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Even Square Semigroups

Research Article

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Abstract: In this article we introduce the notion of even square semigroups and provide some examples. Even square semigroups have non-explicit appearances in the mathematical literatures and the idea to introduce them explicitly has emerged through authors recent work on even square rings.

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1. Introduction

The idea of even square semigroups has originated through [1]. In [1] we have introduced the notion of even square rings and given some results on commutative as well as noncommutative even square rings. There are several examples of even square semigroups of finite as well as infinite order and even square semigroups have implicit presence in the mathematical literatures [2–7]. We notice that even square semigroups have additional structures. In this paper we consider even and even square semigroups and provide some results on even square semigroups containing nil elements. Nil elements are special type of nilpotent elements and have already been introduced and studied in [1]. It is seen that every even semigroup is an even square semigroup however each even square semigroup is not necessarily an even semigroup. We see that each non-zero element of a trivial even square semigroup is obviously a zero divisor however there are non-trivial even square semigroups whose every non-zero element is a zero divisor. It is noticed that if S is a trivial even square semigroup of order n then it has total 2^{n-1} even square subsemigroups and at least one even square subsemigroup of order m for each $0 < m \leq n$. In this case each even square subsemigroup is an ideal of S. It is notable that each element of an even square semigroup is even square element however each element of a subsemigroup of an even square semigroup is not necessarily an even square element. Therefore each subsemigroup of an even square semigroup does not necessarily be an even square subsemigroup. Thus even square property is not a hereditary property of semigroups.

2. Even Square Semigroups

Definition 2.1. *A multiplicative semigroup S together with a unary operation $f : S \to S$ defined by $f(a) = 2a$, $\forall\ a \in S$ is called a unary semigroup.*

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Definition 2.2. *Let S be a unary semigroup. An element $a \in S$ is called an even element of S if $a \in 2S$.*

Definition 2.3. *Let S be a unary semigroup. An element $a \in S$ is called an even square element of S if $a^2 \in 2S$.*

Definition 2.4. *Let S be a unary semigroup. An element $a \in S$ is called a nil element of S if $a^2 = 2a = 0$.*

Definition 2.5. *A unary semigroup S is said to have even square property if every element of S is an even square element.*

Definition 2.6. *Let us call a unary semigroup S an even semigroup if $a \in 2S, \ \forall \ a \in S$.*

Definition 2.7. *Let us call a unary semigroup S an even square semigroup if $a^2 \in 2S, \ \forall \ a \in S$.*

Remark 2.8. *Every even semigroup is an even square semigroup however an even square semigroup is not necessarily an even semigroup.*

Definition 2.9. *Let us call a unary semigroup S a trivial even square semigroup if for each $a, b \in S$; $a^2 = b^2 = ab = ba = 0$. If a unary semigroup S is not a trivial even square semigroup then it is called a non-trivial even square semigroup.*

Remark 2.10. *The least order of a non-trivial even square semigroup is two. Let $S = \{a, b\}$. Define $a^2 = 2a = b$, $b^2 = b = 2a, ab = a$. Then S is a non-trivial even square semigroup of order two.*

Definition 2.11. *Let S be a unary semigroup. An element $a \in S$ is said to annihilate S if $ab = ba = 0, \ \forall \ b \in S$.*

Proposition 2.12. *The set of all even elements in a unary semigroup S forms an even square ideal of S.*

Proposition 2.13. *The set of all even square elements in a commutative unary semigroup S forms an even square ideal of S.*

Corollary 2.14. *Let a and b are any two even square elements of a commutative unary semigroup S then ab is an even square element.*

Proposition 2.15. *The set of all nil elements in a commutative unary semigroup S forms an even square ideal of S.*

Remark 2.16. *If S be an anticommutative unary semigroup then Propositions 2.2 and 2.3 hold good. If S is a noncommutative unary semigroup then the set of all nil elements in S forms an ideal of S provided each nil element annihilates S.*

Proposition 2.17. *If a finite commutative even square semigroup S contains a unique nonzero nil element a then a annihilates S.*

Proof. Let S be a finite commutative even square semigroup and $a \in S$ is a unique nonzero nil element of S. Let b is any element of S. Then $(ab)^2 = 0$ and $2ab = 0$. Therefore ab is a nil element of S. Hence, $ab = ba = 0$. □

Corollary 2.18. *Each nonzero element of S (in Proposition 2.17) is a zero divisor.*

Proposition 2.19. *Let S be an even square semigroup such that $a^2 = b^2 = 0$ for each $a, b \in S$ then S is not necessarily a trivial even square semigroup.*

Proposition 2.20. *Let S be a trivial even square semigroup of order n then it has at least one even square subsemigroup of order m for each $0 < m \leq n$.*

Proposition 2.21. *Let S be a trivial even square semigroup of order n then S has*

(1). one even square subsemigroup of order 1.

(2). $n-1$ *even square subsemigroups of order two.*

(3). $\frac{1}{2}(n-1)(n-2)$ *even square subsemigroups of order three.*

(4). $\frac{1}{6}(n-1)(n-2)(n-3)$ *even square subsemigroups of order four.*

(5). $\frac{1}{24}(n-1)(n-2)(n-3)(n-4)$ *even square subsemigroups of order five and so on.*

(6). total 2^{n-1} *even square subsemigroups and each subsemigroup is an ideal of* S*.*

Proof. Let $S = \{0 = a_1, a_2, ..., a_n\}$ be a trivial even square semigroup of order n. Let $S_1 = \{0\}$, then S_1 is an even square subsemigroup of S. Clearly there is only one even square subsemigroup of order one. Each non-zero element of S together with the zero element forms an even square subsemigroup of S. Thus there are $n-1$ even square subsemigroups of order two. As each even square subsemigroup of S contains the zero element therefore to form an even square subsemigroup of order three we have to select two elements out of $n-1$ elements and this can be done in $\frac{1}{2}(n-1)(n-2)$ ways. Thus there are $\frac{1}{2}(n-1)(n-2)$ even square subsemigroups of order three. Similarly there are $\frac{1}{6}(n-1)(n-2)(n-3)$ and $\frac{1}{24}(n-1)(n-2)(n-3)(n-4)$ even square subsemigroups of order four and five respectively. Proceeding similarly one can get the number of even square subsemigroups of order six and more. Hence there are total 2^{n-1} even square subsemigroups. It can be easily seen that each even square subsemigroup of S is an ideal of S. □

Proposition 2.22. *Let* S *be a finite even square semigroup of order* 2^t*,* t *is a positive integer. Then each element of* S *is nilpotent provided* S *contains the zero element and the reducing modulo* m *is a divisor of* 2^t *or* 2^{t+1}*.*

Proof. Let S be a finite even square semigroup of order 2^t, where t is a positive integer. Let $0 \in S$. If $t = 1$ then each element of S will be nil and therefore each element is a nilpotent element (see Example 2.25). Let $t > 1$ and $x \neq 0$ is any element of S and each element of S is non-nil. Then using the even square property we have $x^{2t} = 2^{t+1}y$ for some $y \in S$. Clearly, $2^{t+1} \equiv 0\,(mod m)$ if m divides 2^t or 2^{t+1}. Hence each nonzero element of S is a nilpotent element of index $\leq 2t$. □

Remark 2.23. *Proposition 2.22 holds for non-commutative even square semigroups also.* m *is a divisor of* 2^t *or* 2^{t+1} *is sufficient but not a necessary condition for each element of a finite even square semigroup of order* 2^t *to be nilpotent.*

Proposition 2.24. *If an even square semigroup* S *contains more than two nil elements then each nil element does not necessarily annihilate* S*.*

Example 2.25. *Let* $S = \{0, 2, 4\}$*.* S *is an even square semigroup under multiplication modulo* 8*. Clearly* $2 \in S$ *is an even square element of* S *but* 2 *is not an even element. However* 4 *is even as well as even square element. In this Example 4 is a nil element.*

Example 2.26. *Let* $Z = \{\cdots, -3 - 2, -1, 0, 1, 2, 3, \cdots\}$*. Then* Z *is a unary semigroup and* $2 \in Z$ *is an even as well as even square element of* Z*.*

Example 2.27. *Let* $E = \{\cdots, -4 - 2, 0, 2, 4, \cdots\}$*. Clearly* E *is the set of all even square elements of* Z*. The set* E *of even square integers is an even square semigroup of infinite order. For each* $m \neq 0, 1, -1$*;* $S = \{mx : x \in E\}$ *is a proper even square subsemigroup of* E*.*

Example 2.28. *Consider the set Q of all rational numbers. It is easy to see that Q is an even square semigroup under multiplication. Let $S_1 = \{0, 1\}, S_2 = \{0, 1, -1\}, S_3 = \{1, -1\}$. Clearly S_1, S_2 and S_3 all are subsemigroups of Q but none of these are even square semigroups. The set Z of integers is a non- even square subsemigroup of Q of infinite order. Thus even square property is not a hereditary property of semigroups.*

Example 2.29. *Let $S = \{0, a, b, c\}$. Let us define $a^2 = 2a = b, b^2 = 2b = 0, c^2 = b = 2a$, $ab = bc = 0$ and $ac = b$. Then S is a nontrivial even square semigroup of order 4. Each element of S is nilpotent and b annihilates S. b is a unique nonzero nil element of S.*

Example 2.30. *Let $S = \{0, a, b, c\}$. If we take $a^2 = b^2 = c^2 = 0$, $ab = c$, $ac = 0$, $bc = 0$, $2a = 2b = 2c = 0$. Then S is a nontrivial even square semigroup and each element of S is nil element and c annihilates S.*

Example 2.31. *Let $S = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \right\}$. Clearly S is an even square semigroup of order five under multiplication modulo 6 and $\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$ is a unique nonzero nil element of S and it annihilates S.*

Example 2.32. *The set of all rational, real and complex numbers possess the even square property however we introduce the set E_Q and E_R as the set of even square rational numbers and real numbers respectively. It may be noted that a non-empty set A is called an even square set if $a^2 \in 2A$, $\forall a \in A$. A set A possesses even square property iff it is an even square set. Let us define $E_Q = \left\{ \frac{p}{q} : (p, q) = 1, q \neq 0, p \in E, q \in Z \right\}$ and $E_R = E_Q \bigcup I_R$, where I_R is the set of those irrational numbers so that E_R possesses even square property. Here E and Z are the set of even square integers and integers respectively. Clearly $\sqrt{2} \notin E_R$ but $2\sqrt{2} \in E_R$.*
We define the set of even square complex numbers over E_Q and E_R as $E_C^Q = \{a + ib : a, b \in E_Q\}$ and $E_C^R = \{a + ib : a, b \in E_R\}$ respectively. It is easy to see that E_Q, E_R, E_C^Q and E_C^R provide examples of even square semigroups of infinite order.

3. Concluding Remarks

This article introduces the notion of even square semigroups and provides some results and examples of even square semigroups. However a further study is desirable to develop the possible theory of even square semigroups.

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