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Converging Solution of Quadratic Fractional Integral Equation With Q Function

Research Article

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Abstract: In this paper, we will find the solution to the quadratic fractional integral equation involving the Q function which is the generalization of Mittag-Leffler function and this solution we will obtain with the help of the approximate solutions of this integral equation, we will form the sequence of solutions converging to the solution of the fractional integral equation involving the Q function. For a nonlinear quadratic fractional integral equation with the new Q function which is the generalization of Mittag-Leffler function on a closed and bounded interval of the real line the existence and convergence of successive approximations with the help of some hybrid conditions we will study.

MSC: 45G10, 47H09, 47H10.

Keywords: Quadratic fractional integral equation, Fractional derivatives and Integrals, approximate solution.

1. Introduction

The quadratic integral equations have been a topic of interest since long time because of their occurrence in the problems of some natural and physical processes of the universe. See Argyros [10], Deimling [13], Chandrasekher [11] and the references therein. The study gained momentum after the formulation of the hybrid fixed point principles in Banach algebras due to Dhage [14–17]. The existence results for such quadratic operators equations are generally proved under the mixed Lipschitz and compactness type conditions together with a certain growth condition on the nonlinearities involved in the quadratic operator or functional equations. The hybrid fixed point theorems in Banach algebras find numerous applications in the theory of nonlinear quadratic differential and integral equations. See Dhage [15–17], Dhage and Dhage [22, 23] and the references therein. The Lipschitz and compactness hypotheses are considered to be very strong conditions in the theory of nonlinear differential and integral equations but nevertheless do not yield any algorithm to determine the numerical solutions. Therefore, it is of interest to relax or weaken these conditions in the existence and approximation theory of quadratic integral equations. This is the main motivation of the present paper. In this paper we prove the existence as well as approximations of the solutions of a certain generalized quadratic integral equation via an algorithm based on successive approximations under weak partial Lipschitz and compactness type conditions. Given a closed and bounded interval $J = [0, T]$ of the real line R for some $T > 0$, we consider the quadratic fractional integral equation (in short QFIE)

$$x(t) = f(t, x(t))\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{(q-1)} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)g(s, x(s))ds\right) \tag{1}$$

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where $f, g : J \times \mathbb{R} \to \mathbb{R}$ and $q : J \to \mathbb{R}$ are continuous functions, $1 \leq q < 2$ and Γ is the Euler gamma function, and $Q^{\gamma,q,r}_{\alpha,\beta,\delta}(x)$ is generalized mittag leffler function.

By a *solution* of the QFIE (1) we mean a function $x \in C(J, \mathbb{R})$ that satisfies the equation (1) on J, where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J.

2. Auxiliary Results

Unless otherwise mentioned, throughout this paper that follows, let E denote a partially ordered real normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$. It is known that E is **regular** if $\{x_n\}_{n\in\mathbb{N}}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. Clearly, the partially ordered Banach space $C(J, \mathbb{R})$ is regular and the conditions guaranteeing the regularity of any partially ordered normed linear space E may be found in Heikkilä and Lakshmikantham [28] and the references therein.

In this section,we present some basic definitions and preliminaries which are useful in further discussion.

Definition 2.1 (Mittag-Leffler Function [3])**.** *The Mittag - Leffler function of one parameter is denoted by $E_\alpha(z)$ and defined as,*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k \tag{2}$$

where $z, \alpha \in C,\ Re(\alpha) > 0.$

If we put $\alpha = 1$, then the above equation becomes

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. \tag{3}$$

Definition 2.2 (Mittag-Leffler Function for two parameters)**.** *The generalization of $E_\alpha(z)$ was studied by Wiman (1905) [9] , Agarwal [1] and Humbert and Agarwal [5] defined the function as,*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} z^k \tag{4}$$

where $z, \alpha, \beta \in C,\ Re(\alpha) > 0, Re(\beta) > 0,$

In 1971, The more generalized function is introduced by Prabhakar [**?**] as

$$E^{\gamma}_{\alpha,\beta}((z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{\Gamma(\alpha k + \beta)}. \tag{5}$$

where $z, \alpha, \beta, \gamma \in C,\ Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$, where $\gamma \neq 0, \gamma)_k = \gamma(\gamma+1)(\gamma+2)...(\gamma+k-1)$ is the Pochhammer symbol [7], and

$$(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$$

In 2007, Shulka and Prajapati [7] introduced the function which is defined as,

$$E^{\gamma,q}_{\alpha,\beta}((z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{k!\Gamma(\alpha k + \beta)}. \tag{6}$$

where $z, \alpha, \beta, \gamma \in C,\ min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0$, and $q \in (0,1) \cup N$. In 2012, further generalization of Mittag - Leffler function was defined by Salim [8] and Chauhan [2] as,

$$E^{\gamma,\delta,q}_{\alpha,\beta}((z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} z^k}{(\delta)_(qk)\Gamma(\alpha k + \beta)}. \tag{7}$$

where $z, \alpha, \beta, \gamma \in C,\ min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0$, and $q \in (0,1) \cup N$

$$(\gamma)_{qk} = \frac{\Gamma(\gamma+qk)}{\Gamma(\gamma)} \text{ and } (\delta)_{qk} = \frac{\Gamma(\delta+qk)}{\Gamma(\delta)}$$

denote the generalized Pochhammer symbol [7],

Definition 2.3 ([6]). *The generalization of Mittag - Leffler function denoted by $Q^{\gamma,q,r}_{\alpha,\beta,\delta}(x)$ and defined by*

$$\begin{aligned} Q^{\gamma,q,r}_{\alpha,\beta,\delta}(x) &= Q^{\gamma,q,r}_{\alpha,\beta,\delta}(a_1,a_2,...,a_r,b_1,b_2,...,b_r,x) \\ &= \sum_{s=0}^{\infty} \frac{\Pi_{n=1}^{r}\beta(b_n,s)(\gamma)_{qs}}{\Pi_{n=1}^{r}\beta(a_n,s)(\delta)_{qs}\Gamma(\alpha s+\beta)} x^s, \end{aligned} \tag{8}$$

where $x,\alpha,\beta,\gamma,\delta,a_i,b_i \in C$, $\min\{Re(\alpha),Re(\beta),Re(\gamma)\} > 0$, and $q \in (0,1) \cup N$,

$$(\gamma)_{qk} = \frac{\Gamma(\gamma+qk)}{\Gamma(\gamma)} \text{ and } (\delta)_{qk} = \frac{\Gamma(\delta+qk)}{\Gamma(\delta)}$$

Definition 2.4. *A mapping $\mathcal{T} : E \to E$ is called* **isotone** *or* **nondecreasing** *if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$.*

Definition 2.5 ([19]). *A mapping $\mathcal{T} : E \to E$ is called* **partially continuous** *at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.*

Definition 2.6. *A mapping $\mathcal{T} : E \to E$ is called* **partially bounded** *if $\mathcal{T}(C)$ is bounded for every chain C in E. $\mathcal{T}$ is called* **uniformly partially bounded** *if all chains $\mathcal{T}(C)$ in E are bounded by a unique constant. $\mathcal{T}$ is called* **bounded** *if $\mathcal{T}(E)$ is a bounded subset of E.*

Definition 2.7. *A mapping $\mathcal{T} : E \to E$ is called* **partially compact** *if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called* **uniformly partially compact** *if $\mathcal{T}(C)$ is a uniformly partially bounded and partially compact on E. $\mathcal{T}$ is called* **partially totally bounded** *if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called* **partially completely continuous** *on E.*

Definition 2.8 ([19]). *The order relation $\preceq$ and the metric d on a non-empty set E are said to be* ***compatible*** *if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the original sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.*

Definition 2.9 ([16]). *A upper semi-continuous and monotone nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function provided $\psi(r) = 0$ iff $r = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \to E$ is called* ***partially nonlinear $\mathcal{D}$-Lipschitz*** *if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|) \tag{9}$$

for all comparable elements $x, y \in E$. If $\psi(r) = k\,r$, $k > 0$, then $\mathcal{T}$is called a partially Lipschitz with a Lipschitz constant k.

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\}$$

and

$$\mathcal{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } \; u, v \in E^+\}. \tag{10}$$

The elements of $\mathcal{K}$ are called the positive vectors of the normed linear algebra E. The following lemma follows immediately from the definition of the set $\mathcal{K}$ and which is often times used in the applications of hybrid fixed point theory in Banach algebras.

Lemma 2.10 ([17]). *If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1 u_2 \preceq v_1 v_2$.*

Definition 2.11. *An operator $\mathcal{T} : E \to E$ is said to be positive if the range $R(\mathcal{T})$ of $\mathcal{T}$ is such that $R(\mathcal{T}) \subseteq \mathcal{K}$.*

Theorem 2.12 ([20]). *Let $\big(E, \preceq, \|\cdot\|\big)$ be a regular partially ordered complete normed linear algebra such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain of E. Let $\mathcal{A}, \mathcal{B} : E \to \mathcal{K}$ be two nondecreasing operators such that*

(a) $\mathcal{A}$ is partially bounded and partially nonlinear $\mathcal{D}$-Lipschitz with $\mathcal{D}$-functions $\psi_{\mathcal{A}}$,

(b) $\mathcal{B}$ is partially continuous and uniformly partially compact, and

(c) $M\psi_{\mathcal{A}}(r) < r$, $r > 0$, where $M = \sup\{\|\mathcal{B}(C)\| : C$ is a chain in $E\}$, and

(d) there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{A}x_0\, \mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0\, \mathcal{B}x_0$.

Then the operator equation

$$\mathcal{A}x\, \mathcal{B}x = x \tag{11}$$

has a solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\, \mathcal{B}x_n$, $n = 0, 1, \ldots,$ converges monotonically to x^*.

3. Main Result

The QFIE (1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{12}$$

and

$$x \leq y \iff x(t) \leq y(t) \tag{13}$$

for all $t \in J$ respectively. Clearly, $C(J, \mathbb{R})$ is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation $\leq$. The following lemma in this connection follows by an application of Arzelá-Ascoli theorem.

Lemma 3.1. *Let $\big(C(J, \mathbb{R}), \leq, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (12) and (13) respectively. Then $\|\cdot\|$ and $\leq$ are compatible in every partially compact subset of $C(J, \mathbb{R})$.*

The lemma mentioned in Dhage [20], but the proof appears in Dhage [21].

Definition 3.2. *A function $v \in C(J,\mathbb{R})$ is said to be a lower solution of the QFIE (1) if it satisfies*

$$v(t) \leq f(t,v(t))\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{(q-1)} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q) g(s,v(s))ds\right)$$

for all $t \in J$. Similarly, a function $u \in C(J,\mathbb{R})$ is said to be an upper solution of the QFIE (1) if it satisfies the above inequalities with reverse sign.

We consider the following set of assumptions in what follows:

(A$_1$) The functions $f,g : J\times\mathbb{R} \to \mathbb{R}_+, q : J \to \mathbb{R}_+$ where q is continuous function.

(A$_2$) There exists constants $M_f, M_g > 0$ such that $0 \leq f(t,x) \leq M_f$ and $0 \leq g(t,x) \leq M_g$ for all $t \in J$ and $x \in \mathbb{R}$.

(A$_3$) There exists a $\mathcal{D}$-function ψ_f such that $0 \leq f(t,x) - f(t,y) \leq \psi_f(x-y)$ for all $t \in J$ and $x, y \in \mathbb{R}$,$x \leq y$.

(A$_4$) $g(t,x)$ is nondecreasing in x for all $t \in J$.

(A$_5$) The QFIE (1) has a lower solution $v \in C(J,\mathbb{R})$.

Theorem 3.3. *Assume that hypotheses (A$_1$)-(A$_5$) holds then the QFIE (1) has a solution x^* defined on J and the sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of successive approximations defined by*

$$x_{n+1}(t) = \left[f(t,x_n(t))\right]\left(\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\, g(s,x_n(s))\, ds\right) \tag{14}$$

for all $t \in J$, where $x_0 = v$, converges monotonically to x^*.

Proof. Set $E = C(J,\mathbb{R})$. Then, from Lemma 3.1 it follows that every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\leq$ in E. Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = f(t,x(t)),\ t \in J, \tag{15}$$

$$\mathcal{B}x(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\, g(s,x(s))\, ds,\ t \in J, \tag{16}$$

From the continuity of the integral and the hypotheses (A$_1$)-(A$_5$), it follows that $\mathcal{A}$ and $\mathcal{B}$ define the maps $\mathcal{A},\mathcal{B} : E \to \mathcal{K}$. Now by definitions of the operators $\mathcal{A}$ and $\mathcal{B}$, the QFIE (1) is equivalent to the operator equation

$$\mathcal{A}x(t)\mathcal{B}x(t) = x(t),\ t \in J. \tag{17}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.12. This is achieved in the series of following steps.

Step I: *$\mathcal{A}$ and $\mathcal{B}$ are nondecreasing on E.*

Let $x, y \in E$ be such that $x \leq y$. Then by hypothesis (A$_3$)and (A$_4$), we obtain

$$\mathcal{A}x(t) = f(t,x(t)) \leq f(t,y(t)) = \mathcal{A}y(t),$$

and

$$\begin{aligned}
\mathcal{B}x(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\, g(s,x(s))\, ds,\ t \in J,\\
&\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\, g(s,y(s))\, ds,\ t \in J,\\
&= \mathcal{B}y(t)
\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing operators on E into E. Thus, $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing positive operators on E into itself.

Step II: *$\mathcal{A}$ is partially bounded and partially $\mathcal{D}$-Lipschitz on E.*

Let $x \in E$ be arbitrary. Then by (A$_2$),

$$|\mathcal{A}x(t)| \leq \big|f(t, x(t)))\big| \leq M_f,$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{A}x\| \leq M_f$ and so, $\mathcal{A}$ is bounded. This further implies that $\mathcal{A}$ is partially bounded on E. Now, let $x, y \in E$ be such that $x \leq y$. Then, by hypothesis (A$_3$),

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &= \big|f(t, x(t)) - f(t, y(t))\big| \\ &\leq \psi_f |x(t) - y(t)| \\ &\leq \psi_f(\|x - y\|), \end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \leq \psi_f(\|x - y\|)$$

for all $x, y \in E$ with $x \leq y$. Hence $\mathcal{A}$ is partially nonlinear $\mathcal{D}$-Lipschitz operators on E which further implies that it is also a partially continuous on E into itself.

Step III: *$\mathcal{B}$ is a partially continuous operator on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C of E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned} \lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\, g(s, x_n(s))\, ds, \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\, \Big[\lim_{n\to\infty} g(s, x_n(s))\Big]\, ds \\ &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^q)\, g(s, x(s))\, ds \\ &= \mathcal{B}x(t), \end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges monotonically to $\mathcal{B}x$ pointwise on J. Next, we will show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then

$$\begin{aligned} \Big|Bx_n(t_2) - Bx_n(t_1)\Big| \leq & \Bigg|\frac{1}{\Gamma(q)} \int_0^{t_2} (t_2-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2-s)^q)\, g(s, x_n(s))\, ds \\ & - \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q)\, g(s, x_n(s))\, ds\Bigg| \\ \leq & \frac{1}{\Gamma(q)} \Bigg|\int_0^{t_2} (t_2-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2-s)^q) g(s, x_n(s)) ds \\ & - \int_0^{t_2} (t_2-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) g(s, x_n(s)) ds\Bigg| \\ & + \Bigg|\int_0^{t_2} (t_2-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) g(s, x_n(s)) ds \\ & - \int_0^{t_1} (t_2-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) g(s, x_n(s)) ds\Bigg| \\ & + \Bigg|\int_0^{t_1} (t_2-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) g(s, x_n(s)) ds \\ & - \int_0^{t_1} (t_1-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) g(s, x_n(s)) ds\Bigg| \end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^{t_2} (t_2-s)^{q-1} \left| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2-s)^q) - Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) \right| |g(s,x_n(s))| \, ds \\
&+ \left| \int_{t_1}^{t_2} (t_2-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) g(s,x_n(s)) ds \right| \\
&+ \int_0^{t_1} \left| (t_2-s)^{q-1} - (t_1-s)^{q-1} \right| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) \, |g(s,x_n(s))| \, ds \\
&\leq \int_0^{T} (t_2-s)^{q-1} \left| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2-s)^q) - Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) \right| M_g ds \\
&+ \int_0^{T} \left| (t_2-s)^{q-1} - (t_1-s)^{\alpha-1} \right| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) M_g ds \\
&+ \int_0^{T} \left| (t_2-s)^{q-1} - (t_1-s)^{q-1} \right| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) M_g ds \\
&\leq M_g \left(\int_0^T \left| (t_2-s)^{q-1} \right|^2 ds \right)^{1/2} \left(\int_0^T \left| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2-s)^q) - Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) \right|^2 ds \right)^{1/2} \\
&+ 2 \left(\int_0^T \left| (t_2-s)^{q-1} - (t_1-s)^{q-1} \right|^2 ds \right)^{1/2} \left(\int_0^T \left| Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) \right|^2 ds \right)^{1/2} M_g \qquad (18)
\end{aligned}
$$

Since the functions $Q^{\gamma,q,r}_{\alpha,\beta,\delta}$, q are continuous on compact interval J and interval is continuous on compact set $J \times J$, they are uniformly continuous there. Therefore, from the above inequality (18) it follows that

$$|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| \to 0 \quad \text{as} \quad n \to \infty$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and hence $\mathcal{B}$ is partially continuous on E.

Step IV: *$\mathcal{B}$ is uniformly partially compact operator on E.*

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set in E. First we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$ be such that $y = \mathcal{B}x$. Now, by hypothesis (A_1),

$$
\begin{aligned}
|y(t)| &\leq \left| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2-s)^q) \; g(s,x(s)) \, ds \right| \\
&\leq r
\end{aligned}
$$

for all $t \in J$. Taking the supremum over t, we obtain $\|y\| \leq \|\mathcal{B}x\| \leq r$ for all $y \in \mathcal{B}(C)$. Hence, $\mathcal{B}(C)$ is a uniformly bounded subset of E. Moreover, $\|\mathcal{B}(C)\| \leq r$ for all chains C in E. Hence, $\mathcal{B}$ is a uniformly partially bounded operator on E. Next, we will show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for any $y \in \mathcal{B}(C)$, one has

$$
\begin{aligned}
\left| Bx(t_2) - Bx(t_1) \right| &\leq \left| \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2-s)^q) \; g(s,x(s)) \, ds \right. \\
&\qquad \left. - \frac{1}{\Gamma(q)} \int_0^{t_1} (t_1-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) \; g(s,x(s)) \, ds \right| \\
&\leq \frac{1}{\Gamma(q)} \left| \int_0^{t_2} (t_2-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2-s)^q) g(s,x(s)) ds \right. \\
&\qquad \left. - \int_0^{t_2} (t_2-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) g(s,x(s)) ds \right| \\
&+ \left| \int_0^{t_2} (t_2-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) g(s,x(s)) ds \right. \\
&\qquad \left. - \int_0^{t_1} (t_2-s)^{q-1} Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q) g(s,x(s)) ds \right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\int_0^{t_1}(t_2-s)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q)g(s,x(s))ds\right.\\
&\qquad\left.-\int_0^{t_1}(t_1-s)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q)g(s,x(s))ds\right|\\
&\le\int_0^{t_2}(t_2-s)^{q-1}\left|Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2-s)^q)-Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q)\right||g(s,x(s))|\,ds\\
&+\left|\int_{t_1}^{t_2}(t_2-s)^{q-1}Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q)g(s,x(s))ds\right|\\
&+\int_0^{t_1}\left|(t_2-s)^{q-1}-(t_1-s)^{q-1}\right|Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q)\,|g(s,x(s))|\,ds\\
&\le\int_0^{T}(t_2-s)^{q-1}\left|Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2-s)^q)-Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q)\right|M_g ds\\
&+\int_0^{T}\left|(t_2-s)^{q-1}-(t_1-s)^{\alpha-1}\right|Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q)M_g ds\\
&+\int_0^{T}\left|(t_2-s)^{q-1}-(t_1-s)^{q-1}\right|Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q)M_g ds\\
&\le M_g\left(\int_0^T\left|(t_2-s)^{q-1}\right|^2ds\right)^{1/2}\left(\int_0^T\left|Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2-s)^q)-Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q)\right|^2ds\right)^{1/2}\\
&+2\left(\int_0^T\left|(t_2-s)^{q-1}-(t_1-s)^{q-1}\right|^2ds\right)^{1/2}\left(\int_0^T\left|Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_1-s)^q)\right|^2ds\right)^{1/2}M_g\\
&\longrightarrow 0\quad\text{as}\quad t_1\to t_2,
\end{aligned}
$$

uniformly for all $y\in\mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is an equicontinuous subset of E. Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set of functions in E, so it is compact. Consequently, $\mathcal{B}$ is a uniformly partially compact operator on E into itself.

Step V: *v satisfies the operator inequality $v\le\mathcal{A}v\,\mathcal{B}v$.*

By hypothesis (A_5), the QFIE (1) has a lower solution v defined on J. Then, we have

$$v(t)\le f(t,v(t))\left(\frac{1}{\Gamma(q)}\int_0^t(t-s)^{(q-1)}Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t_2-s)^q)g(s,v(s))ds\right)\tag{19}$$

for all $t\in J$. From the definitions of the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ it follows that $v(t)\le\mathcal{A}v(t)\,\mathcal{B}v(t)$ for all $t\in J$. Hence $v\le\mathcal{A}v\,\mathcal{B}v$.

Step VI: *The $\mathcal{D}$-functions $\psi_{\mathcal{A}}$ satisfy the growth condition $M\psi_{\mathcal{A}}(r)<r$, for $r>0$.*

Finally, the $\mathcal{D}$-function $\psi_{\mathcal{A}}$ of the operator $\mathcal{A}$ satisfy the inequality given in hypothesis (d) of Theorem 2.12, viz.,

$$M\psi_{\mathcal{A}}(r)<r$$

for all $r>0$. Thus $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.12 and we conclude that the operator equation $\mathcal{A}x\,\mathcal{B}x=x$ has a solution. Consequently the QFIE (1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n\in\mathbb{N}}$ of successive approximations defined by (14) converges monotonically to x^*. This completes the proof. □

Example 3.4. *Given a closed and bounded interval $J=[0,1]$, consider the QFIE,*

$$x(t)\;=\;\frac{1}{2}\left[2+\tan^{-1}x(t)\right]\left(\frac{1}{\Gamma(5/4)}\int_0^t(t-s)^{1/4}Q^{\gamma,q,r}_{\alpha,\beta,\delta}((t-s)^{5/4})\cdot\frac{[1+\tanh x(s)]}{4}\,ds\right)$$

for $t\in J$.

4. conclusion

Finally, An algorithm for the solutions is developed and it is shown that the sequence of successive approximations converges monotonically to the positive solution of related quadratic fractional integral equation under some suitable mixed hybrid conditions. Some of the results along this line will be further studied.

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