tag{25}$$

Proof. The proof (25) follows from (6) and (15). □

Theorem 3.3. *Let $\ell > 0$ and $k \in [\ell, \infty)$. Then $\underset{\alpha(\ell)}{L}(k_\ell^{(n)}) = \dfrac{(-1)^{n+1}\ell^{n+1} n! e^{-ns\ell}}{(e^{-s\ell}-\alpha)^{n+1}}$.*

Proof. Taking $u(k) = k_\ell^{(n)}$ in (4) we get the proof. □

Corollary 3.4. *Let $\ell > 0$ and $k \in [\ell, \infty)$. Then $\underset{\alpha(\ell)}{L}(k^n) = \sum\limits_{q=0}^{n} \dfrac{S_q^n \ell^{n-q}(-\ell)^{q+1} q! e^{-qs\ell}}{(e^{-s\ell}-\alpha)^{q+1}}$.*

Proof. The proof follows from (6) and Theorem 3.3. □

Proposition 3.5. *If $\underset{\alpha(\ell)}{L}(u(k)) = \underset{\alpha(\ell)}{\bar{u}}(s)$ and $\underset{\alpha(\ell)}{L}(v(k)) = \underset{\alpha(\ell)}{\bar{v}}(s)$, then*

$$\underset{\alpha(\ell)}{L}(au(k)+bv(k)) = a \underset{\alpha(\ell)}{\bar{u}}(s) + b \underset{\alpha(\ell)}{\bar{v}}(s) \quad \text{and} \quad \underset{\alpha(\ell)}{L}(u(ak)) = \frac{1}{a} \underset{\alpha(\ell)}{\bar{u}}\left(\frac{s}{a}\right). \tag{26}$$

Proof. From (4), we have $\underset{\alpha(\ell)}{L}(u(ak)) = \underset{\alpha(\ell)}{\overset{-1}{\Delta}} u(ak)e^{-sk}\big|_{k=0}^{\infty}$. By substituting ak by t we get the proof of (26). □

Proposition 3.6. *If $\underset{\alpha(\ell)}{L}(u(k)) = \underset{\alpha(\ell)}{\bar{u}}(s)$, then $\underset{\alpha(\ell)}{L}(e^{-ak}u(k)) = \underset{\alpha(\ell)}{\bar{u}}(s+a)$*

Proof. The proof follows by taking $u(k) = e^{-ak}u(k)$ in (4). □

Theorem 3.7. *Let $k \in (0, \infty)$, $\ell > 0$ and $s > 0$. If $e^{(s\pm na)\ell} \neq \alpha$, then we have $\underset{\alpha(\ell)}{L}(e^{\pm nak}) = \dfrac{\ell e^{(s\pm na)\ell}}{e^{(s\pm na)\ell}-\alpha}$.*

Proof. The proof follows by taking $u(k) = e^{\pm nak}$ in (4). □

Theorem 3.8. *If $e^{-2s\ell} - 2\alpha e^{-s\ell}\cos a\ell + \alpha^2 \neq 0$, then we have*

$$(i) \quad \underset{\alpha(\ell)}{L}(\sin ak) = \frac{\ell e^{-s\ell}\sin a\ell}{e^{-2s\ell} - 2\alpha e^{-s\ell}\cos a\ell + \alpha^2},$$

$$(ii) \quad \underset{\alpha(\ell)}{L}(\cos ak) = \frac{\ell(\alpha - e^{-s\ell}\cos a\ell)}{e^{-2s\ell} - 2\alpha e^{-s\ell}\cos a\ell + \alpha^2}. \tag{27}$$

Proof. The proof of (27) follows from (4). □

Theorem 3.9. *If $S_1 = \sin(n-2r)a\ell$, $C_1 = \cos(n-2r)a\ell$ and $e^{-2s\ell} - 2\alpha e^{-s\ell}C_1 + \alpha^2 \neq 0$, then we have the following relations*

$$\underset{\alpha(\ell)}{L}(\sin^n ak) = \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}} \sum_{r=0}^{[n/2]} (-1)^r \binom{n}{r} \frac{\ell e^{-s\ell} S_1}{e^{-2s\ell} - 2\alpha e^{-s\ell}C_1 + \alpha^2}, \qquad n \text{ is odd}, \tag{28}$$

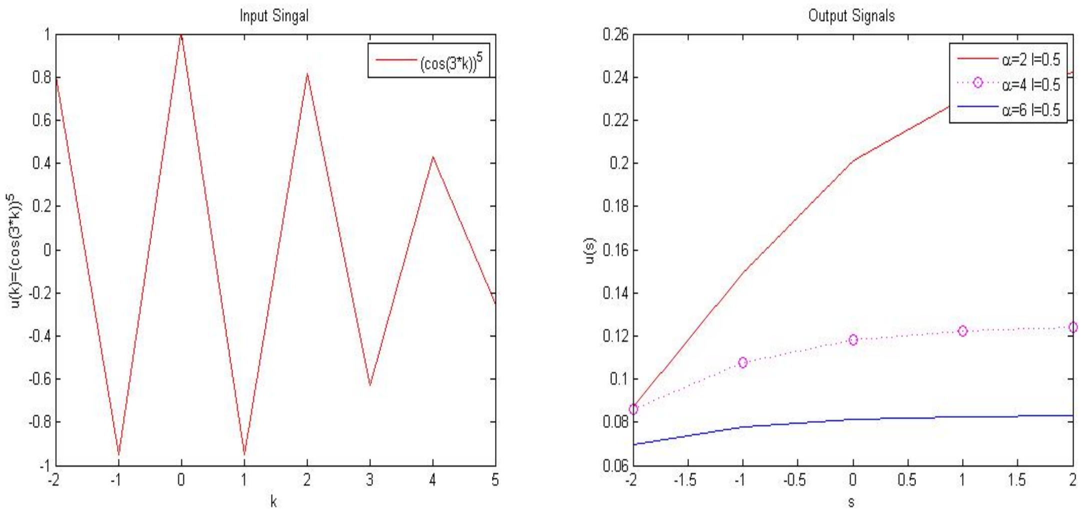
$$\underset{\alpha(\ell)}{L}(\sin^n ak) = \frac{(-1)^{\frac{n}{2}}}{2^{n-1}} \sum_{r=0}^{[n/2]-1} \binom{n}{r} \frac{(-1)^r \ell(\alpha - e^{-s\ell}C_1)}{e^{-2s\ell} - 2\alpha e^{-s\ell}C_1 + \alpha^2} + \binom{n}{\frac{n}{2}} \frac{2^{-n}(-1)^{\frac{n}{2}}}{(\alpha - e^{-s\ell})}, \; n \text{ is even}, \tag{29}$$

$$\underset{\alpha(\ell)}{L}(\cos^n ak) = \frac{1}{2^{n-1}} \sum_{r=0}^{[n/2]} \binom{n}{r} \frac{\ell(\alpha - e^{-s\ell}C_1)}{e^{-2s\ell} - 2\alpha e^{-s\ell}C_1 + \alpha^2}, \qquad n \text{ is odd}, \tag{30}$$

$$\underset{\alpha(\ell)}{L}(\cos^n ak) = \frac{1}{2^{n-1}} \sum_{r=0}^{[n/2]-1} \binom{n}{r} \frac{\ell(\alpha - e^{-s\ell}C_1)}{e^{-2s\ell} - 2\alpha e^{-s\ell}C_1 + \alpha^2} + \binom{n}{\frac{n}{2}} \frac{2^{-n}}{(\alpha - e^{-s\ell})}, \quad n \text{ is even}. \tag{31}$$

Proof. From $\sin^n ak = \frac{(-1)^{\frac{n-1}{2}}}{2^{n-1}} \sum_{r=0}^{[n/2]} (-1)^r \binom{n}{r} \sin(n-2r)ak$, by using (4) and (27) we get the proof of (28). Similarly we can obtain the proof of (29), (30) and (31). □

Example 3.10. *From (30), we arrive* $\underset{\alpha(\ell)}{L}(\cos^5 ak) = \frac{1}{2^4} \sum_{r=0}^{2} \binom{5}{r} \frac{\ell(\alpha - e^{-s\ell}C_1)}{e^{-2s\ell} - 2\alpha e^{-s\ell}C_1 + \alpha^2}$. *In particular, we take $\alpha = 2, \ell = 4$, $s = 5$, $a = 3$ and here we provide MATLAB coding for verification of Alpha Laplace Transform. syms r*
$symsum(4.*2.\wedge(-(r+1)).*((cos(12*r)).\wedge 5).*(exp(-20*r)), r, 0, inf) = 4.*(2-exp(-20).*(cos(5)*12))./(16.*(exp(-40) - 4*exp(-20).*(cos(5)*12)+4)) + 5.*4.*(2-exp(-20).*(cos(3)*12))./(16.*(exp(-40) - 4*exp(-20).*(cos(3)*12)+4)) + 10.*4.*(2-exp(-20).*(cos(1)*12))./(16.*(exp(-40) - 4*exp(-20).*(cos(1)*12)+4))$.



Theorem 3.11. *If $e^{-(s\pm a)\ell} \neq \alpha$ and $s > 0$, then we have α-Laplace for hyperbolic functions*

$$\underset{\alpha(\ell)}{L}(\sinh ak) = \frac{\ell}{2}\Big(\frac{e^{-s\ell}(e^{a\ell} - e^{-a\ell})}{(e^{-(s+a)\ell} - \alpha)(e^{-(s-a)\ell} - \alpha)}\Big), \tag{32}$$

$$\underset{\alpha(\ell)}{L}(\cosh ak) = \frac{\ell}{2}\Big(\frac{2\alpha - e^{-s\ell}(e^{a\ell} + e^{-a\ell})}{(e^{-(s+a)\ell} - \alpha)(e^{-(s-a)\ell} - \alpha)}\Big). \tag{33}$$

Proof. From (4), we have $\underset{\alpha(\ell)}{L}(\sinh ak) = (1/2) \underset{\alpha(\ell)}{\overset{-1}{\Delta}} e^{-sk}(e^{ak} - e^{-ak})$. Which completes the proof of (32). Similarly we can obtain (33). □

Theorem 3.12. *Let $E_1 = e^{-(s-(n-2r)a)\ell} \neq 0$ and $E_2 = e^{-(s+(n-2r)a)\ell} \neq 0$. Then we have*

$$\underset{\alpha(\ell)}{L}(\sinh^n ak) = \frac{\ell}{2^n} \sum_{r=0}^{[n/2]} \binom{n}{r} \Big(\frac{(-1)^r(E_1 - E_2)}{(E_1 - \alpha)(E_2 - \alpha)}\Big), \qquad n \text{ is odd}. \tag{34}$$

$$\underset{\alpha(\ell)}{L}(\sinh^n ak) = \frac{\ell}{2^n} \sum_{r=0}^{[n/2]-1} \binom{n}{r} \Big(\frac{(-1)^{r+1}(E_1 + E_2 - 2\alpha)}{(E_1 - \alpha)(E_2 - \alpha)}\Big) + \binom{n}{\frac{n}{2}} \frac{2^{-n}(-1)^{\frac{n}{2}}\ell}{(\alpha - e^{-s\ell})}, \; n \text{ is even}. \tag{35}$$

$$\underset{\alpha(\ell)}{L}(\cosh^n ak) = \frac{\ell}{2^n}\sum_{r=0}^{[n/2]}\binom{n}{r}\Big(\frac{2\alpha - E_1 - E_2}{(E_1-\alpha)(E_2-\alpha)}\Big), \qquad n \textit{ is odd.} \tag{36}$$

$$\underset{\alpha(\ell)}{L}(\cosh^n ak) = \frac{\ell}{2^n}\sum_{r=0}^{[n/2]-1}\binom{n}{r}\Big(\frac{2\alpha - E_1 - E_2}{(E_1-\alpha)(E_2-\alpha)}\Big) + \binom{n}{\frac{n}{2}}\frac{2^{-n}\ell}{(\alpha - e^{-s\ell})}, \qquad n \textit{ is even.} \tag{37}$$

Proof. From $\sin h^n ak = \frac{1}{2^{n-1}}\sum_{r=0}^{[n/2]}(-1)^r\binom{n}{r}\sin h(n-2r)ak$, using (4) and (32) we get the proof of (34). Similarly, we can obtain the proof of (35), (36) and (37). $\square$

Proposition 3.13. *If* $\underset{k(\ell)}{L}(u(k)) = \underset{k(\ell)}{\bar{u}}(s)$ *and* $\underset{k(\ell)}{L}(v(k)) = \underset{k(\ell)}{\bar{v}}(s)$, *then we have*

$$\underset{k(\ell)}{L}(au(k)+bv(k)) = a\underset{k(\ell)}{\bar{u}}(s) + b\underset{k(\ell)}{\bar{v}}(s) \quad \textit{and} \quad \underset{k(\ell)}{L}(u(ak)) = \frac{1}{a}\underset{k(\ell)}{\bar{u}}\Big(\frac{s}{a}\Big), a \neq 0. \tag{38}$$

Proof. From (5), we have $\underset{k(\ell)}{L}(u(ak)) = \underset{k(\ell)}{\overset{-1}{\Delta}} u(ak)e^{-sk}|_{k=0}^{\infty}$. By substituting ak by t we get the proof of (26). $\square$

Proposition 3.14. *If* $\underset{k(\ell)}{L}(u(k)) = \underset{k(\ell)}{\bar{u}}(s)$, *then* $\underset{k(\ell)}{L}(e^{-ak}u(k)) = \underset{k(\ell)}{\bar{u}}(s+a)$

Proof. The proof follows by replacing $u(k)$ by $e^{-ak}u(k)$ in (5). $\square$

Theorem 3.15. *For* $k \in [0,\infty)$ *and* $\ell > 0$, *we have*

$$(i) \quad \underset{k(\ell)}{\overset{-1}{\Delta}} e^{-sk}(e^{-s\ell}(k+\ell)^n - k^{n+1}) = k^n e^{-sk} \tag{39}$$

$$(ii) \quad \underset{k(\ell)}{\overset{-1}{\Delta}} e^{-sk}(e^{-s\ell}(k+\ell)_\ell^{(n)} - kk_\ell^{(n)}) = k_\ell^{(n)} e^{-sk} \tag{40}$$

$$(iii) \quad \underset{k(\ell)}{\overset{-1}{\Delta}} e^{-sk}(e^{-s\ell}\cos^n(k+\ell)^n - k\cos^n k) = e^{-sk}\cos^n k \tag{41}$$

$$(iv) \quad \underset{k(\ell)}{\overset{-1}{\Delta}} e^{-sk}(e^{-s\ell}\sin^n(k+\ell)^n - k\sin^n k) = e^{-sk}\sin^n k \tag{42}$$

Proof. From (3), we have $\underset{k(\ell)}{\Delta} k^0 e^{-sk} = e^{-s(k+\ell)} - ke^{-sk}$. Now applying $\underset{k(\ell)}{\overset{-1}{\Delta}}$ we get $\underset{k(\ell)}{\overset{-1}{\Delta}} e^{-sk}(e^{-s\ell}(k+\ell) - k) = e^{-sk}$. Again by applying (3), we have $\underset{k(\ell)}{\Delta} ke^{-sk} = (k+\ell)e^{-s(k+\ell)} - k^2e^{-sk}$. By using $\underset{k(\ell)}{\overset{-1}{\Delta}}$, we get $\underset{k(\ell)}{\overset{-1}{\Delta}} e^{-sk}(e^{-s\ell}(k+\ell) - k^2) = ke^{-sk}$. By repeating this process, we get the proof of (39). Similarly we can get the proof (40), (41) and (42). $\square$

Example 3.16. *From (10) and (39), and* $j = k - [\frac{k}{\ell}]\ell = \hat{\ell}(k)$ *we get,*

$$\sum_{r=0}^{[\frac{k}{\ell}]} k_\ell^{(r)} e^{-s(k-r\ell)}(e^{-s\ell}(k-(r-1)\ell) - (k-r\ell)^{n+1}) = (k+\ell)^n e^{-s(k+\ell)} - k_\ell^{([\frac{k}{\ell}]+1)} j^n e^{-sj}.$$

In particular, for $n = 2, k = 5, \ell = 3$, *and* $s = 10$, *we provide MATLAB coding for verification*
$>> symsum(5.\wedge r.*exp(-10.*(5-r*3)).*(exp(-10*3).*(5-(r-1).*3).^2-(5-r*3).\wedge 3), r, 0, 1) = (8).\wedge 2.*exp(-10.*(8))-40*exp(-20)$.

Theorem 3.17. *For* $k \in [0,\infty)$ *and* $\ell > 0$, *we have the identities*

$$\underset{k(\ell)}{\overset{-1}{\Delta}}\Big(\frac{1}{(k+\ell)^n} - \frac{1}{k^{n-1}}\Big) = \frac{1}{k^n}, \quad \underset{k(\ell)}{\overset{-1}{\Delta}}\Big(\frac{k^{n-1}e^{-s\ell} - (k+\ell)^n}{k^{n-1}(k+\ell)^n}\Big)e^{-sk} = \frac{e^{-sk}}{k^n} \tag{43}$$

$$\textit{and} \quad \underset{k(\ell)}{\overset{-1}{\Delta}}\Big(k_\ell^{\lceil k/\ell\rceil}e^{-sk}\Big)((k+\ell)e^{-s\ell} - k) = k_\ell^{\lceil k/\ell\rceil}e^{-sk} \tag{44}$$

Proof. The proof of (43) and (44) follows by taking $\underset{k(\ell)}{\Delta}$ for $\frac{1}{k^n}, \frac{e^{-sk}}{k^n}$ and $k_\ell^{\lceil k/\ell\rceil}e^{-sk}$. $\square$

Theorem 3.18. *For* $k \in [0,\infty)$ *and* $\ell > 0$, *we have* $\underset{\alpha(\ell)}{\overset{-1}{\Delta}}\underset{k(\ell)}{\overset{-1}{\Delta}} e^{-sk}(e^{-s\ell} - k) = \frac{e^{-sk}}{e^{-s\ell} - \alpha}$.

Proof. From (3) we get $\underset{k(\ell)}{\Delta} e^{sk} = e^{sk}(e^{s\ell} - k)$, the proof follows by taking $\underset{\alpha(\ell)}{\Delta}$ on given expression. $\square$

Theorem 3.19. *For $k \in [0, \infty)$ and $\ell > 0$, we have*

$$\sum_{r=1}^{[k/\ell]} \alpha^{r-1} e^{-s(k-r\ell+\ell)} + e^{-sj} \sum_{r_1=1}^{n} \sum_{m=0}^{r_1} \frac{s_{r_1}^{n} \ell^{n-r_1} \binom{r_1}{m} (-1)^m \ell^m (k^{r_1-m} - \alpha^{[k/\ell]} j^{r_1-m}) (\ell_m^{(\alpha)})}{(1-\alpha)^{m+1}} \tag{45}$$

$$- e^{-sj} \sum_{r=1}^{[k/\ell]} \alpha^{r-1} (k - r\ell)_{\ell}^{(n)} = \frac{e^{-s(k+\ell)}}{(e^{-s\ell} - \alpha)} - \alpha^{[k/\ell]} \frac{e^{-s(j+\ell)}}{(e^{-s\ell} - \alpha)}. \tag{46}$$

Proof. From (5) we get $\overset{-1}{\underset{k(\ell)}{\Delta}} e^{-sk}(e^{-s\ell} - k) = e^{-sk}$. Now applying (10), we have

$$\sum_{r=1}^{[k/\ell]} k_{\ell}^{(r)} e^{-s(k-r\ell)} (e^{-s\ell} - k + r\ell) = e^{-s(k+\ell)} - k_{\ell}^{([k/\ell]+1)} e^{-sj}.$$

Taking $\overset{-1}{\underset{\alpha(\ell)}{\Delta}}$ on both sides and using (2), (15), we get the proof of (46). □

Theorem 3.20. *For $k \in [0, \infty)$ and $\ell > 0$, we have the relation*

$$\sum_{r=0}^{n} \sum_{m=0}^{n-r} \frac{\binom{n}{r} \ell^r \binom{n-r}{m} (-1)^m \ell^m \overset{-1}{\underset{k(\ell)}{\Delta}} k^{n-r-m} (\ell_m^{(\alpha)})}{(1-\alpha)^{m+1}} - \sum_{m=0}^{n+1} \frac{\binom{n+1}{m} (-1)^m \ell^m \overset{-1}{\underset{k(\ell)}{\Delta}} k^{n+1-m} (\ell_m^{(\alpha)})}{(1-\alpha)^{m+1}}$$

$$= \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m \ell^m k^{n-m}}{(1-\alpha)^{m+1}} (\ell_m^{(\alpha)}). \tag{47}$$

Proof. From (3) we get $\underset{k(\ell)}{\Delta} k^n = (k+\ell)^n - k^{n+1} \Longrightarrow \sum_{r=0}^{n} \binom{n}{r} \ell^r \overset{-1}{\underset{k(\ell)}{\Delta}} k^{n-r} = k^n + \overset{-1}{\underset{k(\ell)}{\Delta}} k^{n+1}$.

Taking $\overset{-1}{\underset{\alpha(\ell)}{\Delta}}$ on both sides and using (15), we get the proof of (47). □

4. Conclusion

The above outcomes prove the fact that better outcomes can be achieved by replacing the usual Laplace by the newly derived α and k-Laplace Transform. Which tunes the input signals by varying the value of α.

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