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I-Rough Product Topology

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Abstract: In this paper, the concept of product topology is extended to I-rough topological spaces. The properties of the proposed I-rough product topology are explored. Akin to the classical product topology, the I-rough product topology makes each projection mapping an I-rough continuous function and it is found to be the weakest topology in the product rough universe having this property. Also, the projection functions are shown to be open mappings. Further, the I-rough interior of an I-rough set on the product space is expressed as the product of the corresponding I-rough interiors of the component I-rough sets.

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Keywords: I-rough functions, I-rough sets, I-rough topology, I-rough product topology, Projection functions.

1. Introduction

Pawlak's theory of rough sets has facilitated the efficient handling of incomplete or imperfect information in a novel manner [9]. The prime feature of this theory is an equivalence relation defined on the universal set X under consideration. In order to enhance the applications of rough set theoretical concepts to real life problems, generalized rough sets have been proposed and detailed studies are available [14–16]. A parallel theory of rough sets was proposed by Iwinski, in which approximation spaces are replaced by rough universes which are complete subalgebras of the power set of the universal set [2]. Iwinski type rough sets, usually called I-rough sets, are defined as pairs of elements of the subalgebra without the use of any binary relations on X. All these approaches to rough sets have been constantly correlated with topology theory [1, 4, 5, 12]. In [13], the concept of rough topology on an approximation space is proposed as a rough subset of the power set of X, in a way similar to the definition of rough sets by Pawlak. The products of approximation spaces determined by equivalence relations are discussed and the product rough topology on product approximation spaces are studied in [11]. In [6], the authors defined and investigated the properties of I-rough topology. The related concepts like I-rough compactness, I-rough connectedness, I-rough continuous functions are also presented in the context of rough universes [7, 8].

The present work is an investigation into the properties of the product of two rough universes. The concept of product topology is extended to I-rough topological spaces. The proposed I-rough product topology is proved to be the weakest topology that make the projection functions I-rough continuous. Moreover, the I-rough interior of the product of two I-rough sets is obtained as the product of the I-rough interiors of the individual I-rough sets. The structure of the rest of the paper is as follows: section 2 recalls some basic concepts and results. Section 3 discusses the I-rough product topology and section 4 concludes the paper.

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2. Preliminaries

Definition 2.1 ([2]). *A rough universe is a pair (X, Ω), where X represents a nonempty set and Ω denotes a complete subalgebra of the Boolean algebra $P(X)$. The elements of the relation $R_\Omega = \{A = (A_1, A_2) : A_1, A_2 \in \Omega, A_1 \subseteq A_2\}$ are called I-rough sets and the elements of Ω are called exact sets. Every exact set $A_1 \in \Omega$ determines an I-rough set (A_1, A_1).*

Definition 2.2 ([2]). *Let (X, Ω) be a rough universe and $(A_1, A_2), (B_1, B_2) \in R_\Omega$. Then,*

(1). $(A_1, A_2) = (B_1, B_2) \Leftrightarrow A_1 = B_1, A_2 = B_2$ (I-rough equality)

(2). $(A_1, A_2) \subseteq (B_1, B_2) \Leftrightarrow A_1 \subseteq B_1, A_2 \subseteq B_2$ (I-rough inclusion)

(3). $(A_1, A_2) \cup (B_1, B_2) = (A_1 \cup B_1, A_2 \cup B_2)$ (I-rough union)

(4). $(A_1, A_2) \cap (B_1, B_2) = (A_1 \cap B_1, A_2 \cap B_2)$ (I-rough intersection)

(5). $(A_1, A_2)^C = ({A_2}^C, {A_1}^C)$ (I-rough complement)

Proposition 2.3 ([2]). *Let (X, Ω) be a rough universe. Then, the De-Morgan's laws are satisfied by the I-rough operations and $(R_\Omega, \cup, \cap)$ is a complete distributive lattice with $(\emptyset, \emptyset)$ as the zero element and (X, X) as the unit element .*

Definition 2.4 ([8]). *Let (X_1, Ω_1) and (X_2, Ω_2) be rough universes. Then a function $f : X_1 \to X_2$ is said to be an I-rough function if exact sets are mapped to exact sets and inverse images of exact sets are exact sets. This means that for all $A \subseteq X_1, B \subseteq X_2, A \in \Omega_1 \Rightarrow f(A) \in \Omega_2$ and $B \in \Omega_2 \Rightarrow f^{-1}(B) \in \Omega_1$. Also, $f(A_1, A_2) = f(A_1), f(A_2))$ and $f^{-1}(B_1, B_2) = (f^{-1}(B_1), f^{-1}(B_2))$.*

Definition 2.5 ([6]). *Let (X, Ω) be a rough universe. A family $\tau \subseteq R_\Omega$ is called an I-rough topology on X if $(\emptyset, \emptyset) \in \tau$, $(X, X) \in \tau$ and τ is closed under arbitrary I-rough unions and finite I-rough intersections. (X, Ω, τ) is called an I-rough topological space and the elements of τ are called I-rough open sets and an I-rough set (A_1, A_2) is said to be I-rough closed if and only if its complement ${A_2}^C, {A_1}^C)$ is I-rough open .*

Definition 2.6 ([8]). *The I-rough interior of an I-rough set (A_1, A_2) is the largest I-rough open set contained in it and the I-rough closure is the smallest I-rough closed set containing it. They are respectively denoted by $(A_1, A_2)^\circ$ and $\overline{(A_1, A_2)}$.*

Definition 2.7 ([6]). *A family β of I-rough sets is said to be an I-rough covering for (X, X) if (X, X) is contained in the I-rough union of members of β and β is said to be an I-rough open covering if $\beta \subseteq \tau$. Moreover, a family $\beta \subseteq \tau$ is said to be an I-rough base for τ, if every I-rough open set in τ can be written as the I-rough union of some elements of β.*

Theorem 2.8 ([6]). *An I-rough covering β of X is an I-rough base for an I-rough topology on (X, Ω) if and only if the I-rough intersection of any two elements of β can be expressed as the I-rough union of some elements of β.*

Definition 2.9 ([6]). *A family $\mathcal{S}$ of I-rough sets on (X, Ω, τ) is said to be an I-rough subbase for τ , if the family of all finite I-rough intersections of elements of $\mathcal{S}$ forms an I-rough base for τ.*

Definition 2.10 ([7]). *The space (X, Ω, τ) is said to be I-rough compact if every I-rough open covering of (X, X) has a finite I-rough sub covering and (X, Ω, τ) is said to be I-rough connected if (X, X) cannot be expressed as the I-rough union of two disjoint I-rough open sets .*

Definition 2.11 ([8]). *Consider two I-rough topological spaces (X_1, Ω_1, τ_1) and (X_2, Ω_2, τ_2). Then, an I-rough function $f : (X_1, \Omega_1) \to (X_2, \Omega_2)$ is said to be I-rough continuous if and only if $(B_1, B_2) \in \tau_2 \Rightarrow f^{-1}(B_1, B_2) \in \tau_1$. Also, the inverse images of I-rough closed sets are I-rough closed.*

Theorem 2.12 ([7]). *Let (X_1, Ω_1, τ_1) and (X_2, Ω_2, τ_2) be I-rough topological spaces and $f : (X_1, \Omega_1) \to (X_2, \Omega_2)$ be an onto function, which is also I-rough continuous. Then, if X_1 is I-rough compact, then X_2 is I-rough compact. Also, if X_1 is I-rough connected then X_2 is I-rough connected.*

Definition 2.13 ([10]). *Let Ω_1 and Ω_2 be Boolean algebras of subsets of two non-empty sets X_1 and X_2 respectively. For each $A \in \Omega_i$, define $A^* = \{(x_1, x_2) \in X_1 \times X_2 : x_i \in A\}$. Let Ω_i^* denote the collection of all such subsets of $X_1 \times X_2$. The subalgebra Ω of $P(X_1 \times X_2)$, generated by $\Omega^* = \Omega_1^* \cup \Omega_2^*$ is called the product of the Boolean algebras Ω_1 and Ω_2. In fact, $A^* = p_i^{-1}(A) \in \Omega, \forall\, A \in \Omega_i$.*

Remark 2.14 ([10]). *Since Ω is the smallest subalgebra of $P(X_1 \times X_2)$ containing $\Omega^* = \Omega_1^* \cup \Omega_2^*$, the elements $W \in \Omega$ will be of the form $W = \bigcup_m (\bigcap_n W_{m,n}^*)$, where $W_{m,n}^* \in (\Omega_1)^*$, $W_{m,n}^* \in (\Omega_2)^*$, $(W_{m,n}^*)^C \in (\Omega_1)^*$ or $(W_{m,n}^*)^C \in (\Omega_2)^*$ for all values of m and n.*

3. I-Rough Product Topology

Consider two rough universes (X_1, Ω_1) and (X_2, Ω_2) and let R_{Ω_1} and R_{Ω_2} denote the respective families of I-rough sets on (X_1, Ω_1) and (X_2, Ω_2). Then, the product rough universe of these two rough universes is defined as follows:

Definition 3.1. *The product rough universe of (X_1, Ω_1) and (X_2, Ω_2) is the rough universe $(X_1 \times X_2, \Omega)$, where $X_1 \times X_2$ is the cartesian product of X_1 and X_2 and Ω represents the product of the Boolean algebras Ω_1 and Ω_2.*

The following theorem provides a natural way to define the product of two I-rough sets on rough universes.

Theorem 3.2. *Let (X_1, Ω_1) and (X_2, Ω_2) be two rough universes and $(X_1 \times X_2, \Omega)$ denote the product rough universe. Let $A = (A_1, A_2) \in R_{\Omega_1}$ and $B = (B_1, B_2) \in R_{\Omega_2}$ be I-rough sets on X_1 and X_2 respectively. Then, $(A_1 \times B_1, A_2 \times B_2)$, where $\times$ denote the cartesian product of sets, is an I-rough set on the product of the rough universes.*

Proof. We have, $(A_1, A_2) \in R_{\Omega_1} \Rightarrow A_1, A_2 \in \Omega_1, A_1 \subseteq A_2$ and $(B_1, B_2) \in R_{\Omega_2} \Rightarrow B_1, B_2 \in \Omega_2, B_1 \subseteq B_2$. By the properties of cartesian products, $A_1 \subseteq A_2, B_1 \subseteq B_2 \Rightarrow A_1 \times B_1 \subseteq A_2 \times B_2$. From the construction of Ω, it follows that, $A_1, A_2 \in \Omega_1 \Rightarrow p_1^{-1}(A_1) \in \Omega$, $p_1^{-1}(A_2) \in \Omega$. Similarly, $B_1, B_2 \in \Omega_2 \Rightarrow p_2^{-1}(B_1) \in \Omega$, $p_2^{-1}(B_2) \in \Omega$.
Then, $p_1^{-1}(A_1) \in \Omega$, $p_2^{-1}(B_1) \in \Omega \Rightarrow p_1^{-1}(A_1) \cap p_2^{-1}(B_1) \in \Omega$ since Ω is a subalgebra. But, $p_1^{-1}(A_1) = A_1 \times X_2$ and $p_2^{-1}(B_1) = X_1 \times B_1$. So, $(A_1 \times X_2) \cap (X_1 \times B_1) \in \Omega$. Hence, $(A_1 \cap X_1) \times (X_2 \cap B_1) \in \Omega$. Thus, $A_1 \times B_1 \in \Omega$.
Similarly, $A_2 \times B_2 \in \Omega$. Therefore, $(A_1 \times B_1, A_2 \times B_2)$ is an I-rough set on the product of the rough universes. □

Definition 3.3. *Let (X_1, Ω_1) and (X_2, Ω_2) be rough universes and $A = (A_1, A_2) \in R_{\Omega_1}$ and $B = (B_1, B_2) \in R_{\Omega_2}$ be I-rough sets on X_1 and X_2 respectively. Then, the product of the I-rough sets A and B is the I-rough set on $X_1 \times X_2$, given by $(A_1, A_2) \times (B_1, B_2) = (A_1 \times B_1, A_2 \times B_2)$.*

The projection mappings are very important in the study of product spaces. In the context of I-rough set theory, I-rough functions are considered. The next theorem proves that each one of the projection functions defined from the product space to the individual spaces is an I-rough function.

Theorem 3.4. *Let (X_1, Ω_1) and (X_2, Ω_2) be rough universes and $(X_1 \times X_2, \Omega)$ be the product rough universe. Then, the projection mappings are I-rough functions.*

Proof. Consider the projection mapping $p_1 : X_1 \times X_2 \to X_1$, given by $p_1(x_1, x_2) = x_1, \forall (x_1, x_2) \in X_1 \times X_2$. From Remark 2.14, $W \in \Omega \Rightarrow W = \bigcup_m (\bigcap_n W_{m,n}^*)$, where for all m and n, $W_{m,n}^* \in (\Omega_1)^*$, $W_{m,n}^* \in (\Omega_2)^*$, $(W_{m,n}^*)^C \in (\Omega_1)^*$ or

$(W^*_{m,n})^C \in (\Omega_2)^*$. Then, $p_1(W) = p_1(\bigcup_m(\bigcap_n W^*_{m,n})) = \bigcup_m(p_1(\bigcap_n W^*_{m,n}))$.

If $W^*_{m,n} \in (\Omega_1)^*$, then $W^*_{m,n} = p_1^{-1}(W_{m,n})$ for some $W_{m,n} \in \Omega_1$.

If $(W^*_{m,n})^C \in (\Omega_1)^*$, then $(W^*_{m,n})^C = p_1^{-1}(W_{m,n})$ for some $W_{m,n} \in \Omega_1$. Then, $W^*_{m,n} = (p_1^{-1}(W_{m,n}))^C = p_1^{-1}((W_{m,n})^C)$, where $(W_{m,n})^C \in \Omega_1$ since Ω_1 is a subalgebra.

Similarly, if $W^*_{m,n} \in (\Omega_2)^*$, then $W^*_{m,n} = p_2^{-1}(W_{m,n})$ for some $W_{m,n} \in \Omega_2$.

Also, if $(W^*_{m,n})^C \in (\Omega_2)^*$, then $(W^*_{m,n})^C = p_2^{-1}(W_{m,n})$ for some $W_{m,n} \in \Omega_2$. Then, $W^*_{m,n} = (p_2^{-1}(W_{m,n}))^C = p_2^{-1}((W_{m,n})^C)$, where $(W_{m,n})^C \in \Omega_2$ since Ω_2 is a subalgebra.

Thus, each $W^*_{m,n}$ is the inverse image of some element of Ω_1 or Ω_2 under the corresponding projection functions.

Also, $p_i^{-1}(U) \cap p_i^{-1}(V) = p_i^{-1}(U \cap V)$. Therefore, the intersection of all those $W^*_{m,n}$ which are inverse images of some elements of Ω_1 will be of the form $p_1^{-1}(A_m)$ where $A_m \in \Omega_1$.

Similarly, the intersection of all those $W^*_{m,n}$ which are inverse images of some elements of Ω_2 will be of the form $p_2^{-1}(B_m)$ where $B_m \in \Omega_2$.

Thus, $\bigcap_n W^*_{m,n} = p_1^{-1}(A_m) \cap p_2^{-1}(B_m) = (A_m \times X_2) \cap (X_1 \times B_m) = (A_m \cap X_1) \times (X_2 \cap B_m) = A_m \times B_m$.

Therefore, $p_1(\bigcap_n W^*_{m,n}) = A_m \in \Omega_1$. Hence, $p_1(W) = \bigcup_m(p_1(\bigcap_n W^*_{m,n}) = \bigcup_m(A_m)) \in \Omega_1$, as Ω_1 is a subalgebra.

Thus, p_1 maps exact sets to exact sets.

Now, if $A_1 \in \Omega_1$, then $p_1^{-1}(A_1) \in {\Omega_1}^*$. Hence, $p_1^{-1}(A_1) \in \Omega$. So, inverse images of exact sets under p_1 are exact sets. Therefore, p_1 is an I-rough function. Similarly, p_2 is an I-rough function. □

Next, it is verified that the family consisting of all products of I-rough open sets from (X_1, Ω_1, τ_1) and (X_2, Ω_2, τ_2) constitutes an I-rough base for an I-rough topology on $(X_1 \times X_2, \Omega)$, so that we can define it as the I-rough product topology on $(X_1 \times X_2, \Omega)$.

Theorem 3.5. *Let (X_1, Ω_1, τ_1) and (X_2, Ω_2, τ_2) be I-rough topological spaces. Then, the family of I-rough sets on the rough universe $(X_1 \times X_2, \Omega)$, given by $\beta = \{(A_1, A_2) \times (B_1, B_2) : (A_1, A_2) \in \tau_1, (B_1, B_2) \in \tau_2\}$ is an I-rough base for an I-rough topology on $X_1 \times X_2$.*

Proof. Consider the family $\beta = \{(A_1, A_2) \times (B_1, B_2) : (A_1, A_2) \in \tau_1, (B_1, B_2) \in \tau_2\}$. Since $(X_1, X_1) \in \tau_1, (X_2, X_2) \in \tau_2$, we get, $(X_1 \times X_2, X_1 \times X_2) \in \beta$. Therefore, β forms an I-rough covering of $(X_1 \times X_2, X_1 \times X_2)$. Now, if $(A_1, A_2) \times (B_1, B_2) \in \beta$ and $(G_1, G_2) \times (H_1, H_2) \in \beta$, then $(A_1, A_2) \in \tau_1, (B_1, B_2) \in \tau_2, (G_1, G_2) \in \tau_1$ and $(H_1, H_2) \in \tau_2$. Also,

$$\begin{aligned}
((A_1, A_2) \times (B_1, B_2)) \cap ((G_1, G_2) \times (H_1, H_2)) &= ((A_1 \times B_1, A_2 \times B_2) \cap (G_1 \times H_1, G_2 \times H_2)) \\
&= ((A_1 \times B_1) \cap (G_1 \times H_1), (A_2 \times B_2) \cap (G_2 \times H_2)) \\
&= ((A_1 \cap G_1) \times (B_1 \cap H_1), (A_2 \cap G_2) \times (B_2 \cap H_2)) \\
&= (A_1 \cap G_1, A_2 \cap G_2) \times (B_1 \cap H_1, B_2 \cap H_2)) \\
&= ((A_1, A_2) \cap (G_1, G_2)) \times ((B_1, B_2) \cap (H_1, H_2)).
\end{aligned}$$

Since τ_1 and τ_2 are I-rough topologies, $(A_1, A_2) \cap (G_1, G_2) \in \tau_1$ and $(B_1, B_2) \cap (H_1, H_2) \in \tau_2$. By the definition of the I-rough product topology, $(A_1, A_2) \cap (G_1, G_2) \times ((B_1, B_2) \cap (H_1, H_2) \in \beta$. That is, $((A_1, A_2) \times (B_1, B_2)) \cap ((G_1, G_2) \times (H_1, H_2)) \in \beta$. Hence, β is closed under I-rough intersection. Therefore, from Theorem 2.8, β is an I-rough base for an I-rough topology on $(X_1 \times X_2, \Omega)$. □

Definition 3.6. *Let (X_1, Ω_1, τ_1) and (X_2, Ω_2, τ_2) be I-rough topological spaces. Then, the I-rough topology τ on $X_1 \times X_2$ generated by the family $\beta = \{(A_1, A_2) \times (B_1, B_2) : (A_1, A_2) \in \tau_1, (B_1, B_2) \in \tau_2\}$ is called the I-rough product topology of τ_1 and τ_2.*

Lemma 3.7. *Let (X_1, Ω_1) and (X_2, Ω_2) be rough universes and $(X_1 \times X_2, \Omega)$ be the product rough universe. Then,*

(1). $p_i^{-1}(G_1, G_2) \cap p_i^{-1}(H_1, H_2) = p_i^{-1}((G_1, G_2) \cap (H_1, H_2)), \forall (G_1, G_2), (H_1, H_2) \in \Omega_i, for \quad i = 1, 2.$

(2). $p_1^{-1}(A_1, A_2) \cap p_2^{-1}(B_1, B_2) = (A_1, A_2) \times (B_1, B_2), \forall (A_1, A_2) \in \Omega_1$ and $\forall (B_1, B_2) \in \Omega_2$.

Proof. Consider $(G_1, G_2), (H_1, H_2) \in \Omega_i$ for $i = 1, 2$ and $(A_1, A_2) \in \Omega_1$ and $(B_1, B_2) \in \Omega_2$

(1).
$$\begin{aligned} p_i^{-1}(G_1, G_2) \cap p_i^{-1}(H_1, H_2) &= (p_i^{-1}(G_1), p_i^{-1}(G_2)) \cap (p_i^{-1}(H_1), p_i^{-1}(H_2)). \\ &= (p_i^{-1}(G_1) \cap p_i^{-1}(H_1), p_i^{-1}(G_2) \cap p_i^{-1}(H_2)) \\ &= (p_i^{-1}(G_1 \cap H_1), p_i^{-1}(G_2 \cap H_2)) \\ &= p_i^{-1}((G_1, G_2) \cap (H_1, H_2)) \end{aligned}$$

(2).
$$\begin{aligned} p_1^{-1}(A_1, A_2) \cap (p_2^{-1}(B_1, B_2)) &= (p_1^{-1}(A_1), p_1^{-1}(A_2)) \cap (p_2^{-1}(B_1), p_2^{-1}(B_2)) \\ &= (A_1 \times X_2, A_2 \times X_2) \cap (X_1 \times B_1, X_1 \times B_2) \\ &= ((A_1 \times X_2) \cap (X_1 \times B_1), (A_2 \times X_2) \cap (X_1 \times B_2)) \\ &= ((A_1 \cap X_1) \times (X_2 \cap B_1), (A_2 \cap X_1) \times (X_2 \cap B_2)) \\ &= (A_1 \times B_1, A_2 \times B_2) \\ &= (A_1, A_2) \times (B_1, B_2) \end{aligned}$$
□

Theorem 3.8. *Let $(X_1 \times X_2, \Omega, \tau)$ be the I-rough product space of (X_1, Ω_1, τ_1) and (X_2, Ω_2, τ_2) Then, the family $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of I-rough sets on $X_1 \times X_2$, where $\mathcal{S}_1 = \{p_1^{-1}(A_1, A_2) : (A_1, A_2) \in \tau_1\}$ and $\mathcal{S}_2 = \{p_2^{-1}(B_1, B_2) : (B_1, B_2) \in \tau_2\}$ is an I-rough subbase for τ.*

Proof. It is enough to prove that the family of all finite I-rough intersections of elements of the family $\mathcal{S}$ coincides with $\beta = (A_1, A_2) \times (B_1, B_2) : (A_1, A_2) \in \tau_1, (B_1, B_2) \in \tau_2$. Consider a finite subfamily $\mathcal{S}^*$ of $\mathcal{S}$.
In the case that $\mathcal{S}^*$ contains elements from both $\mathcal{S}_1$ and $\mathcal{S}_2$, $\mathcal{S}^*$ will contain some elements of the form $p_1^{-1}(A_1, A_2)$, where $(A_1, A_2) \in \tau_1$ and some elements of the form $p_2^{-1}(B_1, B_2)$, where $(B_1, B_2) \in \tau_2$. Using Lemma 3.7, the intersection of the elements of the form $p_1^{-1}(A_1, A_2)$ will be of the form $p_1^{-1}(U_1, U_2)$, where $(U_1, U_2) \in \tau_1$, since τ_1 is an I-rough topology. Similarly, the intersection of elements of the form $p_2^{-1}(B_1, B_2)$ will be of the form $p_2^{-1}(V_1, V_2)$, where $(V_1, V_2) \in \tau_2$.
Hence, $\bigcap \mathcal{S}^* = p_1^{-1}(U_1, U_2) \cap p_2^{-1}(V_1, V_2) = (U_1, U_2) \times (V_1, V_2)$, using Lemma 3.7. Since, $(U_1, U_2) \in \tau_1$ and $(V_1, V_2) \in \tau_2$, we get $(U_1, U_2) \times (V_1, V_2) \in \beta$.
If $\mathcal{S}^*$ contains elements from $\mathcal{S}_1$ only or $\mathcal{S}_2$ only, then $\bigcap \mathcal{S}^*$ will be of one of the forms $p_1^{-1}(U_1, U_2)$ where $(U_1, U_2) \in \tau_1$ or $p_2^{-1}(V_1, V_2)$, where $(V_1, V_2) \in \tau_2$. That is, $\bigcap \mathcal{S}^* = (U_1, U_2) \times (X_2, X_2) \in \beta$ or $\bigcap \mathcal{S}^* = (X_1, X_1) \times (V_1, V_2) \in \beta$. Thus, intersection of a finite number of elements of $\mathcal{S}$ is an element of β.
Now let $(A_1, A_2) \times (B_1, B_2) \in B$. Then, $(A_1, A_2) \in \tau_1, (B_1, B_2) \in \tau_2$. So, $p_1^{-1}(A_1, A_2) \in \mathcal{S}_1$ and $p_2^{-1}(B_1, B_2) \in \mathcal{S}_2$. Also, from Lemma 3.7, $(A_1, A_2) \times (B_1, B_2) = p_1^{-1}(A_1, A_2) \cap p_2^{-1}(B_1, B_2)$. Hence, elements of β can be expressed as I-rough intersections of elements of $\mathcal{S}$. Thus, B consists of all the finite I-rough intersections of elements of $\mathcal{S}$. Hence, $\mathcal{S}$ is an I-rough subbase for τ. □

Theorem 3.9. *Let (X_1, Ω_1, τ_1) and (X_2, Ω_2, τ_2) be I-rough topological spaces. Then, the I-rough product topology τ on $X_1 \times X_2$ is the weakest I-rough topology on $X_1 \times X_2$, that makes the projection functions I-rough continuous.*

Proof. The projection function $p_1 : X_1 \times X_2 \to X_1$ is given by $p_1(x_1, x_2) = x_1, \forall (x_1, x_2) \in X_1 \times X_2$. Let $A = (A_1, A_2) \in R_{\Omega_1}$ be an I-rough open set in τ_1. Then, $p_1^{-1}(A_1) = A_1 \times X_2$ and $p_1^{-1}(A_2) = A_2 \times X_2$. So, $p_1^{-1}(A) = (A_1 \times X_2, A_2 \times X_2) =$

$(A_1, A_2) \times (X_2, X_2) \in \tau$ as $(A_1, A_2) \in \tau_1$ and $(X_2, X_2) \in \tau_2$. Thus, inverse image of every I-rough open set is I-rough open. Hence, p_1 is continuous. Similarly, p_2 is continuous. Now let τ' be any I-rough topology on $X_1 \times X_2$ that makes each projection function I-rough continuous.

To prove that $\tau \subseteq \tau'$, it is enough to prove that every basic I-rough open set in τ is I-rough open in τ'.

Let $A \times B \in B$. Then, $A = (A_1, A_2) \in \tau_1, B = (B_1, B_2) \in \tau_2$. Since each p_i is continuous with respect to τ', $p_1^{-1}(A) \in \tau'$ and $p_2^{-1}(B) \in \tau'$. Hence, $p_1^{-1}(A) \cap p_2^{-1}(B) \in \tau'$. That is, $p_1^{-1}(A_1, A_2) \cap p_2^{-1}(B_1, B_2) \in \tau'$. Using Lemma 3.7, $(A_1, A_2) \times (B_1, B_2) \in \tau'$. Thus, $A \times B \in \tau'$. Hence, $B \subseteq \tau$. Therefore, $\tau \subseteq \tau'$. Thus, the I-rough product topology is the weakest topology that makes each projection function I-rough continuous. □

Since the projection functions are onto, the next two corollaries are direct consequences of the above theorem and Theorem 2.12.

Corollary 3.10. *If $X_1 \times X_2$ is I-rough compact with respect to the product topology, then, both X_1 and X_2 are I-rough compact with respect to the individual I-rough topologies.*

Corollary 3.11. *If $X_1 \times X_2$ is I-rough connected with respect to the product topology, then, both X_1 and X_2 are I-rough connected with respect to the individual I-rough topologies.*

Theorem 3.12. *Let (X_1, Ω_1, τ_1) and (X_2, Ω_2, τ_2) be I-rough topological spaces and τ be the I-rough product topology on $X_1 \times X_2$. Then, the projection functions are I-rough open mappings.*

Proof. Let $W \in \tau$. Then, $W = \bigcup_i (A_i \times B_i)$ for some $A_i = (A_{i1}, A_{i2}) \in \tau_1, B_i = (B_{i1}, B_{i2}) \in \tau_2$. Then,

$$\begin{aligned} p_1(W) &= p_1\left(\bigcup_i (A_i \times B_i)\right) \\ &= \bigcup_i (p_1(A_i \times B_i)) \\ &= \bigcup_i (p_1((A_{i1}, A_{i2}) \times (B_{i1}, B_{i2}))) \\ &= \bigcup_i (p_1(A_{i1} \times B_{i1}, A_{i2} \times B_{i2})) \\ &= \bigcup_i (p_1(A_{i1} \times B_{i1}), p_1(A_{i2} \times B_{i2})) \\ &= \bigcup_i (A_{i1}, A_{i2}) \\ &= \bigcup_i A_i \in \tau_1, \end{aligned}$$

since τ_1 is an I-rough topology. Therefore, p_1 is an I-rough open mapping. Similarly, p_2 is an I-rough open mapping. □

In what follows, some properties of I-rough closed sets, I-rough interior with respect to the I-rough product space are discussed.

Theorem 3.13. *Let (X_1, Ω_1, τ_1) and (X_2, Ω_2, τ_2) be I-rough topological spaces and τ be the I-rough product topology on $X_1 \times X_2$. Then, the product of any two I-rough closed sets in τ_1 and τ_2 is I-rough closed in the I-rough product topology.*

Proof. Let $C = (C_1, C_2)$ and $D = (D_1, D_2)$ be two I-rough closed sets in τ_1 and τ_2 respectively. Then, their I-rough complements are I-rough open sets. That is, $(C_1, C_2)^C \in \tau_1, (D_1, D_2)^C \in \tau_2$. We have, $C \times D = (C_1 \times D_1, C_2 \times D_2)$. Also,

$$(C \times D)^C = ((C_2 \times D_2)^C, (C_1 \times D_1)^C)$$

$$= ((C_2{}^C \times X_2) \cup (X_1 \times D_2)^C), (C_1{}^C \times X_2) \cup (X_1 \times D_1)^C))$$
$$= ((C_2{}^C \times X_2), (C_1{}^C \times X_2)) \cup ((X_1 \times D_2)^C), (X_1 \times D_1)^C))$$
$$= ((C_2{}^C, C_1^C) \times (X_2, X_2)) \cup ((X_1, X_1) \times (D_2{}^C, D_1{}^C))$$
$$= ((C_1, C_2)^C \times (X_2, X_2)) \cup ((X_1, X_1) \times (D_1, D_2)^C)$$

Now, $(C_1, C_2)^C \in \tau_1, (X_2, X_2) \in \tau_2 \Rightarrow (C_1, C_2)^C \times (X_2, X_2) \in \tau$. Similarly, $(X_1, X_1) \in \tau_1, (D_1, D_2)^C \in \tau_2 \Rightarrow (X_1, X_1) \times (D_1, D_2)^C \in \tau$. Therefore, $(C \times D)^C \in \tau$. Thus, $(C \times D)^C$ is I-rough open. Hence, $C \times D$ is I-rough closed. □

Theorem 3.14. *Let (X_1, Ω_1, τ_1) and (X_2, Ω_2, τ_2) be I-rough topological spaces and τ be the I-rough product topology on$X_1 \times X_2$. Then,*

$$((A_1, A_2) \times (B_1, B_2))^\circ = (A_1, A_2)^\circ \times (B_1, B_2)^\circ, \forall\ (A_1, A_2) \in R_{\Omega_1}, (B_1, B_2) \in R_{\Omega_2}.$$

Proof. Consider $(A_1, A_2) \in R_{\Omega_1}$ and $(B_1, B_2) \in R_{\Omega_2}$. The I-rough interior of an I-rough set is the largest I-rough open set contained in it. Hence, $(A_1, A_2)^\circ \subseteq (A_1, A_2)$ and $(B_1, B_2)^\circ \subseteq (B_1, B_2)$. So, $(A_1, A_2)^\circ \times (B_1, B_2)^\circ \subseteq (A_1, A_2) \times (B_1, B_2)$. Since $(A_1, A_2)^\circ$ and $(B_1, B_2)^\circ$ are I-rough open sets, $(A_1, A_2)^\circ \times (B_1, B_2)^\circ$ is I-rough open. Hence, $(A_1, A_2)^\circ \times (B_1, B_2)^\circ \subseteq ((A_1, A_2) \times (B_1, B_2))^\circ$. Now let $W = \bigcup_i ((A_{i1}, A_{i2}) \times (B_{i1}, B_{i2}))$ be any I-rough open set contained in $(A_1, A_2) \times (B_1, B_2)$, where each (A_{i1}, A_{i2}) and (B_{i1}, B_{i2}) are basic I-rough open sets.

$$\begin{aligned}
\bigcup_i ((A_{i1}, A_{i2}) \times (B_{i1}, B_{i2})) \subseteq (A_1, A_2) \times (B_1, B_2) &\Rightarrow (A_{i1}, A_{i2}) \times (B_{i1}, B_{i2}) \subseteq (A_1, A_2) \times (B_1, B_2), \forall i \\
&\Rightarrow (A_{i1}, A_{i2}) \subseteq (A_1, A_2), (B_{i1}, B_{i2}) \subseteq (B_1, B_2), \forall i \\
&\Rightarrow (A_{i1}, A_{i2}) \subseteq (A_1, A_2)^\circ, (B_{i1}, B_{i2}) \subseteq (B_1, B_2)^\circ, \forall i \\
&\Rightarrow (A_{i1}, A_{i2}) \times (B_{i1}, B_{i2}) \subseteq (A_1, A_2)^\circ \times (B_1, B_2)^\circ, \forall i \\
&\Rightarrow \bigcup_i ((A_{i1}, A_{i2}) \times (B_{i1}, B_{i2})) \subseteq (A_1, A_2)^\circ \times (B_1, B_2)^\circ
\end{aligned}$$

That is, $W \subseteq (A_1, A_2)^\circ \times (B_1, B_2)^\circ$. Thus, $(A_1, A_2)^\circ \times (B_1, B_2)^\circ$ is the largest I-rough open set contained in $(A_1, A_2) \times (B_1, B_2)$. Therefore, $((A_1, A_2) \times (B_1, B_2))^\circ = (A_1, A_2)^\circ \times (B_1, B_2)^\circ$. □

4. Conclusion

In the present paper, the concept of product topology has been extended to I-rough topological spaces. The projection functions are found to be I-rough functions. Several properties of the classical product spaces are extended to the proposed I-rough product topology. It has been shown that the I-rough product topology is the weakest topology which makes make the projection functions I-rough continuous. The I-rough interior of an I-rough set on the product space has been expressed as the product of the corresponding I-rough interiors of the component I-rough sets respectively. The results presented here can be extended to the product of a finite number of I-rough topological spaces.

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