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Another New Proof of the Butterfly Theorem

Research Article

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Abstract: In this article we present a new proof of butterfly theorem.

Keywords: Cyclic Quadrilateral, Butterfly theorem, Ptolemy's theorem.

1. Introduction

Butterfly Theorem: *Let XY be a chord of a circle with midpoint P. Suppose AC and BD are two other chords that pass through P. Let L and N be the intersections of AB and CD with XY. Then $PL = PN$ (see Figure 1).*

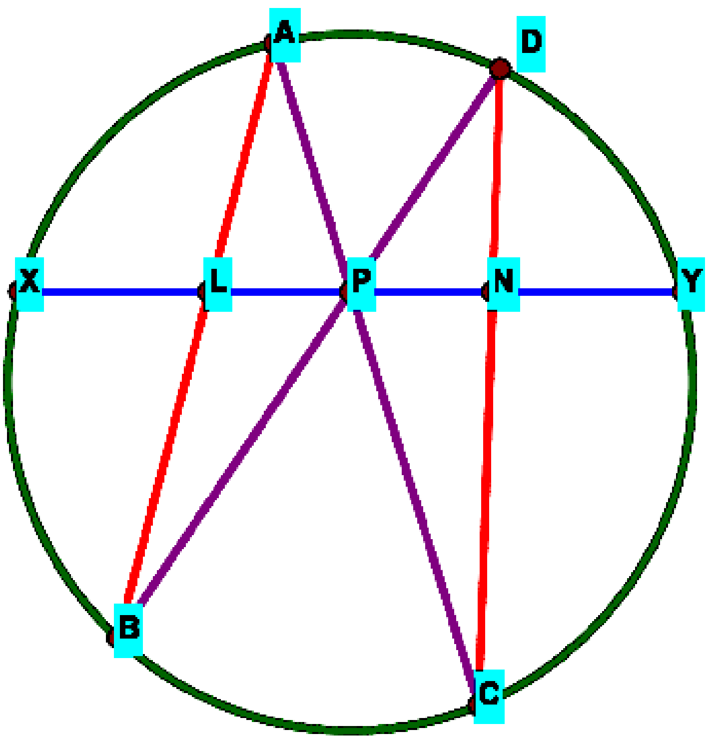

Figure 1.

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Butterfly theorem has gained a lot of interest in the past, in particular, In the last quarter of the 20^{th} century, see the list of references, in particular [5], Recently In the articles [2], [6] and [7] Cesare Donolato, Martin Celli and Tran Quang Hung gave new synthetic proofs of this theorem, Much to my delight, the proof I came up with, used a lemma related to the cyclic quadrilateral. A variety of proofs can even be found on the internet [1].

2. Main Result

Lemma 2.1. *Let $ABCD$ is a cyclic quadrilateral and Q be the point of intersection of diagonals AC, BD. If M be any point in the plane of a quadrilateral then the following are true.*

$$\frac{CQ}{AC}AM^2 + \frac{AQ}{AC}CM^2 = \frac{DQ}{BD}BM^2 + \frac{BQ}{BD}DM^2 \tag{1}$$

$$\frac{AC}{BD} = \frac{BC\,.\,CD\,.\,AM^2 + AB\,.\,AD\,.\,CM^2}{AD\,.\,CD\,.\,BM^2 + AB\,.\,BC\,.\,DM^2} \tag{2}$$

Proof. The proof of the above lemma can be found in [3]. □

Theorem 2.2 (The Butterfly Theorem)**.** *Through the midpoint P of a chord XY of a circle, two other chords AC and BD are drawn. Chords AB and CD intersect XY at points L and N, respectively. Then P is also the midpoint of LN.*

Proof. From the chords AB, CD and XY, we have

$$\frac{AL}{BL} = \frac{AX\,.\,AY}{BX\,.\,BY}, \; \frac{DN}{CN} = \frac{DX\,.\,DY}{CX\,.\,CY} \tag{3}$$

and

$$\frac{PX}{PY} = \frac{AX\,.\,CX}{AY\,.\,CY} = \frac{BX\,.\,DX}{BY\,.\,DY}.$$

Now Since P is the midpoint of the chord XY, we can notice that

$$\frac{AX\,.\,BX}{CY\,.\,DY} = \frac{AY\,.\,BY}{CX\,.\,DX} = \frac{AX\,.\,BX + AY\,.\,BY}{CX\,.\,DX + CY\,.\,DY} \tag{4}$$

By applying Lemma 2.1 (1), for the cyclic quadrilateral AXBY, we get

$$\frac{AL}{AB}BM^2 + \frac{BL}{AB}AM^2 = \frac{XL}{XY}YM^2 + \frac{YL}{XY}XM^2$$

Since it is true for any M, let us fix M as X then

$$\frac{AL}{AB}BX^2 + \frac{BL}{AB}AX^2 = XL.YX \tag{5}$$

In the similar manner by applying Lemma 2.1 (1), for the cyclic quadrilateral DYCX and also by fixing M as Y we get

$$\frac{DN}{CD}CY^2 + \frac{CN}{CD}DY^2 = YL.YX \tag{6}$$

Now from (5) and (6), for proving the butterfly theorem it is enough to prove (5) = (6). That is we need to prove

$$\frac{AL}{AB}BX^2 + \frac{BL}{AB}AX^2 = \frac{DN}{CD}CY^2 + \frac{CN}{CD}DY^2$$

Using (3). It can be rewritten as

$$\frac{AX\,.\,AY\,.\,BX^2+BX.\,BY\,.\,AX^2}{CX\,.\,CY\,.\,DY^2+DX\,.\,DY\,.\,CY^2}=\frac{AX.AY+BX.BY}{CX.CY+DX.DY} \tag{7}$$

Now using Lemma 2.1 (2), for the cyclic quadrilaterals AXBY and XDYC, we have

$$\frac{AB}{XY}=\frac{AX\,.\,AY\,.\,BM^2+BX.\,BY\,.\,AM^2}{AX\,.\,BX\,.\,YM^2+AY\,.\,BY\,.\,XM^2} \tag{8}$$

and

$$\frac{XY}{CD}=\frac{CX\,.\,DX\,.\,YM^2+CY.\,DY\,.\,XM^2}{DX\,.\,DY\,.\,CM^2+CX\,.\,CY\,.\,DM^2} \tag{9}$$

Since (8) is true for any M, let us fix M as X as well as S (circumcenter). We get

$$\frac{AB}{XY}=\frac{AX\,.\,AY\,.\,BX^2+BX.\,BY\,.\,AX^2}{AX\,.\,BX\,.\,YX^2}=\frac{AX\,.\,AY\,+BX.\,BY}{AX\,.\,BX\,+AY\,.\,BY}$$

Further simplified as

$$\frac{AX\,.\,AY\,.\,BX^2+BX.\,BY\,.\,AX^2}{AX\,.\,AY\,+BX.\,BY}=\frac{AX\,.\,BX\,.\,YX^2}{AX\,.\,BX\,+AY\,.\,BY} \tag{10}$$

In the similar manner, since (9) is true for any M, let us fix M as Y as well as S (circumcenter). We get

$$\frac{DX\,.\,DY\,.\,CY^2+CX\,.\,CY\,.\,DY^2}{CX\,.\,CY\,+DX.\,DY}=\frac{CY\,.\,DY\,.\,YX^2}{CX\,.\,DX\,+CY\,.\,DY} \tag{11}$$

Now using (4), we can prove (10)=(11) it implies (7). Which proves $XL=YN$ and it implies $PL=PN$. It concludes the proof of Butterfly Theorem. □

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