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Efficiently Dominating (γ, ed)-Number of Graphs

Research Article

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Abstract: In this paper, we introduce the new concept connected and independent edge detour domination number of a graph and obtain the connected and independent edge detour domination number for some well known graphs.

Keywords: Edge detour domination, efficient domination, efficient(γ,ed)-number of graphs.

1. Introduction

The concept of domination was introduced by Ore and Berge [8]. Let G be a finite, undirected connected graph with neither loops nor multiple edges. A subset D of V(G) is a dominating set of G if every vertex in V-D is adjacent to at least one vertex in D. The minimum cardinality among all dominating sets of G is called the domination number $\gamma(G)$ of G. We consider connected graphs with at least two vertices. For basic definitions and terminologies, we refer Harary [3]. For vertices u and v in a connected graph G, the detour distance D(u,v) is the length of longest u-v path in G. A u-v path of length D(u,v) is called a u - v detour. A subset S of V is called a detour set if every vertex in G lie on a detour joining a pair of vertices of S. The detour number dn(G) of G is the minimum order of a detour set and any detour set of order dn(G) is called a detour basis of G. These concepts were studied by chartrand [4]. A subset S of V is called an edge detour set of G if every edge in G lie on a detour joining a pair of vertices of S. The edge detour number dn_1(G) of G is the minimum order of its edge detour sets and any edge detour set of order dn_1 is an edge detour basis. A graph G is called an edge detour graph if it has an edge detour set. Edge detour graphs were introduced and studied by Santhakumaran and Athisayanathan [10]. Let G be a connected graph with at least two vertices. An edge detour dominating set is a subset S of V(G) which is both a dominating and an edge detour set of G. An edge detour dominating set is said to be minimal edge detour dominating set of G if no proper subset of S is an edge detour dominating set of G. An edge detour dominating set S is said to be minimum edge detour dominating set of G if there exists no edge detour dominating set S' such that S'. The smallest cardinality of an edge detour dominating set of G is called the edge detour domination n umber of G. It is denoted by (γ, eD). Any edge detour dominating set G of minimum cardinality is called γ_{eD}-set of G.

The edge detour domination number of graphs were introduced and studied by A.Mahalakshmi, K.Palani and

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S.Somasundaram [7]. A subset S of V(G) is called an efficient dominating if for every v∈V(G), $|N[v] \cap S| = 1$. A graph G is efficient if G has an efficient dominating set. Efficient dominating graphs were introduced and studied by D.W.Bange, A.E.Barkauskas and P.J.Slater[1]. Let G be a connected graph and S be an edge detour dominating set of G. Then, S is an efficiently dominating (γ, eD)-set of G if for every $v \in V(G)$, $|N[v] \cap\ S| = 1$. The minimum cardinality of S is the efficiently dominating (γ, eD)-set of G and is denoted by eγ_{eD}(G). An efficiently dominating (γ, eD)-set of minimum cardinality is called a eγ_{eD}-set of G. A graph G is said to be an efficiently dominating graph if it has an efficiently dominating (γ, eD)-set. In our paper [7], we need the symbol γ_{eD}-set to represent any edge detour dominating set which we now changed to (γ, eD)-set.

Theorem 1.1 ([6]). *The domination number of some standard graphs are given as follows.*

(1). $\gamma(P_p) = \lceil \frac{p}{3} \rceil$, $p \geq 3$.

(2). $\gamma(C_p) = \lceil \frac{p}{3} \rceil$, $p \geq 3$.

(3). $\gamma(K_p) = \gamma(W_p) = \gamma(K_{1,n}) = 1$.

(4). $\gamma(K_{m,n} = 2$ *if* $m, n \geq 2$.

The following results are from [10].

Theorem 1.2. *Every end vertex of an edge detour graph G belongs to every edge detour set of G. Also, if the set of all end vertices of G is an edge detour set, then S is the unique edge detour basis for G.*

Theorem 1.3. *If G is an edge detour graph of order* $p \geq 3$ *such that* $\{u, v\}$ *is an edge detour basis of G, then u and v are not adjacent.*

Theorem 1.4. *If T is a tree with k end vertices, then* $dn_1(T) = k$.

The following theorems are by A.Mahalakshmi, K.Palani, S.Somasundaram [7].

Theorem 1.5. K_p *is an edge detour dominating graph and for* $p \geq 3$; $\gamma_{eD}(K_p) = 3$.

Theorem 1.6. $\gamma_{eD}(K_{1,n}) = n$.

Theorem 1.7.

$$\gamma_{eD}(P_n) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

Definition 1.8. *A subset S of V is called an independent set of G if no two vertices of S are adjacent.*

Definition 1.9. *A subdivision of an edge* $e = uv$ *of a graph G is the replacement of the edge e by a path* $\{u, v, w\}$*. If every edge of G is subdivided exactly once, then the resulting graph is called the subdivision graph S(G).*

Definition 1.10. *Let* G_1 *and* G_2 *be two graphs with disjoint vertex sets* V_1 *and* V_2 *and edge sets* E_1 *and* E_2 *respectively. Then their union* $G = G_1 \cup G_2$ *is a graph with vertex set* $V = V_1 \cup V_2$ *and edge set* $E = E_1 \cup E_2$*. Clearly,* $G_1 \cup G_2$ *has* $p_1 + p_2$ *vertices and* $q_1 + q_2$ *edges.*

Definition 1.11. *If* G_1 *and* G_2 *are disjoint graphs, then the join of* G_1 *and* G_2 *is denoted by* $G_1 + G_2$ *and is defined as* $V(G_1 + G_2) = V_1 \cup V_2$ *and* $E(G_1 + G_2) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\}$.

In the next section, we introduce the definition of efficiently dominating (γ, eD)-graph and find its values for different graphs.

2. Efficiently Dominating (γ, eD)-graph

Definition 2.1. *Let G be a connected graph. An efficiently dominating (γ, eD)-set of G is an edge detour dominating set of G such that for every $v \in V(G)$, $|N[v] \cap S| = 1$. The minimum cardinality of among all efficiently dominating (γ, eD) is the efficiently dominating (γ, eD)-number of G and is denoted by $e\gamma_{eD}(G)$. An efficiently dominating (γ, eD)-set of minimum cardinality $e\gamma_{eD}(G)$ is called a $e\gamma_{eD}$-set of G. A graph G is said to be an efficiently dominating (γ, eD)-graph if it has an efficiently dominating (γ, eD)-set.*

Example 2.2. *Consider, the graph G in Figure 2.1 (a). Here, $S = \{v_1, v_4\}$ is the minimum edge detour dominating set of G and $|N[v_i]\cap| = 1$ for all $v_i \in V(G)$, $1 \leq i \leq 5$. Therefore, S is the minimum efficiently dominating (γ, eD)-set of G and $e\gamma_{eD}(G) = 2$. Therefore, the graph G is an efficiently dominating (γ, eD)-graph.*

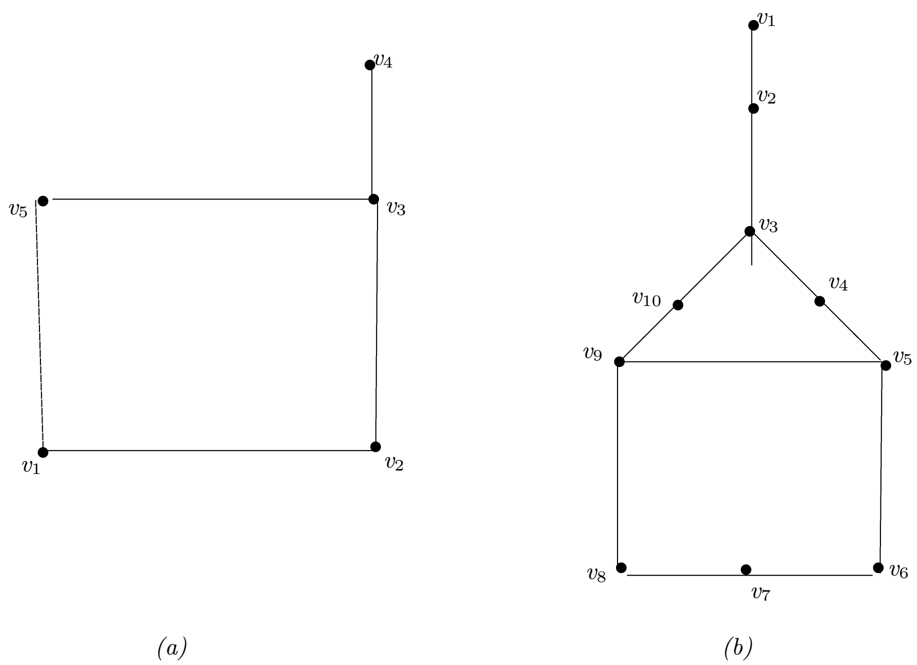

Figure 2.1

For the graph G in Figure 2.1 (b), $S = \{v_1, v_4, v_7, v_{10}\}$ is the minimum edge detour dominating set of G and $|N[v_i] \cap S| = 1$ for all $v_i \in V(G)$, $1 \leq i \leq 10$. Therefore, S is the minimum efficiently dominating (γ, eD)-set of G and $e\gamma_{eD}(G) = 4$. Therefore, the graph G is an efficiently dominating (γ, eD)-graph.

Observation 2.3.

(1) Any efficiently dominating (γ,eD)-set contains only isolated vertices.

(2) Every edge detour dominating set need not be an efficiently dominating (γ, eD)-set.

Theorem 2.4. *The path P_n, $n \equiv 1$ (mod 3) is an efficiently dominating (γ, eD)-graph and $e\gamma_{eD}(P_n) = \gamma_{eD}(P_n) = \lceil \frac{n-4}{3} \rceil + 2$.*

Proof. Let $P_n = (v_1, v_2, v_3, ..., v_{3k+1})$, $k > 0$. $S = \{v_1, v_4, v_7, ..., v_{3k+1}\}$ is a unique γ_{eD}-set of P_n. Further, $|N[v] \cap S| = 1$

for all $v \in P_n$. Hence, S is the unique efficiently dominating (γ, eD)-set of P_n. Therefore, $e\gamma_{eD}(P_n) = \gamma_{eD}(P_n)$. By Theorem 1.8, $\gamma_{eD}(P_n) = \lceil \frac{n-4}{3} \rceil + 2$. □

Theorem 2.5. *The graph $G = C_{3n}$, $n > 1$ is an efficiently dominating (γ, eD)-graph and $e\gamma_{eD}(C_{3n}) = n$.*

Proof. Let $V(C_{3n}) = \{v_1, v_2, ..., v_{3n}\}$. The only γ_{eD}-sets of C_{3n} are $S_1 = \{v_1, v_4, ..., v_{3(n-1)+1}\}$, $S_2 = \{v_2, v_5, ..., v_{3(n-1)+2}\}$ and $S_3 = \{v_3, v_6, ..., v_{3n}\}$. Further, $|N[v] \cap S_i| = 1$ for all $v \in C_{3n}$ and $i = 1, 2, 3$. Hence, C_{3n} is an efficiently dominating (γ, eD) graph and S_i, $i = 1, 2, 3$ are efficiently dominating (γ, eD)-sets of C_{3n}. Therefore, $e\gamma_{eD}(C_{3n}) = \gamma_{eD}(C_{3n})$. Therefore, by Theorem 1.9, $e\gamma_{eD}(C_{3n}) = \gamma_{eD}(C_{3n}) = \lceil \frac{3n}{3} \rceil = n$. □

Theorem 2.6. *Every efficiently dominating (γ, eD)-set of a graph G is independent.(i.e), No two vertices of an efficiently dominating (γ, eD)-set are adjacent.*

Proof. Let G be a graph and S be a $e\gamma_{eD}$-set of G. Suppose u, v∈S such that u and v are adjacent. Then, $\{u, v\} \subseteq N[u] \cap S$. Therefore, $|N[u] \cap S| \neq 1$. So, S is not an efficiently dominating $e\gamma_{eD}$-set of G, which is a contradiction. Therefore, u and v are not adjacent. Since u and v are arbitrary, no two vertices are adjacent. Hence, every efficiently dominating (γ, eD)-set is independent. □

Proposition 2.7. *Complete graph K_n, $n > 2$ are not efficiently dominating (γ, eD)-graphs.*

Proof. By Observation 2.3 (ii), any efficiently dominating (γ, eD)-set is also a (γ, eD)-set. Further, any (γ, eD)-set of K_n contains at least two vertices. As being vertices of K_n, they are adjacent. Therefore, by Observation 2.3(ii), K_p has no efficiently dominating (γ, eD)-graph. □

Corollary 2.8. *Complete bipartite graphs are not efficiently dominating (γ, eD)-graphs.*

Proof. Let V_1, V_2 be the bipartition of $V(K_{m,n})$. Any, γ_{eD}-set of $K_{m,n}$ contains at least one vertex from both V_1, and V_2. Obviously, they are adjacent. Therefore, by Theorem 2.6, $K_{m,n}$ are not efficiently dominating (γ, eD)-graphs. □

Observation 2.9.

(1) A graph G has no efficiently dominating (γ, eD)-set if it has a vertex with more than one pendant edge.

(2) For any graph $e\gamma_{eD}(G) \leq \gamma_{eD}(G)$.

(3) If G_1 and G_2 are efficiently dominating (γ, eD)-graphs then, $G_1 \cup G_2$ is also an efficiently dominating (γ, eD)-graph and $e\gamma_{eD}(G_1 \cup G_2) = \gamma_{eD}(G_1) + \gamma_{eD}(G_2)$.

Proposition 2.10. *An edge detour dominating set S is an efficiently dominating (γ, eD)-set if and only if $d(u, v) \geq 3$.*

Proof. Let G be a graph. S is an efficiently dominating (γ, eD)-set of G. Suppose, for all $u, v \in S$ such that $d(u, v) < 3$.
Case 1: $d(u, v) = 1$. $\{u, v\} \subseteq N[u] \cap S$. Therefore, $N[u] \cap S \geq 1$.
Case 2: $d(u, v) = 2$. Let uwv be a smallest u-v path in G. Then, $N[w] \cap S \geq 2$. In both the cases, $N[x] \cap S \neq 1$ for at least one $x \in S$. Therefore, S is not an efficiently dominating (γ, eD)-set of G, which is a contradiction. Therefore, $d(u, v) \geq 3$.
Conversely, $d(u, v) \geq 3$. Let S be an edge detour dominating set of G. To prove that S is efficiently dominating (γ, eD)-set of G. Suppose not. Then, $S \cap N[x] \neq 1$ for at least one $x \in S$. Therefore, there exist one vertex say u in S such that x is adjacent to u. Then, obviously $d(u, x) = 1 < 3$. Which is a contradiction. Therefore, S is an efficiently dominating (γ, eD)-set of G. □

Lemma 2.11. *Wheel graph $W_{1,n}$ is an edge detour dominating graph,* $\gamma_{eD}(W_{1,n}) = \begin{cases} 2 if n = 6 \\ 3 otherwise \end{cases}$

Proof. Let G be a wheel graph with central vertex v.

Case 1: When $n = 6$, $S_i = \{v_i, v_{i+1}\}$, $i = 1, 2, 3$. Forms an edge detour dominating set of G. Therefore, $\gamma_{eD}(G) = 2$.

Case 2: Let $n > 6$. Now v along with any two of the rim vertices forms a edge detour dominating set. Therefore, $\gamma_{eD}(G) \leq 3$.

Claim: $\gamma_{eD}(G) \neq 2$.

Let S be any two element subset of $V(G)$. Suppose $S = \{v, v_i\}$, then the edge vv_i does not lie on any detour joining v and v_i. If $S = \{v_i v_j\}$ two cases arises.

Sub Case 2a: If v_i and v_j are adjacent. Here, the edge $v_i v_j$ does not lie on any detour joining v_i and v_j.

sub Case 2b: If v_i and v_j are non-adjacent. Since $n > 6$, some vertices of G are either not dominated by v_i and v_j or they do not lie on any detour joining v_i and v_j. Therefore, S cannot be a detour joining set of G. Since, S is arbitrary, $\gamma_{eD}(G) \neq 2$. Hence, $\gamma_{eD}(G) = 3$. □

Corollary 2.12. *Wheel graph is a non - efficiently dominating (γ, eD)-graph.*

Proof. Any edge detour dominating set of W_p contains at least two of the rim vertices. Obviously, $d(u, v) = 2 < 3$, for any two rim vertices u and v. Therefore, by Proposition 2.10, any edge detour dominating set is not an efficiently dominating (γ, eD)-set. Therefore, wheel is not an efficiently dominating (γ, eD)-graph. □

3. Efficiently Dominating (γ, eD)-Number of Subdivision Graphs

Theorem 3.1. *The graph $S(P_n)$, subdivision of path P_n, $n \equiv 1$ (mod 3) is an efficiently dominating (γ, eD)-graph.*

Proof. The graph $S(P_n)$, $n \equiv 1$ (mod 3) is again a path P_m with $m \equiv 1$ (mod 3). Therefore by Theorem 2.4, $S(P_n)$ where $n \equiv 1$ (mod 3) is an efficiently dominating (γ, eD)-graph. □

Theorem 3.2. *The graph $S(C_{3n})$, subdivision of the cycle C_{3n} where $n \geq 1$ is an efficiently dominating (γ, eD)-graph and $e\gamma_{eD}(S(C_{3n})) = 2n$.*

Proof. The graph $S(C_{3n})$ where $n \geq 1$, is again a cycle $C_{2(3n)}$. Therefore, by Theorem 2.5, $S(C_{3n})$ is an efficiently dominating (γ, eD)-graph and also $e\gamma_{eD}(S(C_{3n})) = e\gamma_{eD}(C_{3(2n)}) = 2n$. □

Theorem 3.3. *The subdivision graph $S(K_n)$ is an efficiently dominating (γ, eD)-graph if and only if $n = 3$.*

Proof. Let $n = 3$. $S(K_n) = C_6$. Hence, by Theorem 2.5, $\gamma(S(K_3) = 2$. Let $n \geq 4$. Any (γ, eD)-set should contain at least three points. Therefore, it must contain at least two points such that $d(u, v) = 2$. Hence, the middle vertex in any shortest $u - v$ path is dominated by both u and v. Therefore, any (γ, eD)-set cannot be an efficiently dominating (γ, eD)-set. Hence, $S(K_n)$ has no efficiently dominating (γ, eD)-set. □

Corollary 3.4. *The subdivision of a star graph $S(K_{1,n})$ is not an efficiently dominating (γ, eD)-graph.*

Proof. Any efficiently dominating (γ, eD)-set S of $S(K_{1,n})$ contain at least $n + 1$ vertices. Therefore, it must be at least two vertices u and v in S such that $d(u, v) \leq 2$.

Case 1: $d(u, v) = 1$, then $|N(u) \cap S| = |N(v) \cap S| = 2$. Then, S is not an efficiently dominating (γ, eD)-set.

Case 2: $d(u, v) = 2$, Hence, the middle vertex in any shortest $u - v$ path is dominated by both u and v. Therefore, any (γ, eD)-set cannot be an efficiently dominating (γ, eD)-set. Hence, $S(K_{1,n})$ has no efficiently dominating (γ, eD)-set. □

Remark 3.5. *Let G be a connected graph with $p \geq 2$ vertices. If every (γ, eD)-set of G should contains at least two vertices of shortest length ≤ 2, then G contains no efficiently (γ, eD)-set.*

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