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Numerical Solution of Intuitionistic Fuzzy Differential Equation by Milne's Predictor - Corrector Method Under Generalised Differentiability

Research Article

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Abstract: Nowadays many real life problems are identified with Fuzzy set theory. The Fuzzy set theory is a useful tool to describe the situation in which data are imprecise or vague or uncertain. This set theory is completely described by its membership function. A membership function of a classical fuzzy set assigns to each element of the universe of discourse a number from the interval $[0, 1]$ to indicate the degree of belongingness to the set under consideration. The degree of non belongingness is just automatically the complement to "1" of the membership degree. But many times, a human being does not express the degree of non membership as the complement to "1".There may be some hesitation about the belongingness and non-belongingness. This missing data or hesitation is accomplished by a set known as intuitionistic fuzzy set. In this paper, Milne's Predictor - Corrector method is used for finding numerical solution of an intuitionistic fuzzy differential equation (IFDE). The proposed method is based on the concept of generalized differentiability. IFDE is transformed into four ordinary differential systems and then Milne's Predictor - Corrector method is applied.Also,the convergence and stability of the proposed method is given and its applicability is illustrated by solving a first order IFDE.

MSC: 34A12,65L05,65L06.

Keywords: Intuitionistic fuzzy differential equation, Numerical methods, Milne's Predictor - Corrector, Generalised differentiability.

1. Introduction

In 1965, Zadeh introduced the fuzzy set theory [38] and Atanassov extended the concept of fuzzy set theory to intuitionistic fuzzy set (IFS) theory[9].Fuzzy differential equation (FDE) models have wide range of applications in many branches of engineering and in the field of medicine. Many research papers have been focused on numerical solutions of fuzzy initial value problems (FIVPs).Ming Ma et al introduced Euler method for solving FDEs numerically under H-derivative[24]. Numerical solutions for FIVPs using H-derivatives have been studied and can be found out in [1, 3, 5, 16, 27, 31].But they have some disadvantages that the diameter of the solution becomes infinite as the independent variable increases.To over come this disadvantage,Bede and Gal introduced the strongly generalized differentiability to FDEs[12] and first order fuzzy differential equation has been studied under generalised differentiablity in [13].Following Bede and Gal[12],Chalco-Cano and Roman-Flores studied numerical solution of fuzzy differential equations by lateral H-derivative[14].This opened a way to study numerical solutions of FDEs under generalised differentiability concept.Bede[11]

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has also proved a characterization theorem which states that under certain conditions a FDE under H-derivative is equivalent to a system of ordinary differential equations(ODEs)[11].So any numerical method which is used to solve ODEs,can be extended to solve FDEs.Using this characterization theorem,numerical solutions of FDEs have been studied in [6, 7, 19]and the numerical solutions studied through generalised differentiability can be found out in [4, 8, 10, 26, 34, 35, 37] .

Differential and partial differential equations under intuitionistic fuzzy environment have been discussed by Melliani and Chadli[21, 22].Abbasbandy and Allah Viranloo have discussed numerical solution of FDE by Runge-Kutta method with intuitionistic treatment [2].A time dependent intuitionistic fuzzy linear differential equation has been introduced by Sneh Lata and Amit Kumar and they have proposed a method to solve it[36]. Sankar Prasad Mondal and Tapan Kumar Roy have discussed strong and weak solution of intuitionistic fuzzy ordinary differential equation [32] and they have studied system of differential equations with initial value as triangular intuitionistic fuzzy number[33].Mondal and Roy have stuided second order linear differential equations with generalized trapezoidal intuitionistic Fuzzy boundary value[25]. Numerical solutions of IFDE under generalised differentiability have been studied through Euler method, modified Euler method and fourth order Runge-Kutta method respectively in [28, 29, 30].Successive approximations method has been used for finding solution of IFDEs in [15].In this paper, intuitionistic fuzzy Cauchy problem is solved numerically by Milne's Predictor - Corrector method under generalised differentiability concept.

This paper is arranged as follows: Section 2 is related to derivatives of intuitionistic fuzzy functions. Intuitionistic fuzzy Cauchy problem is given in Section 3. Milne's Predictor - Corrector method for IFDE is presented in Section 4. The convergence and stability of the proposed method is presented in Section 5. Section 6 consists of a numerical example and conclusion of the paper is in section 7.

2. Preliminaries

Definition 2.1 ([17]). *An intuitionistic fuzzy number(IFN) is as an intuitionistic fuzzy set defined over the real axis R.(i.e)An intuitionistic fuzzy number is given by* $N = \{(x, \mu_N(x), \nu_N(x))\backslash x \in R\}$ *such that* $\mu_N(x)$ *and* $(1-\nu_N)(x) = 1-\nu_N(x)$, $\forall x \in R$, *are fuzzy numbers. Therefore an IFN N is a conjecture of two fuzzy numbers,namely* N^+ *with a membership function* $\mu_{N^+}(x) = \mu_N(x)$ *and* N^- *with a membership function* $\mu_{N^-}(x) = 1-\nu_N(x)$.

Definition 2.2 ([17]). *The α-cut of an IFN N is defined as follows:*

$$N = \{(x, \mu_N(x), \nu_N(x))\backslash x \in R, \mu_N(x) \geq \alpha, \nu_N(x) \leq 1-\alpha\}, \quad \forall\, x \in [0,1].$$

The α-cut representation of IFN N is given by $[N]_\alpha = \{[\underline{N}^+(\alpha), \overline{N}^+(\alpha)], [\underline{N}^-(\alpha), \overline{N}^-(\alpha)]\}$.

Definition 2.3 ([20]). *A Triangular Intuitionistic Fuzzy Number(TIFN) T is an intuitionistic fuzzy set in R with the following membership function* $\mu_T(x)$ *and non-membership function* $\nu_T(x)$ *given as follows:*

$$\mu_T(x) = \begin{cases} \frac{x-a_l}{a_2-a_l}, & a_l \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3 \\ 0, otherwise \end{cases} \quad and \quad \nu_T(x) = \begin{cases} \frac{a_2-x}{a_2-a_l^{'}}, & a_l^{'} \leq x \leq a_2 \\ \frac{x-a_2}{a_3^{'}-a_2}, & a_2 \leq x \leq a_3^{'} \\ 1, otherwise \end{cases}$$

where $a_1^{'} \leq a_1 \leq a_2 \leq a_3 \leq a_3^{'}$ *and TIFN is denoted by* $N = (a_1, a_2, a_3; a_1^{'}, a_2, a_3^{'})$.

For arithmetic operations over TIFNs ,we refer to [20]. By Definition(2.8),we can define α-cut of an intuitionistic fuzzy function as follows:

Definition 2.4 ([28]). *Let $f : I \to W$ be an intuitionistic fuzzy function for some interval I.The α - cut of f is given by* $[f(t)]_\alpha = \{[\underline{f^+}(t;\alpha), \overline{f^+}(t;\alpha)], [\underline{f^-}(t;\alpha), \overline{f^-}(t;\alpha)]\}$, *where*

$$\underline{f^+}(t;\alpha) = Min\{f^+(t;\alpha) \mid t \in I, 0 \le \alpha \le 1\},$$

$$\overline{f^+}(t;\alpha) = Max\{f^+(t;\alpha) \mid t \in I, 0 \le \alpha \le 1\}$$

$$\underline{f^-}(t;\alpha) = Min\{f^-(t;\alpha) \mid t \in I, 0 \le \alpha \le 1\},$$

$$\overline{f^-}(t;\alpha) = Max\{f^-(t;\alpha) \mid t \in I, 0 \le \alpha \le 1\}$$

Let W be the class of all intuitionistic fuzzy subsets over R.Then W can be written as $W = [E^+, E^-]$ where E^+ and E^- are two spaces of fuzzy numbers.So, we can define metric structures on E^+ and E^- as follows: The metric structure on E^+ can be defined as $D^+ : E^+ \times E^+ \to R_+ \cup \{0\}$ such that $D^+(u^+, v^+) = sup_{0\le r\le 1} max\{|\underline{u^+} - \underline{v^+}|, |\overline{u^+} - \overline{v^+}|\}$. The metric structure on E^- can be defined as $D^- : E^- \times E^- \to R_+ \cup \{0\}$ such that $D^-(u^-, v^-) = sup_{0\le r\le 1} max\{|\underline{u^-} - \underline{v^-}|, |\overline{u^-} - \overline{v^-}|\}$, where $u^+ = [\underline{u^+}, \overline{u^+}], v^+ = [\underline{v^+}, \overline{v^+}] \in E^+$ and $u^- = [\underline{u^-}, \overline{u^-}], v^- = [\underline{v^-}, \overline{v^-}] \in E^-$. Clearly (E^+, D^+) and (E^-, D^-) are complete metric spaces [18]. A study on intuitionistic metric spaces can also be found out in [23]. Extending the generalised differentiability concept of fuzzy functions [12] to intuitionistic fuzzy function ,we have:(only two cases have been taken):

Definition 2.5 ([28]). *Let $F : (a, b) \to W$ and $t_0 \in (a, b)$.It is said that F is strongly generalized differentiable on t_0, if $\exists\, F^{+\prime}(t_0) \in E^+, F^{-\prime}(t_0) \in E^-$, such that*

(i) for all $h > 0$ sufficiently small, $\exists\, F^+(t_0 + h) - F^+(t_0)$, $F^+(t_0) - F^+(t_0 - h)$and the limits(in the metric D^+)

$$\lim_{h\searrow 0} \frac{F^+(t_0 + h) - F^+(t_0)}{h} = \lim_{h\searrow 0} \frac{F^+(t_0) - F^+(t_0 - h)}{h} = F^{+\prime}(t_0) \ \ (OR)$$

(ii) for all $h > 0$ sufficiently small, $\exists\, F^+(t_0) - F^+(t_0 + h), F^+(t_0 - h) - F^+(t_0)$ and the limits

$$\lim_{h\searrow 0} \frac{F^+(t_0) - F^+(t_0 + h))}{-h} = \lim_{h\searrow 0} \frac{F^+(t_0 - h) - F^+(t_0)}{-h} = F^{+\prime}(t_0) \ \ and$$

(ia) for all $h > 0$ sufficiently small,$\exists F^-(t_0 + h) - F^-(t_0)$, $F^-(t_0) - F^-(t_0 - h)$and the limits(in the metric D^-)

$$\lim_{h\searrow 0} \frac{F^-(t_0 + h) - F^-(t_0)}{h} = \lim_{h\searrow 0} \frac{F^-(t_0) - F^-(t_0 - h)}{h} = F^{-\prime}(t_0) \ \ (OR)$$

(iia) for all $h > 0$ sufficiently small, $\exists\, F^-(t_0) - F^-(t_0 + h), F^-(t_0 - h) - F^-(t_0)$ and the limits

$$\lim_{h\searrow 0} \frac{F^-(t_0) - F^-(t_0 + h))}{-h} = \lim_{h\searrow 0} \frac{F^-(t_0 - h) - F^-(t_0)}{-h} = F^{-\prime}(t_0)$$

where, [h and $(-h)$ at denominators mean $\frac{1}{h}\odot$ and $\frac{-1}{h}\odot$, respectively].

Remark 2.6. *A function that is strongly differentiable as in cases(i)[and (ia)] and (ii)[(iia)] of definition (2.5),will be referred as (1)- differentiable and (2) differentiable, respectively.*

With respect to Chalco-Cano & Roman-Flores [14], the intuitionistic fuzzy lateral H-derivative for an intuitionistic fuzzy mapping $F : (a, b) \to W$ is defined as follows:

Definition 2.7 ([28]). *Let $F : (a, b) \to W$ and $t_0 \in (a, b)$. We say that, if $\exists\, F^{+\prime}(t_0) \in E^+, F^{-\prime}(t_0) \in E^-$, such that*

(i) for all $h > 0$ sufficiently small, $\exists\, F^+(t_0+h)-F^+(t_0)$, $F^+(t_0)-F^+(t_0-h)$, $F^-(t_0+h)-F^-(t_0)$, $F^-(t_0)-F^-(t_0-h)$and the limits (in the metric D^+ and in D^-,respectively)

$$\lim_{h\to 0^+} \frac{F^+(t_0 + h) - F^+(t_0)}{h} = \lim_{h\to 0^+} \frac{F^+(t_0) - F^+(t_0 - h)}{h} = F^{+\prime}(t_0)$$

$$\lim_{h\to 0^+} \frac{F^-(t_0+h)-F^-(t_0)}{h} = \lim_{h\to 0^+} \frac{F^-(t_0)-F^-(t_0-h)}{h} = F^{-\prime}(t_0)$$

(OR)

(ii) for all $h<0$ sufficiently small, $\exists\ F^+(t_0+h)-F^+(t_0)$, $F^+(t_0)-F^+(t_0-h)$, $F^-(t_0+h)-F^-(t_0)$, $F^-(t_0)-F^-(t_0-h)$ and the limits (in the metric D^+ and in D^-, respectively)

$$\lim_{h\to 0^-} \frac{F^+(t_0+h)-F^+(t_0)}{h} = \lim_{h\to 0^-} \frac{F^+(t_0)-F^+(t_0-h)}{h} = F^{+\prime}(t_0)$$

$$\lim_{h\to 0^-} \frac{F^-(t_0+h)-F^-(t_0)}{h} = \lim_{h\to 0^-} \frac{F^-(t_0)-F^-(t_0-h)}{h} = F^{-\prime}(t_0)$$

Theorem 2.8 ([28]). *Let $F:(a,b)\to W$ and $t\in(a,b)$. $[F(t)]_\alpha = \{[\underline{F^+}(t;\alpha),\overline{F^+}(t;\alpha)],[\underline{F^-}(t;\alpha),\overline{F^-}(t;\alpha)]\}, 0\le\alpha\le 1$.*
(i) If F is (1)-differentiable,then $\underline{F^+}^{\prime}, \overline{F^{+\prime}}, \underline{F^-}^{\prime}, \overline{F^{-\prime}}$ are differentiable functions and

$$[F'(t)]_\alpha = \{[\underline{F^+}^{\prime}, \overline{F^{+\prime}}],[\underline{F^-}^{\prime}, \overline{F^{-\prime}}]\}.$$

(ii) If F is (2)-differentiable,then $\underline{F^+}^{\prime}, \overline{F^{+}}^{\prime}, \underline{F^-}^{\prime}, \overline{F^{\prime -}}$ are differentiable functions and

$$[F'(t)]_\alpha = \{[\overline{F^{+\prime}}, \underline{F^+}^{\prime}],[\overline{F^{-\prime}}, \underline{F^-}^{\prime}]\}.$$

3. Intuitionistic Fuzzy Cauchy Problem

Following Ming Ma et al [24], an intuitionistic fuzzy Cauchy problem of first order is defined as follows:

$$y^{'}(t) = f(t,y(t)), y(t_0) = y_0 \, and \, t\in I=[a,b] \tag{1}$$

where the initial value $y(t_0)=y_0$ is an intiutionistic fuzzy number and $f: I\times W\to W$.

Theorem 3.1. *Let $f^+: I\times E^+\to E^+$ and $f^-: I\times E^-\to E^-$ be two continuous fuzzy functions such that there exist $k_1>0$ and $k_2>0$ respectively such that $D^+(f^+(t,x_1),f^+(t,z_1))\le k_1D^+(x_1,z_1)$ and $D^-(f^-(t,x_2),f^-(t,z_2))\le k_2D^-(x_2,z_2)$, $\forall\, t\in I, x_1,z_1\in E^+, x_2,z_2\in E^-$. Then the problem in Equation (1) has two solutions [(1)-differentiable and (2)-differentiable] on I.*

By the theorem (2.8),equation (1) can be replaced by four equivalent systems when $y(t)$ is (1)-differentiable as follows: $y^{'}(t) = \{[\underline{y^{'+}}(t), \overline{y^{'+}}(t)],[\underline{y^{'-}}(t),\overline{y^{'-}}(t)]\}$,where

$$\underline{y^{'+}}(t) = \underline{f^+}(t,y^+) = min\{f^+(t,u)\mid u\in[\underline{y^+},\overline{y^+}]\} = F(t,\underline{y^+},\overline{y^+}), \underline{y^+}(t_0) = \underline{y_0^+} \tag{2}$$

$$\overline{y^{'+}}(t) = \overline{f^+}(t,y^+) = max\{f^+(t,u)\mid u\in[\underline{y^+},\overline{y^+}]\} = G(t,\underline{y^+},\overline{y^+}), \overline{y^+}(t_0) = \overline{y_0^+} \tag{3}$$

$$\underline{y^{'-}}(t) = \underline{f^-}(t,y^-) = min\{f^-(t,u)\mid u\in[\underline{y^-},\overline{y^-}]\} = H(t,\underline{y^-},\overline{y^-}), \underline{y^-}(t_0) = \underline{y_0^-} \tag{4}$$

$$\overline{y^{'-}}(t) = \overline{f^-}(t,y^-) = max\{f^-(t,u)\mid u\in[\underline{y^-},\overline{y^-}]\} = I(t,\underline{y^-},\overline{y^-}), \overline{y^-}(t_0) = \overline{y_0^-} \tag{5}$$

The parametric forms of the system of equations given in Equations (2 to 5) are as follows:

$$\underline{y^{'+}}(t;r) = F(t,\underline{y^+}(t;r),\overline{y^+}(t;r)), \underline{y^+}(t_0;r) = \underline{y_0}^+(r)$$

$$\overline{y'^{+}}(t;r) = G(t, \underline{y^{+}}(t;r), \overline{y^{+}}(t;r)), \overline{y^{+}}(t_0;r) = \overline{y_0}^{+}(r)$$

$$\underline{y^{'-}}(t;r) = H(t, \underline{y^{-}}(t;r), \overline{y^{-}}(t;r)), \underline{y^{-}}(t_0;r) = \underline{y_0}^{-}(r)$$

$$\overline{y'^{-}}(t;r) = I(t, \underline{y^{-}}(t;r), \overline{y^{-}}(t;r)), \overline{y^{-}}(t_0;r) = \overline{y_0}^{-}(r)$$

for $r \in [0,1]$. Again, by the Theorem (2.8), Equation (1) can be replaced by four equivalent systems when $y(t)$ is (2)-differentiable as follows: $y^{'}(t) = \{[\overline{y'^{+}}(t), \underline{y^{'+}}(t)], [\overline{y'^{-}}(t), \underline{y^{'-}}(t)]\}$, where

$$\underline{y^{'+}}(t;r) = G(t, \underline{y^{+}}(t;r), \overline{y^{+}}(t;r)), \underline{y^{+}}(t_0;r) = \underline{y_0}^{+}(r)$$

$$\overline{y'^{+}}(t;r) = F(t, \underline{y^{+}}(t;r), \overline{y^{+}}(t;r)), \overline{y^{+}}(t_0;r) = \overline{y_0}^{+}(r)$$

$$\underline{y^{'-}}(t;r) = I(t, \underline{y^{-}}(t;r), \overline{y^{-}}(t;r)), \underline{y^{-}}(t_0;r) = \underline{y_0}^{-}(r)$$

$$\overline{y'^{-}}(t;r) = H(t, \underline{y^{-}}(t;r), \overline{y^{-}}(t;r)), \overline{y^{-}}(t_0;r) = \overline{y_0}^{-}(r)$$

for $r \in [0,1]$.

4. The Milne's Predictor-Corrector Method

In this section, we will present the Milne's Predictor - Corrector Method for finding the numerical solutions of the intuitionistic fuzzy differential equations. In the interval $I = [a,b]$ we consider a set of discrete equally spaced grid points $a = t_0 < t_1 < t_2 < \cdots < t_N = b$ at which two exact solutions $Y_1(t) = \{[\underline{Y_1^{+}}(t), \overline{Y_1^{+}}(t)], [\underline{Y_1^{-}}(t), \overline{Y_1^{-}}(t)]\}$ and $Y_2(t) = \{[\underline{Y_2^{+}}(t), \overline{Y_2^{+}}(t)], [\underline{Y_2^{-}}(t), \overline{Y_2^{-}}(t)]\}$ are approximated by some $y_1(t) = \{[\underline{y_1^{+}}(t), \overline{y_1^{+}}(t)], [\underline{y_1^{-}}(t), \overline{y_1^{-}}(t)]\}$ and $y_2(t) = \{[\underline{y_2^{+}}(t), \overline{y_2^{+}}(t)], [\underline{y_2^{-}}(t), \overline{y_2^{-}}(t)]\}$,respectively.The grid points at which the solutions are calculated are $t_n = t_0 + nh$ where $h = \frac{b-a}{N}$. The exact and approximate solutions at t_n, $0 \le n \le N$ are denoted by $Y_{1(n)}(r)$, $Y_{2(n)}(r)$, $y_{1(n)}(r)$ and $y_{2(n)}(r)$, respectively. Rewriting the Milne's Predictor-Corrector method to the IFDE given in Equation (1) when $y(t)$ is (1)-differentiable, we have (6 to 9):

$$\underline{y}^{+}_{1(n+4,P)}(r) = \underline{y}_1(n)^{+}(r) + \frac{4h}{3}[2\underline{f}^{+}_{1(n+1)}(r) - \overline{f}^{+}_{1(n+2)}(r) + 2\underline{f}^{+}_{1(n+3)}(r)];$$

$$\underline{y}^{+}_{1(n+4,C)}(r) = \underline{y}^{+}_{1(n+2)}(r) + \frac{h}{3}[\underline{f}^{+}_{1(n+2)}(r) + 4\underline{f}^{+}_{1(n+3)}(r) + \underline{f}^{+}_{1(n+4)}(r)]; \quad (6)$$

$$\overline{y}^{+}_{1(n+4,P)}(r) = \overline{y}^{+}_{1(n)}(r) + \frac{4h}{3}[2\overline{f}^{+}_{1(n+1)}(r) - \underline{f}^{+}_{1(n+2)}(r) + 2\overline{f}^{+}_{1(n+3)}(r)];$$

$$\overline{y}^{+}_{1(n+4,C)}(r) = \overline{y}^{+}_{1(n+2)}(r) + \frac{h}{3}[\overline{f}^{+}_{1(n+2)}(r) + 4\overline{f}^{+}_{1(n+3)}(r) + \overline{f}^{+}_{1(n+4)}(r)]; \quad (7)$$

$$\underline{y}^{-}_{1(n+4,P)}(r) = \underline{y}^{-}_{1(n)}(r) + \frac{4h}{3}[2\underline{f}^{-}_{1(n+1)}(r) - \overline{f}^{-}_{1(n+2)}(r) + 2\underline{f}^{-}_{1(n+3)}(r)];$$

$$\underline{y}^{-}_{1(n+4,C)}(r) = \underline{y}^{-}_{1(n+2)}(r) + \frac{h}{3}[\underline{f}^{-}_{1(n+2)}(r) + 4\underline{f}^{-}_{1(n+3)}(r) + \underline{f}^{-}_{1(n+4)}(r)]; \quad (8)$$

$$\overline{y}^{-}_{1(n+4,P)}(r) = \overline{y}^{-}_{1(n)}(r) + \frac{4h}{3}[2\overline{f}^{-}_{1(n+1)}(r) - \underline{f}^{-}_{1(n+2)}(r) + 2\overline{f}^{-}_{1(n+3)}(r)];$$

$$\overline{y}^{-}_{1(n+4,C)}(r) = \overline{y}^{-}_{1(n+2)}(r) + \frac{h}{3}[\overline{f}^{-}_{1(n+2)}(r) + 4\overline{f}^{-}_{1(n+3)}(r) + \overline{f}^{-}_{1(n+4)}(r)]; \quad (9)$$

Again,when $y(t)$ is (2)-differentiable,we have (10 to 13):

$$\underline{y}^{+}_{2(n+4,P)}(r) = \underline{y}^{+}_{2(n)}(r) + \frac{4h}{3}[2\overline{f}^{+}_{2(n+1)}(r) - \underline{f}^{+}_{2(n+2)}(r) + 2\overline{f}^{+}_{2(n+3)}(r)];$$

$$\underline{y}^{+}_{2(n+4,C)}(r) = \underline{y}^{+}_{2(n+2)}(r) + \frac{h}{3}[\overline{f}^{+}_{2(n+2)}(r) + 4\overline{f}^{+}_{2(n+3)}(r) + \overline{f}^{+}_{2(n+4)}(r)]; \tag{10}$$

$$\overline{y}^{+}_{2(n+4,P)}(r) = \overline{y}^{+}_{2(n)}(r) + \frac{4h}{3}[2\underline{f}^{+}_{2(n+1)}(r) - \overline{f}^{+}_{2(n+2)}(r) + 2\underline{f}^{+}_{2(n+3)}(r)];$$

$$\overline{y}^{+}_{2(n+4,C)}(r) = \overline{y}^{+}_{2(n+2)}(r) + \frac{h}{3}[\underline{f}^{+}_{2(n+2)}(r) + 4\underline{f}^{+}_{2(n+3)}(r) + \underline{f}^{+}_{2(n+4)}(r)]; \tag{11}$$

$$\underline{y}^{-}_{2(n+4,P)}(r) = \underline{y}^{-}_{2(n)}(r) + \frac{4h}{3}[2\overline{f}^{-}_{2(n+1)}(r) - \underline{f}^{-}_{2(n+2)}(r) + 2\overline{f}^{-}_{2(n+3)}(r)];$$

$$\underline{y}^{-}_{2(n+4,C)}(r) = \underline{y}^{-}_{2(n+2)}(r) + \frac{h}{3}[\overline{f}^{-}_{2(n+2)}(r) + 4\overline{f}^{-}_{2(n+3)}(r) + \overline{f}^{-}_{2(n+4)}(r)]; \tag{12}$$

$$\overline{y}^{-}_{2(n+4,P)}(r) = \overline{y}^{-}_{2(n)}(r) + \frac{4h}{3}[2\underline{f}^{-}_{2(n+1)}(r) - \overline{f}^{-}_{2(n+2)}(r) + 2\underline{f}^{-}_{2(n+3)}(r)];$$

$$\overline{y}^{-}_{2(n+4,C)}(r) = \overline{y}^{-}_{2(n+2)}(r) + \frac{h}{3}[\underline{f}^{-}_{2(n+2)}(r) + 4\underline{f}^{-}_{2(n+3)}(r) + \underline{f}^{-}_{2(n+4)}(r)]; \tag{13}$$

5. Convergence and Stability

The following lemmas will be applied to show the convergences of the approximations $[\underline{y}^{+}_{1(n+4,C)}(r), \overline{y}^{+}_{1(n+4,C)}(r)]$, $[\underline{y}^{-}_{1(n+4,C)}(r), \overline{y}^{-}_{1(n+4,C)}(r)]$,$[\underline{y}^{+}_{2(n+4,C)}(r), \overline{y}^{+}_{2(n+4,C)}(r)]$ and $[\underline{y}^{-}_{2(n+4,C)}(r), \overline{y}^{-}_{2(n+4,C)}(r)]$ to the exact solutions.

Lemma 5.1. *Let the sequence of numbers* $\{W_n\}_{n=0}^{N}$ *satisfy* $\mid W_{n+1} \mid \leq A \mid W_n \mid +B$,$0 \leq n \leq N-1$,*for some given positive constants* A *and* B,*then* $\mid W_n \mid \leq A^n \mid W_0 \mid +B\frac{A^n-1}{A-1}$,$0 \leq n \leq N-1$.

Lemma 5.2. *Let the sequence of numbers*$\{W_n\}_{n=0}^{N}$,$\{V_n\}_{n=0}^{N}$ *satisfy* $\mid W_{n+1} \mid \leq \mid W_n \mid +Amax\{\mid W_n \mid, \mid V_n \mid\} + B$, $\mid V_{n+1} \mid \leq \mid V_n \mid +Amax\{\mid W_n \mid, \mid V_n \mid\} + B$, *for some positive constants*A *and* B, *and denote* $U_n = \mid W_n \mid + \mid V_n \mid$, $0 \leq n \leq N$.*Then* $U_n \leq (1+2A)U_0 + 2B\frac{[1+2A]^n-1}{[1+2A]-1}$,$0 \leq n \leq N$.

Theorem 5.3. *For arbitrary fixed* r,$0 \leq r \leq 1$ *the approximate solutions given in Equations(6 to 9) converge to the exact solutions* $\underline{Y_1^+}(r)$,$\overline{Y_1^+}(r)$,$\underline{Y_1^-}(r)$ *and* $\overline{Y_1^-}(r)$,*respectively.And the approximate solutions given in Equations(10 to 13) converge to the exact solutions* $\underline{Y_2^+}(r)$,$\overline{Y_2^+}(r)$,$\underline{Y_2^-}(r)$ *and* $\overline{Y_2^-}(r)$,*respectively uniformly in* t, *for* $\underline{Y_1^+}(r)$,$\overline{Y_1^+}(r)$,$\underline{Y_1^-}(r)$, $\overline{Y_1^-}(r)$, $\underline{Y_2^+}(r)$,$\overline{Y_2^+}(r)$,$\underline{Y_2^-}(r)$ *and* $\overline{Y_2^-}(r) \in C^4[t_0, t_N]$.

Definition 5.4. *An m-step method for solving the initial value problem is one whose difference equation for finding the approximation* $y(t_{i+1})$ *at the mesh point* t_{i+1} *can be represented by the following equation:*

$$y(t_{i+1}) = a_{m-1}y(t_i) + a_{m-2}y(t_{i-1}) + ... + a_0 y(t_{i+1-m}) + h\{b_m f(t_{i+1}, y_{i+1}) + b_{m-1}f(t_i, y_i) + ... + b_0 f(t_{i+1-m}, y_{i+1-m})\},$$

for $i = m-1, m, ..., N-1$ *such that* $a = t_0 \leq t_1 \leq .... \leq t_N = b$, $h = \frac{(b-a)}{N}$ *and* $a_0, a_1, a_2, \cdots a_{m-1}, b_0, b_1, ...b_m$ *are constants with the starting values* $y_0 = \alpha_0, y_1 = \alpha_1, ...y_{m-1} = \alpha_{m-1}$

When $b_m = 0$,the method is known as explicit and when $b_m \neq 0$,the method is known as implicit.

Definition 5.5. *Associated with the difference equation*

$$y(t_{i+1}) = a_{m-1}y(t_i) + a_{m-2}y(t_{i-1}) + ... + a_0 y(t_{i+1-m}) + h\{b_m f(t_{i+1}, y_{i+1}) + b_{m-1}f(t_i, y_i) + ... + b_0 f(t_{i+1-m}, y_{i+1-m})\} \tag{14}$$

the characteristic polynomial of the method is defined by

$$P(\lambda) = \lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - ...a_1\lambda - a_0.$$

If $\mid \lambda_i \mid \leq 1$for $i = 1, 2, 3...m$,and all roots with absolute value 1 are simple roots, then the difference method is said to satisfy the root condition.

Theorem 5.6. *A multistep method of the form given in Equation(14) is stable iff it satisfies the root condition.*

Remark 5.7. *A Predictor-Corrector method is stable iff the corresponding Corrector method is stable.*

Theorem 5.8. *The implicit four step method is stable.*

Proof. For the implicit four step method, there exists only one characteristic polynomial $P(\lambda) = \lambda^4 - \lambda^2$. So it satisfies the root condition, and therefore, it is a stable method. □

6. Numerical examples

Example 6.1. *Let us consider the nuclear decay equation:*

$$y^{'}(t) = -\lambda \odot y(t), y(t_0) = y_0, t \in I = [t_0, T] \tag{15}$$

where $y(t)$is the number of radio nuclide present in a given radioactive material,λ is the decay constant and y_0 is the initial number of radio nuclide. In the model, uncertainty is introduced if we have uncertain information on the initial value y_0 of radio nuclide present in the material. Note that the phenomenon of nuclear disintegration is considered a stochastic process, uncertainty being introduced by the lack of information on the radioactive material under study. However, in some situations, there may be hesitation on the number of radio nuclide present in the radioactive material .In order to take into account the uncertainty and hesitation; we consider y_0 being a triangular intuitionistic fuzzy number.

Solution:

Let $\lambda = 1$ and $I = [0, 1]$. The α - cut of the initial value is given by $y(t_0, \alpha) = y_0(\alpha) = \{[5 + 2\alpha, 9 - 2\alpha], [3 + 4\alpha, 11 - 4\alpha]\}$

Case(1): (1)-Differentiability

The exact solution of equation(15) under (1)-differentiability is given by

$$\underline{y^+}(t;\alpha) = (2\alpha - 2)e^t + 7e^{-t}, \overline{y^+}(t;\alpha) = -(2\alpha - 2)e^t + 7e^{-t}$$
$$\underline{y^-}(t;\alpha) = (4\alpha - 4)e^t + 7e^{-t}, \underline{y^-}(t;\alpha) = -(4\alpha - 4)e^t + 7e^{-t}.$$

Approximate solutions were calculated at different r-values for different t. In particular ,errors are estimated for both membership and non-membership functions of equation(15) at t =0.04 with h=0.01 and are given in Table 1.

r	Error (Membership)	Error (Non-Membership)
0	0.000362739	0.000725476
0.2	0.00029019	0.000580382
0.4	0.000217643	0.000435286
0.6	0.000145096	0.00029019
0.8	0.000072547	0.000145096
1	0	0

Table 1. Comparison of error between exact and approximate solutions at t = 0.04

Case(2): (2)-Differentiability

The exact solution of equation(15) under (2)-differentiability is given by $\underline{y^+}(t;\alpha) = (5 + 2\alpha)e^{-t}; \overline{y^+}(t;\alpha) = (9 - 2\alpha)e^{-t}$, $\underline{y^-}(t;\alpha) = (3 + 4\alpha)e^{-t}; \underline{y^-}(t;\alpha) = (11 - 4\alpha)e^{-t}$. The Error estimation for both membership and non-membership functions of equation(15) at t =0.04 with h=0.01 is given in Table 2.

r	Error (Membership)	Error (Non-Membership)
0	0.000348515	0.000697029
0.2	0.00027881	0.000557625
0.4	0.000209109	0.000418218
0.6	0.000139407	0.00027881
0.8	0.000069702	0.000139407
1	0	0

Table 2. **Comparison of error between exact and approximate solutions at t = 0.04**

The solution of equation(15) under (1)-differentiability has an increasing length of its support,which leads us to the conclusion that there is a possibility that,the number of radio nuclide increases and even a non-zero possibility that it is negative. Fortunately, the real situation is different and the number of radio nuclide decreases with time and it cannot be negative.Therefore,(2)-differentiability is suitable for this type of problems. In both cases,the numerical solutions obtained for the equation(15) by the Milne's predictor - corrector method are closer to the exact solutions.However,the errors can be minimised by reducing the step size h.

7. Conclusion

In this work the Milne's predictor - corrector method has been used for finding the numerical solution of IFDEs under generalised differentiabilty.The applicability of the method is illustrated by solving a first order intuitionistic fuzzy differential equation.In future,other predictor-corrector methods will be used to study numerical solution of intuitionistic fuzzy differential equations.

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